Invariant Sets for non Classical Reaction-Diffusion Systems

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Abstract

The question discussed in this study concerns the invariant sets for a system of coupled equations for a binary mixtures of isotropic and homogeneous heat conductors. Conditions are given for the existence of an invariant set in the first quadrant of the phase plan. Furthermore, we show that the positively invariant set is not only an attracting set to $\mathbb{R}^2_+$ but it is also an absorbing set in $\mathbb{R}^2_+$.

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1. Introduction

The study of invariant sets (or invariant regions) for partial differential equations systems was extensively developed in the literature, see for example in [3, 4, 5, 7, 16, 18].

The question on the existence of invariant sets for reaction-diffusion systems is the subject of considerable interest in the study of evolution problems see [17]. The existence of such regions is a very powerful tool, which is used to obtain some interesting properties of the solutions, like their global existence [8, 13, 21], their positivity [2], and some other qualitative properties. Our aim in this study is to treat the same subject but

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for non classical type of dynamical systems which come from the area of thermodynamics in mixtures, and especially we care for coupled equations for a binary mixture of isotropic and homogeneous heat conductors. For the background literature and recent developments we refer for example to [1, 9, 10, 12, 15] and the references cited therein.

In [6] Gurtin and De la Penha introduced the following system of partial differential equations, governing the heat flow through a mixture

\[
(S) \quad \begin{align*}
    a_{11} \partial_t u(t, x) + a_{12} \partial_t v(t, x) &= k_1 \Delta u(t, x) + f(u(t, x), v(t, x)) \\
    a_{21} \partial_t u(t, x) + a_{22} \partial_t v(t, x) &= k_2 \Delta v(t, x) + g(u(t, x), v(t, x))
\end{align*}
\]

for \( t > 0 \) and \( x \) in a bounded domain \( \Omega \) in \( \mathbb{R}^n \) \((n = 1, 2 \text{ or } 3)\), with a smooth boundary \( \partial \Omega \). Here \( u \) and \( v \) are the temperature variations of the constituents, \( \Delta \) is the laplacian operator (the diffusion part), the entries of the matrix \( (a_{ij}) \) and \( k_1, k_2 \) are the specific heats and the thermal conductivities of the constituents (the reaction part), \( f \) and \( g \) are the functions of heat transfer between constituents.

The system \((S)\) is considered with the Neumann boundary conditions

\[
\frac{\partial u(t, x)}{\partial \nu} = \frac{\partial v(t, x)}{\partial \nu} = 0 \quad \text{for} \quad t > 0 \quad \text{and} \quad x \in \partial \Omega,
\]

and the initial conditions

\[
u(0, x) = u_0(x) \quad \text{and} \quad v(0, x) = v_0(x) \quad \text{and} \quad x \in \Omega.
\]

The results on existence and uniqueness of the solutions for \((S)\) are obtained in [15]. For reliability of the model proposed through \((S)\) we have to care about the non negativity of the solutions, since these constraints are imposed by certain physical considerations. For this purpose we present in our study, a mathematical analysis based on the well-known characterization of flow invariance by means of tangential conditions. Conditions are given for the existence of invariant set noted \( \Sigma \) in the first quadrant of the \((u, v)\)-plan. Furthermore we show that \( \Sigma \) is not only a positively invariant but it is also an attracting set in \( \mathbb{R}^2_+ \). We deduce that any neighborhood of \( \Sigma \) is an absorbing set, that is any trajectory of system \((S)\) starting within \( \mathbb{R}^2_+ \) but outside \( \Sigma \) will enter into a neighborhood of \( \Sigma \) in a finite time. Moreover we show that under certain conditions, \( \Sigma \) is an absorbing set in \( \mathbb{R}^2_+ \).

