On the weighted degenerate Carlitz $q$-Bernoulli polynomials and numbers

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Abstract
In this paper, by using the $p$-adic $q$-integral on $\mathbb{Z}_p$ which was defined by Kim, we define the weighted Carlitz $q$-Bernoulli polynomials and investigate some identities of these polynomials. In particular, we define the weighted degenerate Carlitz’s $q$-Bernoulli polynomials and numbers and give some interesting properties that are associated with these numbers and polynomials.

AMS subject classification:
Keywords: $q$-Bernoulli polynomials, weighted degenerate $q$-Bernoulli polynomials, $p$-adic, $q$-integral.

1. Introduction

Let $p$ be a fixed prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{C}$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will, respectively, the ring of $p$-adic integers, the complex field, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The $p$-adic norm $| \cdot |_p$ is normalized by $|p|_p = \frac{1}{p}$. When one talks of $q$-extension, $q$ is variously considered as an determinate a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$.

If $q \in \mathbb{C}$, we assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we assume that $|q - 1|_p < p^{-p-1}$ so that $q^x = \exp(x \log q)$ for $|x|_p < 1$. We use the notation $[x]_q = \frac{1 - q^x}{1 - q}$. Let $UD(\mathbb{Z}_p)$ be the set of all uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x \quad (\text{see } [1]).$$  (1)
It is well-known that
\[ q I_q(f_1) - I_q(f) = (q - 1) f(0) + \frac{q - 1}{\log q} f'(0) \quad \text{(see [1]).} \tag{2} \]

The Carlitz’s \( q \)-Bernoulli polynomials can be represented by \( p \)-adic \( q \)-integral on \( \mathbb{Z}_p \) as follows:
\[ \int_{\mathbb{Z}_p} [x + y]^n q^d \mu_q(y) = \beta_{n,q}(x) \quad (n \geq 0) \quad \text{(see [1]).} \tag{3} \]

Thus, by (3), we get
\[ \int_{\mathbb{Z}_p} e^{x+y} q^d \mu_q(y) = \sum_{n=0}^{\infty} \beta_{n,q}(x) \frac{t^n}{n!} \quad \text{(see [1]).} \tag{4} \]

From (4), we can derive the following equation:
\[ \beta_{n,q}(x) = \frac{1}{(1-q)^n} \sum_{j=0}^{n} (-1)^j q^{\frac{j}{n}} \binom{n}{j} \frac{j + 1}{[j + 1]_q} (n \geq 0) \quad \text{(see [1]).} \tag{5} \]

In this paper, we define the weighted Carlitz’s \( q \)-Bernoulli polynomials and investigate some identities of these polynomials. In particular, we define the weighted degenerate Carlitz’s \( q \)-Bernoulli polynomials and numbers and give some interesting properties that are associated with these polynomials and numbers.

2. The weighted Carlitz’s \( q \)-Bernoulli numbers and polynomials

In this section, we assume that \(|\omega q - 1| \leq p^{-\frac{1}{p-1}}\). We define the weighted \( q \)-Bernoulli numbers as follows:
\[ \int_{\mathbb{Z}_p} \omega^x [x]^n_q d \mu_q(x) = B_{n,\omega,q}. \tag{6} \]

We observe that
\[
B_{n,\omega,q} = \int_{\mathbb{Z}_p} \omega^x [x]^n_q d \mu_q(x)
\]
\[
= \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{pN-1} \omega^x \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (q^{l+1})^x (-1)^l
\]
\[
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N \to \infty} \frac{1}{[pN]_q} \sum_{x=0}^{pN-1} (\omega q^{l+1})^x
\]
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\[
\begin{align*}
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \lim_{N \to \infty} \left( \frac{1-q}{1-q^{PN}} \right) \left( \frac{1-(\omega q^{l+1})^{pN}}{1-q^{l+1}} \right) \\
&= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1-q}{1-\omega q^{l+1}} \lim_{N \to \infty} \frac{1-(\log pN)^{pN}}{1-q^{pN}} \\
&= \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{1-q^{l+1}} \\
&\quad + \frac{1}{(1-q)^{n-1}} \log \omega \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1-\omega q^{l+1}}. \\
\end{align*}
\]

From (7), we obtain the following theorem:

**Theorem 2.1.** For $n \geq 0$, we have

\[
B_{n,\omega,q} = \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{1-\omega q^{l+1}} \\
+ \frac{1}{(1-q)^{n-1}} \log \omega \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{1}{1-\omega q^{l+1}}. \\
\]

