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Hopf bifurcation control of ecological model via PD controller

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Abstract

In this paper, we investigate the problem of bifurcation control for an ecological modle with timedelay. By choosing the timedelay as the bifurcation parameter, we present a Proportional-Derivative (PD) feedback controller to control Hopf bifurcation. We show that the onset of Hopf bifurcation can be delayed or advanced via a PD controller by setting proper controling parameters. Under consideration model as operator Equation, apply orthogonal decomposition, compute the center manifold and normal form we determined the direction and stability of bifurcating periodic solutions. Therefore the Hopf bifurcation of the model became controllable to achieve desirable behaviors which are applicable in certain circumstances.

AMS subject classification: 34H20, 34D23.
Keywords: Hopf bifurcation, bifurcation control, PD controller, ecological modle.

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1. Introduction

The delay differential equation (DDE)

$$\frac{d}{ds}x(s) = ax(s-\tau)e^{-bx(s-\tau)} - cx(s) \tag{1.1}$$

which is one of the important ecological systems, describes the dynamics of Nicholsons blowflies equation. Here $x(s)$ is the size of the population at time s, a is the maximum per capita daily egg production rate, $\frac{1}{b}$ is the size at which the population reproduces at the maximum rate, c is the per capita daily adult death rate, and τ is the generation time, the positive equilibrium $x_* = \left(\frac{1}{b}\right)\ln\left(\frac{a}{c}\right)$. Equation (1.1) has been extensively studied in the literature. The majority of the results on (1.1) deal with the global attractiveness of the positive equilibrium and oscillatory behaviors of solutions.

The aim of bifurcation control is to delay (advance) the onset of an inherent bifurcation, change the parameter value of an existing bifurcation point, the PD control strategy is used to control the bifurcation [1].

2. Existence Hopf bifurcation in uncontrol ecological modle

We consider the delay differential equation (1.1) and let $u(s) = x(\tau s)$ then

$$\frac{d}{ds}u(s) = \frac{d}{ds}x(\tau s) = \tau[ax(\tau s-\tau)e^{-bx(\tau s-\tau)} - cx(\tau s)] \tag{2.2}$$

and equation(1.1) can be rewritten as

$$\frac{d}{ds}u(s) = a\tau u(s-1)e^{-bu(s-1)} - c\tau u(s) \tag{2.3}$$

We consider the timedelay as the bifurcation parameter, The critical point in the system is $u_* = \left(\frac{1}{b}\right)\ln\left(\frac{a}{c}\right)$ and With the change of variables $z(s) = u(s) - u_*$ we have

$$\frac{d}{ds}z(s) = a\tau(z(s-1)+u_*)e^{-b(z(s-1)+u_*)} - c\tau(z(s)+u_*) \tag{2.4}$$

the linearization of equation (2,3) at z(s)=0 is

$$\frac{d}{ds}z(s) = c\tau[(1-bu_*)z(s-1) - z(s)] \tag{2.5}$$

whose characteristic equation is

$$\lambda = c\tau[(1-bu_*)e^{-\lambda} - 1] \tag{2.6}$$

Assume that $\lambda = i\omega$ is a root of the equation (2.6) that $\omega > 0$ then

$$i\omega = c\tau[(1 - bu_*)e^{-i\omega} - 1] \tag{2.7}$$

and

$$i\omega = c\tau[(1 - bu_*)(\cos\omega - i\sin\omega) - 1] \tag{2.8}$$

Separating the real and imaginary parts, we obtain

$$c\tau(1 - bu_*)\cos\omega = c\tau \tag{2.9}$$

and

$$c\tau(1 - bu_*)\sin\omega = -\omega \tag{2.10}$$

then $\omega = \pm\tau c\sqrt{(1 - bu_*)^2 - 1}$ if $(1 - bu_*)^2 - 1 > 0$. Let $\lambda_k(\tau) = \alpha_k(\tau) + i\beta_k(\tau)$ denote a root of equation (2.6) and near $\tau = \tau_k$ such that $\alpha_k(\tau_k) = 0$, $\beta_k(\tau_k) = \omega$ so that

$$\lambda_k(\tau_k) = \alpha_k(\tau_k) + i\beta_k(\tau_k) = i\omega$$

and

$$i\omega = c\tau_k[(1 - bu_*)e^{-i\omega} - 1],$$

$$\begin{aligned} ic\tau_k\sqrt{(1 - bu_*)^2 - 1} = c\tau_k[(1 - bu_*)(cos(c\tau_k\sqrt{(1 - bu_*)^2 - 1}) \\ - isin(c\tau_k\sqrt{(1 - bu_*)^2 - 1}) - 1] \end{aligned}$$