The rest of the paper is organized as follows. In the next section, we reformulate the previous system and we give assumptions and transformations for the existence of invariant sets in the positive quadrant of the phase plane. In section 3 we determine an invariant set with the linear reaction term \( F \) for which the study of the existence of positive solution is made in [12]. In section 4 we prove that \( \Sigma \) is not only an attractive set in \( \mathbb{R}^2_+ \) but also it is an absorbing set in \( \mathbb{R}^2_+ \). Finally, in section 5 we give an example which illustrates our results.
2. Preliminaries

By use of the notation (see [14])

\[ M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad U(t, x) = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \]

and

\[ k_{1}\Delta U(t, x) = \begin{pmatrix} k_{1}\Delta u(t, x) \\ k_{2}\Delta v(t, x) \end{pmatrix}, \quad F(U(t, x)) = \begin{pmatrix} f(u(t, x), v(t, x)) \\ g(u(t, x), v(t, x)) \end{pmatrix} \]

with \((t, x) \in [0, T] \times \Omega\), where \(T > 0\) and \(\Omega\) is a bounded domain of \(\mathbb{R}^n\) \((n = 2\) or \(3))\) with a smooth boundary \(\partial \Omega\), the system \((S)\) can be written as

\[ M(\partial_t U(t, x)) = k_{1}\Delta U(t, x) + F(U(t, x)). \]

If we assume that the entries of matrix \(M\) satisfies the following conditions

\[(C_g) \quad 0 < a_{22}a_{11} - a_{12}a_{21} < \frac{1}{4}(a_{11} + a_{22})^2, \quad a_{22} - |a_{11}| > 0, \quad \text{and} \quad a_{12}a_{21} < 0, \]

then \(M\) is a defined positive matrix. So it has two real non-negative eigenvalues \(\lambda_1\) and \(\lambda_2\), and we are allowed to rewrite the system \((S)\) as

\[(S_1) \quad \partial_t U(t, x) = M^{-1}(k_{1}\Delta U(t, x)) + M^{-1} \circ F(U(t, x)), \]

where \(M^{-1}\) is the inverse matrix of \(M\).

By the following transformations we obtain a simpler formulation of system \((S_1)\). Let \(d = a_{22}a_{11} - a_{12}a_{21}\), and \(k_1, k_2\) be positive real numbers.

For \((t, x) \in [0, T] \times \Omega\), we define \(W(t, x) = (w_1(t, x), w_2(t, x))\) by

\[ w_1(t, x) = u(dt, \sqrt{k_1}x), \quad \text{and} \quad w_2(t, x) = v(dt, \sqrt{k_2}x), \]

where \(U(t, x) = (u(t, x), v(t, x))\) is a solution of \((S)\), and therefore the system \((S_1)\) is transformed to

\[
\begin{align*}
\partial_t w_1 &= a_{22}\Delta w_1 - a_{12}\Delta w_2 + a_{22}f(w_1, w_2) - a_{12}g(w_1, w_2) \\
\partial_t w_2 &= -a_{21}\Delta w_1 + a_{11}\Delta w_2 - a_{21}f(w_1, w_2) + a_{11}g(w_1, w_2)
\end{align*}
\]

which can be written more compactly as

\[(S_2) \quad \partial_t W(t, x) = E(\Delta W(t, x)) + E \circ F(W(t, x)), \]

where \(E\) is the matrix defined by \(dM^{-1}\).
Remark 2.1.

1) Under conditions \((C_g)\) the matrix \(E\) has two left eigenvectors \(v_1 = (A_1, 1)\) and \(v_2 = (A_2, -1)\) corresponding respectively to the eigenvalues \(\lambda_1\) and \(\lambda_2\) with

\[
A_1 = \frac{1}{2} \frac{a_{11} - a_{22} - \delta}{a_{12}} \quad \text{and} \quad A_2 = \frac{1}{2} \frac{-a_{11} + a_{22} - \delta}{a_{12}}
\]

where

\[
\delta = \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}
\]

2) It is important to note here, that because of conditions \((C_g)\), \(\lambda_1\) and \(\lambda_2\) are both positive while \(A_1 < -1\) and \(0 < A_2 < 1\).

3. Invariant sets and positivity of the solutions in the linear case

Some sufficient conditions which assure the existence of positive solutions for system \((S_2)\) when \(F\) is a linear map are given in [8]. By the use of the tangential conditions (see [11]), we determine a (positively) invariant set in the positive quadrant of \((w_1, w_2)\)-plan.