We note that if $\omega = 1$, by (5) with $x = 0$ and (8), the we have

\[
B_{n,1,q} = \frac{1}{(1-q)^{n-1}} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{1-q^{l+1}} \\
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l \frac{l+1}{[l+1]_q} \\
= \beta_{n,q}. \\
\]

Now, we define the weighted Carlitz’s $q$-Bernoulli polynomials as follows:

\[
\int_{\mathbb{Z}_p} \omega^y e^{[x+y]q^t} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\omega,q}(x) \frac{t^n}{n!}, \\
\]

From (10), we have

\[
B_{n,\omega,q}(x) = \int_{\mathbb{Z}_p} \omega^y [x+y]_q^n d\mu_q(y). \\
\]

By (11), we get

\[
B_{n,\omega,q}(x) = \int_{\mathbb{Z}_p} \omega^y [x+y]_q^n d\mu_q(y) \\
\]
\[
B_{n,1,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} \omega^y q^y d\mu_q(y)
\]

From (12) and (5), we note that if \( \omega = 1 \), then we have

\[
B_{n,1,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} \omega^y q^y d\mu_q(y)
\]

\[
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} \omega^y q^y d\mu_q(y)
\]

\[
= \beta_{n,q}(x).
\]

From (12), we also obtain the following theorem:

Theorem 2.2. For \( n \geq 0 \), we have

\[
B_{n,\omega,q}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} \omega^y q^y d\mu_q(y)
\]

\[
= \frac{1}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^l x \int_{\mathbb{Z}_p} \omega^y q^y d\mu_q(y)
\]

\[
= \beta_{n,q}(x).
\]

We note that

\[
[x+y]^n_q = \left( [x]_q + q^x [y]_q \right)^n_q
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [y]_q^l [x]_q^{n-l}.
\]

From (11) and (15), we have

\[
B_{n,\omega,q}(x) = \int_{\mathbb{Z}_p} \omega^y [x+y]^n_q d\mu_q(y)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} \int_{\mathbb{Z}_p} \omega^y [y]_q^l d\mu_q(y)
\]

\[
= \sum_{l=0}^{n} \binom{n}{l} q^{lx} [x]_q^{n-l} B_{l,\omega,q}.\]
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By (16), we obtain the following theorem:

**Theorem 2.3.** For \( n \geq 0 \), we have

\[
B_{n,\omega,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{l\lambda} [x]_q^{n-l} B_{l,\omega,q}. \tag{17}
\]

3. The weighted degenerate Carlitz’s \( q \)-Bernoulli polynomials and numbers

In this section, we assume that \( \lambda, t \in \mathbb{C}_p \) with \( 0 < |\lambda|_p \leq 1, |t|_p < p^{-\frac{1}{p-1}} \). Then, as \( |\lambda t|_p < p^{-\frac{1}{p-1}} \), \( |\log(1 + \lambda t)|_p = |\lambda t|_p \) and hence \( \frac{1}{\lambda} \log(1 + \lambda t)|_p = |t|_p < 1 \).

In the viewpoint of (11), we define the weighted degenerate Carlitz’s \( q \)-Bernoulli polynomials which are given by the generating function to be

\[
\int_{\mathbb{Z}_p} \omega^y (1 + \lambda t)^{\left[ x+y \right]_q} d\mu_q(y) = \sum_{n=0}^{\infty} B_{n,\omega,q}(x|\lambda) t^n. \tag{18}
\]

When \( x = 0 \), \( B_{n,\omega,q}(\lambda) = B_{n,\omega,q}(0|\lambda) \) are called the weighted degenerate Carlitz’s \( q \)-Bernoulli numbers.

We observe that

\[
\int_{\mathbb{Z}_p} \omega^y (1 + \lambda t)^{\left[ x+y \right]_q} d\mu_q(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \omega^y \left( \frac{\left[ x+y \right]_q}{\lambda} \right)^n d\mu_q(y) \lambda^n t^n
\]

\[
= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \omega^y \left( \frac{\left[ x+y \right]_q}{\lambda} \right)^n d\mu_q(y) \lambda^n t^n. \tag{19}
\]

where \( \left( \frac{\left[ x+y \right]_q}{\lambda} \right)^n = \left[ x+y \right]_q \times \left( \frac{\left[ x+y \right]_q}{\lambda} - 1 \right) \times \cdots \times \left( \frac{\left[ x+y \right]_q}{\lambda} - n + 1 \right) \).

We remark that we define \( [x+y]_{n,\lambda} \) as \( [x+y]_{0,\lambda} = 1, [x+y]_{n,\lambda} = [x+y]_q ([x+y]_q - \lambda) \cdots ([x+y]_q - (n-1)\lambda) \) \( (n \geq 1) \).