Definition 3.1. A subset \(\Sigma_w \subset \mathbb{R}^2\) is called a positively invariant set for the system \((S_2)\) if any solution \(W(t, x)\) with initial and boundary values in \(\Sigma_w\) satisfies \(W(t, x) \in \Sigma_w\) for all \((t, x) \in [0, T] \times \Omega\) (where \(T\) is the maximal time of existence of \(W(t, x)\)).

In the case of the reaction term given in [6], for which the existence has been studied in [15], we obtain the invariant set given by the following proposition

Proposition 3.2. If the conditions \((C_g)\) are satisfied and

\[
F(w_1, w_2) = (-c(w_1 - w_2), c(w_1 - w_2))
\]

such that \(c > 0\), the set defined by

\[
\Sigma_w = \{(w_1, w_2) \in \mathbb{R}^2, \text{ such that } w_1 \geq 0 \text{ and } A_2 w_1 \leq w_2 \leq -A_1 w_1\}
\]

is an invariant set for \((S_2)\).

Proof. Let \(G_1\) and \(G_2\) be two smooth real functions defined on \(\mathbb{R}^2\) as follows

\[
\forall W = (w_1, w_2) \in \mathbb{R}^2, \quad G_1(W) = A_1 w_1 + w_2 \quad \text{and} \quad G_2(W) = A_2 w_1 - w_2.
\]

It is easy to see that \(G_1\) and \(G_2\) satisfy the conditions of theorem 4.3 in [2]. Thus, the set which is equal exactly to \(\Sigma_w\), and given by

\[
\{W = (w_1, w_2) \in \mathbb{R}^2, \text{ such that } w_1 \geq 0 \text{ and } G_i(W) \leq 0, \quad i = 1, 2\}
\]

is an invariant set for \((S_2)\).
Definition 3.3. It is said that $F(w_1, w_2)$ is quasi-positive if it satisfies for all $(w_1, w_2) \in \mathbb{R}^2_+ : F_1(0, w_2) \geq 0 \text{ or } F_2(w_1, 0) \geq 0$, where $F_1$ and $F_2$ are the components of $F$.

At first, it is important to point out here, that in spite of the quasi-positivity of the reaction term $F(w_1, w_2)$, the positively invariant set $\Sigma_w$ that we obtained, is strictly included in the first quadrant of the phase plan, which is not the case for the classical systems like in [11]. Furthermore we show that it is not only a positively invariant set, but it is also an attracting set in $\mathbb{R}^2_+$.

Indeed by the help of the maximum principle (via comparison theorem), we prove that all trajectories $W(t) = (w_1(t, x), w_2(t, x))$ which are in the positive quadrant of the $(w_1, w_2)$-plan are attracted by the invariant set $\Sigma_w$.

Definition 3.4. We say that the set $\Sigma_w \subset \mathbb{R}^2_+$ is an attractor if it satisfies

1) $\Sigma_w$ is an invariant set.

2) $\Sigma_w$ admits an open neighborhood $O$ such that, for any solution $W(t)$ of $(S_2)$ with $W(0) \in O$;

$$\text{dist}(W(t), \Sigma_w) \rightarrow 0 \text{ when } t \rightarrow +\infty.$$  

Here $\text{dist}(u, L)$ means the distance between the point $u$ and the set $L$.

Proposition 3.5. Let us assume that assumptions of Proposition 3.2 are satisfied. If $W(t)$ is a positive global solution of $(S_2)$ with initial condition

$$W(0) = (w_1(0, x), w_2(0, x)) \in \mathbb{R}^2_+, \forall x \in \Omega,$$

then it is attracted by the set $\Sigma_w$.

Proof. Under the assumptions of Proposition 3.2, if we take the dot product of $(S_2)$ by the left eigenvector $v_1$ we have

$$\left(v_1, \frac{\partial W}{\partial t}\right) = \lambda_1 \langle v_1, \Delta W \rangle + \lambda_1 \langle v_1, F(W) \rangle.$$

If we take $G_1 = A_1 w_1 + w_2 > 0$, we obtain

$$\frac{\partial G_1}{\partial t} = \lambda_1 G_{1xx} + \lambda_1 (c A_1 (w_1 - w_2) + c (w_1 - w_2))$$

$$= \lambda_1 G_{1xx} + \lambda_1 c (-G_1 + A_1 w_2 + w_1)$$

$$\leq \lambda_1 G_{1xx} + \lambda_1 c (-w_2 - A_1 w_1)$$

$$= \lambda_1 G_{1xx} + \lambda_1 c (-2G_1)$$

thus,

$$\frac{\partial G_1}{\partial t} - \lambda_1 G_{1xx} + 2\lambda_1 c G_1 \leq 0.$$
So if \( J_1(t,x) = G_{10}e^{-2\lambda_1ct} \) with \( G_{10} = \max_{x \in \Omega} G_1(0,x) \) we have

\[
\frac{\partial J_1}{\partial t} = \lambda_1 J_{1,xx} - 2\lambda_1 c G_{10} e^{-2\lambda_1ct} = \lambda_1 J_{1,xx} - 2\lambda_1 c J_1.
\]

Then

\[
\begin{cases}
\frac{\partial G_1}{\partial t} - \lambda_1 G_{1,xx} + 2\lambda_1 c G_1 \leq 0 = \frac{\partial J_1}{\partial t} - \lambda_1 J_{1,xx} + 2\lambda_1 c J_1 \\
\frac{\partial G_1}{\partial v} = 0 = \frac{\partial J_1}{\partial v} \\
G_1(0,x) \leq G_{10} = J_1(0,x)
\end{cases}
\]

Then according to the theorem 10.1 in [19] we can say that

\( G_1(t,x) \leq J_1(t,x) = G_{10}e^{-2\lambda_1ct} \).

In other words \( G_1(t,x) \) converge to 0 when \( t \) tends to \( +\infty \);

\[
G_1(t,x) = A_1 w_1(t,x) + w_2(t,x) \lim_{t \to +\infty} 0 \quad (3.1)
\]

In the same way, taking the dot product of the system \( (S_2) \) by the left eigenvector \( v_2 \), if we assume that \( G_2 = A_2 w_1 - w_2 > 0 \), we obtain a similar result for \( G_2 \)

\[
G_2(t,x) = A_2 w_1(t,x) - w_2(t,x) \lim_{t \to +\infty} 0. \quad (3.2)
\]

Finally, (3.1) and (3.2) imply that \( \Sigma_w \) is an attractive set to \( \mathbb{R}^2_+ \).

According to the compactness of the trajectories, and taking the result of Proposition 3.5 into account, we show that the set \( \Sigma_w \) has some absorbing neighborhoods.

## 4. Absorbing set

The positively invariant set \( \Sigma_w \) is not only an attracting set in \( \mathbb{R}^2_+ \), but we show that it is also an absorbing set in \( \mathbb{R}^2_+ \).

**Definition 4.1.** Let \( \Sigma_w \subset \mathbb{R}^2_+ \) and \( O \) an open set containing \( \Sigma_w \). We say that \( \Sigma_w \) is an absorbing set in \( O \), if the trajectory of all solution of \( (S_2) \) enters in \( \Sigma_w \) after some finite time.

A direct consequence of Proposition 3.5 is that any open neighborhood of the attracting set \( \Sigma_w \) is an absorbing set.

**Corollary 4.2.** Under assumptions of Proposition 3.5, and for all \( \epsilon > 0 \) (small enough), the set defined by

\[
\Sigma_\epsilon = \{(w_1, w_2) \in \mathbb{R}^2, \text{ such that } w_1 \geq 0, \ w_2 \geq 0 \text{ and } A_2 w_1 - \epsilon < w_2 < -A_1 w_1 + \epsilon\}
\]
is an absorbing set in $\mathbb{R}^2_+$. 