From (19), we obtain the following theorem.

**Theorem 3.1.** For \( n \geq 0 \), we have

\[
\int_{\mathbb{Z}_p} \omega^y \left( \frac{\left[ x+y \right]_q}{\lambda} \right)^n d\mu_q(y) = \int_{\mathbb{Z}_p} \omega^y [x+y]_{n,\lambda} d\mu_q(y) = B_{n,\omega,q}(x|\lambda). \tag{20}
\]

Let \( S_1(n, m) \) be the Stirling numbers of the first kind which are defined by

\[
(x)_n = \sum_{l=0}^{n} S_1(n, l)x^l, \quad (n \geq 0). \tag{21}
\]
From (20) and (21), we get
\[
\int Z_p \omega^y \left( \frac{[x + y]_q}{\lambda} \right)_m \lambda^m d \mu_q(y) = \sum_{l=0}^{m} S_1(m, l) \lambda^{m-l} \int Z_p \omega^y [x + y]^l_q d \mu_q(y) \\
= \sum_{l=0}^{m} S_1(m, l) \lambda^{m-l} B_{l, o, q}(x). \tag{22}
\]

Therefore, by (20) and (21), we obtain the following theorem.

**Theorem 3.2.** For \( n \geq 0 \), we have
\[
B_{n, o, q}(x|\lambda) = \sum_{l=0}^{n} S_1(n, l) \lambda^{n-l} B_{l, o, q}(x). \tag{23}
\]

From (23), we note that \( \lim_{\lambda \to 0} B_{n, o, q}(x|\lambda) = B_{n, o, q}(x|\lambda) \) and \( \lim_{|\omega| \to 1} B_{n, o, q}(x|\lambda) = \beta_{n, q}(x|\lambda) \), where \( \beta_{n, q}(x|\lambda) \) are called the degenerate Carlitz’s \( q \)-Bernoulli polynomials defined by
\[
\int Z_p \omega^y \left( 1 + \lambda t \right) \frac{[x + y]_q}{\lambda} d \mu_q(y) = \sum_{n=0}^{\infty} \beta_{n, q}(x|\lambda) \frac{t^n}{n!}. \tag{24}
\]

By (17) and (23), we obtain the following theorem.

**Theorem 3.3.** For \( n \geq 0 \), we have
\[
B_{n, o, q}(x|\lambda) = \sum_{l=0}^{n} \sum_{j=0}^{l} S_1(n, l) \lambda^{n-l} (l^j_q [x]^l_q)^{j-l} B_{j, o, q}. \tag{25}
\]

We observe that
\[
\omega^y \left( 1 + \lambda t \right) \frac{[x + y]_q}{\lambda} = \omega^y e^{\log(1+\lambda t)} \frac{[x + y]_q}{\lambda} n \frac{1}{n!} (\log(1+\lambda t))^n \\
= \sum_{n=0}^{\infty} \omega^y \left( \frac{[x + y]_q}{\lambda} \right)^n \frac{1}{n!} (\log(1+\lambda t))^n \\
= \sum_{m=0}^{\infty} \omega^y \left( \frac{[x + y]_q}{\lambda} \right)^m \frac{1}{m!} m! \sum_{n=m}^{\infty} S_1(n, m) \frac{\lambda^n t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \omega^y [x + y]^m_q \right) \frac{t^n}{n!}. \tag{26}
\]

By (26), we have
\[
\int Z_p \omega^y \left( 1 + \lambda t \right) \frac{[x + y]_q}{\lambda} d \mu_q(y) = \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) \int Z_p \omega^y [x + y]^m_q d \mu_q \right) \frac{t^n}{n!}.
\]
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\[= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) B_{m,\omega,q}(x) \right) \frac{t^n}{n!}. \quad (27)\]

From (27), we obtain the following theorem.

**Theorem 3.4.** For $n \geq 0$, we have

\[B_{n,\omega,q}(x|\lambda) = \sum_{m=0}^{n} \lambda^{n-m} S_1(n, m) B_{m,\omega,q}(x), \quad (28)\]

where $S_1(n, m)$ is the Stirling numbers of first kind.

Replacing $t$ by $\frac{1}{\lambda} (e^{\lambda t} - 1)$ in (18), we get

\[
\int_{\mathbb{Z}_p} \omega^y e^{[x+y]_{q^\lambda}} d\mu_q(y) = \sum_{m=0}^{\infty} B_{m,\omega,q}(x|\lambda) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m \\
= \sum_{m=0}^{\infty} B_{m,\omega,q}(x|\lambda) \lambda^{-m} \sum_{n=m}^{\infty} S_2(n, m) \frac{\lambda^n t^n}{n!} \\
= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} B_{m,\omega,q}(x|\lambda) \lambda^{n-m} S_2(n, m) \right) \frac{t^n}{n!}. \quad (29)\]

where $S_2(n, m)$ are the Stirling numbers of the second kind.