**Proof.** Under assumptions of Proposition 3.5, $\Sigma_w$ is an attracting set in $\mathbb{R}^2_+$, so if we note $W (t, x) = (w_1(t, x), w_2(t, x))$ such that $w_1 > 0$ and $w_2 + A_1 w_1 \geq 0$, we have

$$\text{dist} (W (t, x), \Sigma_w) \rightarrow 0 \quad \text{uniformly for } x \in \Omega.$$ 

Therefore

$$\forall \epsilon > 0, \exists T^* > 0, \text{ such that } \forall t > T^* : \inf_{\sigma_w \in \Sigma_w} \| W (t, x) - \sigma_w \| < \frac{\epsilon}{1 - A_1}, \quad \forall x \in \Omega.$$ 

Then, there exists $\sigma^*_w = (\sigma^*_1, \sigma^*_2) = (\sigma^*_1, -A_1 \sigma^*_1)$, such that

$$\| W (t, x) - \sigma^*_w \| < \frac{\epsilon}{1 - A_1}.$$ 

So if we choose $\| . \|_1$ as a norm in $\mathbb{R}^2$ (all the norms are equivalent in $\mathbb{R}^2$) we have

$$\| W (t, x) - \sigma^*_w \|_1 = \|(w_1, w_2) - (\sigma^*_1, -A_1 \sigma^*_1)\|_1$$

$$= |w_1 - \sigma^*_1| + |w_2 + A_1 \sigma^*_1| < \frac{\epsilon}{1 - A_1}$$

$$\Rightarrow \left\{ \begin{array}{c}
|w_1 - \sigma^*_1| \leq \frac{\epsilon}{1 - A_1} \\
|w_2 + A_1 \sigma^*_1| \leq \frac{\epsilon}{1 - A_1}
\end{array} \right.$$ 

$$\Rightarrow \left\{ \begin{array}{c}
-\frac{\epsilon}{1 - A_1} \leq w_1 - \sigma^*_1 \\
w_2 + A_1 \sigma^*_1 \leq \frac{\epsilon}{1 - A_1}
\end{array} \right.$$ 

$$\Rightarrow \left\{ \begin{array}{c}
-A_1 \sigma^*_1 \leq -A_1 w_1 - A_1 \frac{\epsilon}{1 - A_1} \\
w_2 \leq \frac{\epsilon}{1 - A_1} - A_1 \sigma^*_1
\end{array} \right.$$ 

$$\Rightarrow w_2 \leq -A_1 w_1 - A_1 \frac{\epsilon}{1 - A_1} + \frac{\epsilon}{1 - A_1}$$

$$= -A_1 w_1 + \frac{\epsilon}{1 - A_1} (1 - A_1) = -A_1 w_1 + \epsilon$$

Then

$$\forall \epsilon > 0, \exists T^* > 0, \text{ such that } \forall t > T^* : W (t, x) \in \Sigma_\epsilon.$$ 

So, $\Sigma_\epsilon$ is an absorbing set in $\mathbb{R}^2_+$. 

In the next, we give our main result, proving that any trajectory

$$W (t) = (w_1 (t, x), w_2 (t, x))$$
Lemma 4.3. Let
\[ \Phi(t) = (\Phi_1(t), \Phi_2(t)) = \left( \int_{\Omega} w_1(t, x) \varphi(x) \, dx, \int_{\Omega} w_2(t, x) \varphi(x) \, dx \right), \]
where \( \varphi(x) \) is the eigenfunction corresponding to the principal eigenvalue \( \lambda \) of the laplacian (with Neumann boundary conditions). Then \( \Phi \) is a solution of the following system of ordinary differential equations
\[
A = \begin{pmatrix}
-(\lambda a_{22} + c_1) & (\lambda a_{12} + c_1) \\
(\lambda a_{21} + c_2) & -(\lambda a_{11} + c_2)
\end{pmatrix},
\]
c_1 = c(a_{22} + a_{12}) and c_2 = c(a_{11} + a_{21}). Furthermore
\[
\| \Phi(t) \| \geq \| \Phi_0 \| \, e^{-\mu(-A)t}, \quad \forall t \geq 0,
\]
where \( \mu(-A) \) is the matrix measure of \(-A\).

Proof. Let \( \varphi \) be the solution of
\[
\left\{ \begin{array}{l}
-\varphi_{xx}(x) = \lambda \varphi(x), \quad \varphi(x) > 0, \quad x \in \Omega, \\
\frac{\partial \varphi(x)}{\partial \nu} = 0, \quad x \in \partial \Omega.
\end{array} \right.
\]
According to (S2) we have
\[
\left\{ \begin{array}{l}
\frac{d\Phi_1}{dt} = -\lambda a_{22} \Phi_1 + \lambda a_{12} \Phi_2 + a_{22}c(\Phi_2 - \Phi_1) - a_{12}c(\Phi_1 - \Phi_2), \\
\frac{d\Phi_2}{dt} = \lambda a_{21} \Phi_1 - \lambda a_{11} \Phi_2 - a_{21}c(\Phi_2 - \Phi_1) + a_{11}c(\Phi_1 - \Phi_2)
\end{array} \right. \quad (4.3)
\]
\[
\Rightarrow \frac{d\Phi(t)}{dt} = A(\Phi(t)) \quad \text{and} \quad \Phi(0) = \Phi_0.
\]
Thus, by the Gronwall and Langenhop inequalities ([20] page 47) we obtain
\[
\| \Phi(t) \| \geq \| \Phi(0) \| \, e^{-\mu (-A)t}, \quad \forall t \geq 0.
\]

Lemma 4.4. If we denote by \( \eta := \lambda_1(\lambda + c) \) and we assume that the entries of matrix \( M \) satisfies conditions \((C_g)\) and \( a_{12} + a_{21} \geq 0 \) then \( \eta - \mu < 0 \) \((\mu = \mu(-A))\).

Proof. Since the norms are equivalent in \( \mathbb{R}^2 \), if we choose \( \| . \|_2 \) defined by
\[
\| \Phi(t) \|_2 = \left( \sum_{i=1}^{2} |\Phi_i(t)|^2 \right)^{\frac{1}{2}},
\]
the matrix measure is given by
\[ \mu(A) = \frac{\beta_{\text{max}}(A + A^t)}{2}, \]
such that \(A^t\) is the transpose of the matrix \(A\), and \(\beta_{\text{max}}(A + A^t)\) is the maximum of the eigenvalues of the matrix \(A + A^t\).

Let
\[ B = -A = \begin{pmatrix} \lambda a_{22} + c_1 & - (\lambda a_{12} + c_1) \\ - (\lambda a_{21} + c_2) & \lambda a_{11} + c_2 \end{pmatrix}. \]
Firstly, we calculate the eigenvalues of \(B + B^t\) which are the roots of the characteristic polynomial
\[ P(\beta) = \beta^2 - 2\beta \left[ \lambda (a_{11} + a_{22}) + c_1 + c_2 \right] \\
+ 4 (\lambda a_{22} + c_1) (\lambda a_{11} + c_2) - \left[ \lambda (a_{12} + a_{21}) + c_1 + c_2 \right]^2. \]

According to the conditions \((C_g)\)
\[ (a_{11} - a_{22})^2 + (a_{12} + a_{21})^2 > 0, \]
therefore
\[ 2c_1c_2 \left[ (a_{12} - a_{21})^2 - (a_{11} - a_{22})^2 \right] + 2 (a_{11} - a_{22}) (a_{12} + a_{21}) (c_2^2 - c_1^2) < 0, \]
and hence the discriminant \(\Delta'\) of \(P\) is positive. Then the matrix \(B + B^t\) admits two real eigenvalues.