We note that the left hand side of (29) is given by

\[
\int_{\mathbb{Z}_p} \omega^y e^{[x+y]_{q^\lambda}} d\mu_q(y) = \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \omega^y [x+y]_{q^\lambda} d\mu_q(y) \frac{t^n}{n!} \\
= \sum_{n=0}^{\infty} B_{n,\omega,q}(x) \frac{t^n}{n!}. \quad (30)\]

Therefore, by (29) and (30), we obtain the following theorem.

**Theorem 3.5.** For $n \geq 0$, we have

\[B_{n,\omega,q}(x) = \sum_{m=0}^{n} B_{m,\omega,q}(x|\lambda) \lambda^{n-m} S_2(n, m). \quad (31)\]

Note that

\[\omega^y (1 + \lambda t)^{[x+y]_{q^\lambda}} = \omega^y (1 + \lambda t)^{[x]_{q^\lambda}} (1 + \lambda t)^{[y]_{q^\lambda}} \]

\[= \left( \sum_{m=0}^{\infty} [x]_{m,\lambda} \frac{t^m}{m!} \right) e^{[y]_{q^\lambda}} \log(1 + \lambda t) \]
\[
\begin{align*}
= & \left(\sum_{m=0}^{\infty} \frac{x_m \lambda^m}{m!}\right) \left(\sum_{l=0}^{\infty} \frac{q^{lx} \omega^y[y]_q^l}{\lambda^l} \frac{(\log(1+\lambda l))^l}{l!}\right) \\
= & \left(\sum_{m=0}^{\infty} \frac{x_m \lambda^m}{m!}\right) \left(\sum_{l=0}^{\infty} \frac{q^{lx} \omega^y[y]_q^l}{\lambda^l} \frac{1}{l!} \sum_{k=l}^{\infty} S_1(k, l) \frac{t^k}{k!}\right) \\
= & \left(\sum_{m=0}^{\infty} \frac{x_m \lambda^m}{m!}\right) \left(\sum_{l=0}^{\infty} \sum_{k=0}^{k} \frac{\lambda^{k-l} q^{lx} \omega^y[y]_q^l S_1(k, l)}{k!} \frac{t^k}{k!}\right) \\
= & \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x_{n-k} \lambda^{k-l} q^{lx} \omega^y[y]_q^l S_1(k, l)}{n!} \left(\frac{n}{k}\right) \frac{t^n}{n!}\right). \tag{32}
\end{align*}
\]

Therefore, by (32), we obtain the following theorem.

**Theorem 3.6.** For \(n \geq 0\), we have

\[
B_{n, \omega, q}(x|\lambda) = \sum_{k=0}^{n} \sum_{l=0}^{k} \binom{n}{k} [x]_{n-k, \lambda^{k-l} q^{lx} S_1(k, l) B_{l, \omega, q}}. \tag{33}
\]

Now, we observe that

\[
B_{n, \omega, q}(x|\lambda) = \sum_{l=0}^{\infty} S_1(n, l) \lambda^{n-l} \int_{\mathbb{Z}_p} \omega^y[x + y]_q^l d\mu_q(y) \tag{34}
\]

and

\[
\int_{\mathbb{Z}_p} \omega^y[x + y]_q^l d\mu_q(y) = \frac{1}{[m]_q} \sum_{i=0}^{m-1} q^i[m]_q^i \int_{\mathbb{Z}_p} \omega^y \left[\frac{x + i}{m} + y\right]_q^l d\mu_q^m(y) \\
= [m]_q^{l-1} \sum_{i=0}^{m-1} q^i B_{l, \omega, q}^m \left(\frac{x + i}{m}\right), \tag{35}
\]

where \(l \in \mathbb{N} \cup \{0\}\) and \(m \in \mathbb{N}\).

By (34) and (35), we obtain the following theorem.

**Theorem 3.7.** For \(n \geq 0, m \in \mathbb{N}\), we have

\[
B_{m, \omega, q}(x|\lambda) = \sum_{l=0}^{\infty} \sum_{i=0}^{m-1} S_1(n, l) \lambda^{n-l} \left[m]_q^i q^i B_{l, \omega, q}^m \left(\frac{x + i}{m}\right)\right]. \tag{36}
\]

**References**