It follows that
\[ \mu(-A) = \frac{1}{2} \max \left\{ \lambda(a_{11} + a_{22}) + c_1 + c_2 + \sqrt{\Delta'}, \lambda(a_{11} + a_{22}) + c_1 + c_2 - \sqrt{\Delta'} \right\}, \]
where
\[ \Delta' = \lambda^2 \left[ (a_{11} - a_{22})^2 + (a_{12} + a_{21})^2 \right] \\
+ 2\lambda \left[ (a_{11} - a_{22}) (c_2 - c_1) + (a_{12} + a_{21}) (c_2 + c_1) \right] + c_1^2 + c_2^2. \]
The assumption \(a_{12} + a_{21} \geq 0\) leads to
\[ \mu(-A) = \frac{1}{2} \left( \lambda(a_{11} + a_{22}) + c_1 + c_2 + \sqrt{\Delta} \right). \]
Finally,
\[ \eta - \mu = \lambda (\lambda + c) - \frac{1}{2} \left[ \lambda(a_{11} + a_{22}) + c_1 + c_2 - \sqrt{\Delta} \right] \\
= \frac{1}{2} \left[ a_{11} + a_{22} - \sqrt{(a_{11} + a_{22})^2 - 4 (a_{11} a_{22} - a_{12} a_{21})} \right] (\lambda + c) \\
- \frac{1}{2} \left[ \lambda(a_{11} + a_{22}) + c_1 + c_2 - \sqrt{\Delta} \right] \\
= -\frac{1}{2} \left[ c (a_{12} + a_{21}) + (c + \lambda) \sqrt{(a_{11} + a_{22})^2 - 4 (a_{11} a_{22} - a_{12} a_{21}) + \sqrt{\Delta}} \right] < 0. \]
Remark 4.5. If we denote by $\eta := \lambda_2 (\lambda + c)$ and we assume that the entries of matrix $M$ satisfies conditions ($C_g$) and $a_{12} + a_{21} \geq 0$ with $a_{12} > 0$, then $\eta - \mu < 0$ ($\mu = \mu (-A)$).

Theorem 4.6. Let the assumptions of Proposition 3.5 be satisfied, and $\alpha_0 = |\cos \theta|$ where $\theta$ is the angle between the lines $-A_1 w_1 = w_2$ and $w_1 = 0$. If we assume that $1 < \frac{c\lambda_1 \alpha_0}{(\mu - \eta)}$ and $a_{12} + a_{21} \geq 0$, then $\Sigma_w$ is an absorbing set in $\mathbb{R}^2_+$. 

Proof. Let $(w_1 (t, x), w_2 (t, x))$ be a solution of ($S_2$) in $\mathbb{R}^2_+$ outside $\Sigma_w$ for all $t > 0$. Suppose that 

$$A_1 w_1 (t, x) + w_2 (t, x) > 0, w_1 (t, x) > 0, \forall x \in \Omega, \forall t > 0.$$ 

If we set 

$$V (t) = A_1 \Phi_1 (t) + \Phi_2 (t),$$ 

according to (4.3) we have 

$$\frac{dV (t)}{dt} = -\lambda EV (t) + c EV (t)$$ 

which can be rewritten as 

$$\frac{dV (t)}{dt} = -\lambda_1 (\lambda + c) V (t) + \lambda_1 c \langle (1, A_1), (\Phi_1, \Phi_2) \rangle.$$ \hspace{1cm} (4.4) 

Since $A_1 w_1 + w_2 > 0$, there exists $\alpha > \alpha_0 > 0$ such that 

$$\langle (1, A_1), (\Phi_1, \Phi_2) \rangle = -\alpha \| v_1 \| \| \Phi \|.$$ 

Therefore, according to lemma 4.3 we get 

$$\langle (1, A_1), (\Phi_1, \Phi_2) \rangle \leq -\alpha \| v_1 \| \| \Phi (0) \| e^{-\mu t}.$$ 

Consequently, the equation (4.4) implies 

$$\frac{dV}{dt} \leq -\lambda_1 (\lambda + c) V - c\lambda_1 \alpha \| v_1 \| \| \Phi (0) \| e^{-\mu t}.$$ 

If we put $\kappa = c\lambda_1 \alpha \| v_1 \| \| \Phi (0) \|$, we obtain 

$$\frac{dV}{dt} + \eta V \leq -\kappa e^{-\mu t} \Rightarrow \frac{d}{dt} (e^{\eta t} V (t)) \leq -\kappa e^{(\eta - \mu) t}$$ 

$$\Rightarrow e^{\eta t} V (t) \leq V (0) - \kappa \int_0^t e^{(\eta - \mu) s} ds \Rightarrow e^{\eta t} V (t) \leq V (0) - \frac{\kappa}{\eta - \mu} (e^{(\eta - \mu) t} - 1)$$ 

$$\Rightarrow V (t) \leq V (0) + \frac{\kappa}{\mu - \eta} (e^{(\eta - \mu) t} - 1).$$
The function $h$ defined by
\[
h(t) := V(0) + \frac{\kappa}{\mu - \eta} \left( e^{(\eta - \mu)t} - 1 \right)
\]
is a positive continuous function such that $h(0) = V(0) > 0$. According to Lemma 4.4 we have $\mu - \eta > 0$, so
\[
\lim_{t \to +\infty} h(t) = V(0) - \frac{\kappa}{\mu - \eta} \leq \|v_1\| \|\Phi(0)\| \leq \|v_1\| \|\Phi(0)\| \left( 1 - \frac{c\lambda_1\alpha}{\mu - \eta} \right) < \|v_1\| \|\Phi(0)\| \left( 1 - \frac{c\lambda_1\alpha_0}{\mu - \eta} \right).
\]

By assumptions of the Theorem we have $1 < \frac{c\lambda_1\alpha_0}{\mu - \eta}$, which implies that $\lim_{t \to +\infty} h(t) < 0$. Then, there exists $t_0 > 0$, such that $h(t_0) = 0$, therefore $V(t_0) \leq h(t_0) = 0$ which contradicts the fact that $V(t) > 0, \forall t > 0$. So there exists $t^* > 0$ such that $(w_1(t^*, x), w_2(t^*, x)) \in \Sigma_w$. This proves that $\Sigma_w$ is an absorbing set in $\mathbb{R}_+^2$. \qed

**Remark 4.7.** The complementary set of $\Sigma_w$ in $\mathbb{R}_+^2$ consists of two positive cones, one situated below $\Sigma_w$ (limited by the half-lines $w_2 = 0$, and $w_2 = A_2 w_1$) and the other above $\Sigma_w$ (limited by the half-lines $w_1 = 0$, and $w_2 = -A_1 w_1$). In the proofs of Corollary 4.2 and Theorem 4.6, we considered only the second situation, where the starting point of a solution outside the set $\Sigma_w$, is a point of $\mathbb{R}_+^2$, located above $\Sigma_w$. In the other situation, taking into account the Remark 4.5 the reasoning can be applied in a similar way.

### 5. Example

To illustrate our results, we take as an example the following system
\[
(S) \quad \begin{cases} 
\partial_t v(t, x) = k_1 u_{xx}(t, x) - c(u(t, x) - v(t, x)) \\
-\partial_t u(t, x) + 3\partial_x v(t, x) = k_2 v_{xx}(t, x) + c(u(t, x) - v(t, x))
\end{cases}
\]
where $k_1$ and $k_2$ are two positive real numbers, $c$ is a positive real constant, and $(t, x) \in ]0, T[ \times ]0, 1[$. Here the matrix $M$ is defined by
\[
M = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}.
\]
It can easily be checked that the entries of the matrix $M$ satisfy the conditions (Cg). So, $M$ possesses two real non-negative eigenvalues $\lambda_1 = \frac{3 - \sqrt{5}}{2}$ and $\lambda_2 = \frac{3 + \sqrt{5}}{2}$, and
its inverse matrix $E = M^{-1}$ has two left eigenvectors $v_1 = (A_1, 1)$ and $v_2 = (A_2, -1)$ corresponding respectively to the eigenvalues $\lambda_1$ and $\lambda_2$ with

$$A_1 = \frac{-3 - \sqrt{5}}{2} \quad \text{and} \quad A_2 = \frac{3 - \sqrt{5}}{2}.$$  

Thus, after carrying out the transformations introduced in Section 2, according to Proposition 3.2, $\Sigma_w$ defined by

$$\Sigma_w = \left\{ (w_1, w_2) \in \mathbb{R}^2, \text{ such that } w_1 \geq 0 \text{ and } \frac{3 - \sqrt{5}}{2} w_1 \leq w_2 \leq \frac{3 + \sqrt{5}}{2} w_1 \right\}$$

is an invariant set.

Finally, the condition

$$\eta - \mu = -\frac{1}{2} \left[ \sqrt{5} (\lambda + c) + \sqrt{9 \lambda^2 + 120 \lambda c + 36 c^2} \right] < 0$$

being satisfied, with an appropriate choice of the constant $c$, Theorem 4.6 allows us to assert that $\Sigma_w$ is an absorbing set in $\mathbb{R}_+^2$.

**References**