Then decompose the equationin to real and imaginary parts

$$c\tau_k(1 - bu_*)(cos(c\tau_k\sqrt{(1 - bu_*)^2 - 1}) - c\tau_k = 0 \tag{2.11}$$

and

$$c\tau_k\sqrt{(1 - bu_*)^2 - 1} + c\tau_k(1 - bu_*)sin(c\tau_k\sqrt{(1 - bu_*)^2 - 1}) = 0 \tag{2.12}$$

which lead to

$$\tau_k = \frac{1}{c\sqrt{(1 - bu_*)^2 - 1}}[\arcsin(\frac{\sqrt{(1 - bu_*)^2 - 1}}{(1 - bu_*)}) + 2k\pi] \qquad k = 0, 1, 2, ... \tag{2.13}$$

Lemma 2.1. We have to equation (2.6).

Proof. Differentiating both sides of equation (2.6) with respect to τ, we obtain

$$\frac{d\lambda}{d\tau} = \frac{c[(1 - bu_*)e^{-\lambda} - 1]}{1 + \tau c[(1 - bu_*)e^{-\lambda}]} \tag{2.14}$$

therefore

$$\frac{d\lambda}{d\tau}|_{\tau=\tau_k} = \frac{c(1-bu_*)cos\omega - c - ic(1-bu_*)sin\omega}{1+\tau_k c(1-bu_*)cos\omega - i\tau_k c(1-bu_*)sin\omega} \tag{2.15}$$

this implies that

$$\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau_k} = \frac{c(1-bu_*)cos\omega + \tau_k[c(1-bu_*)]^2}{[1+\tau_k c(1-bu_*)cos\omega]^2 + [\tau_k c(1-bu_*)sin\omega]^2} \tag{2.16}$$

then

$$\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau_k} \frac{c + \tau_k[c(1-bu_*)]^2}{[1+\tau_k c(1-bu_*)cos\omega]^2 + [\tau_k c(1-bu_*)sin\omega]^2} \tag{2.17}$$

so that completing the proof. ■

Theorem 2.2. For the system (1.1), the following statements are true:

(i) if $c < a < ce^2$,then $x = x_*$ is asymptotically stable.

(ii) if $a > ce^2(bx_* > 2)$, then $x = x_*$ is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$

(iii) Equation (1.1) undergoes a Hopf bifurcation at $x = x_*$ when $\tau = \tau_k$ for $k = 0, 1, 2, \ldots$ [1].

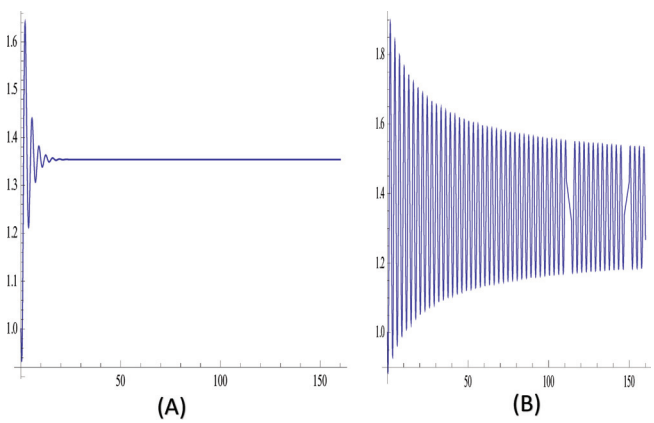

Figure 1: the numerical solution of uncontrol ecological modle with corresponding to (A) $a = 30, b = 2, c = 2, \tau = 0.4$ and (B) $a = 30, b = 2, c = 2, \tau = 0.8$.

3. Hopf bifurcation in controlled model based on PD controller

In this section,we focus on designing a controller to control the Hopf bifurcation in model based on the PD control strategy. The PD controller uses a single-input and single-output system as follows: the input error signal is defined as $e(t) = z(s) - z_*$ and the output control law $u(t)$ is defined as

$$u(t) = k_p z(s) + k_d(\frac{d}{ds}z(s)) \tag{3.18}$$

where k_p is proportional control parameter and k_d is derivative control parameter and $k_p < 1$; $k_d < 1$. Apply the PD controller to system (2.4), we get

$$\frac{d}{ds}z(s) = a\tau(z(s-1)+u_*)e^{-b(z(s-1)+u_*)} - c\tau(z(s)+u_*)+k_p z(s)+k_d\frac{d}{ds}z(s) \tag{3.19}$$

Therefore, the controlled model can bee expressed as follows

$$\frac{d}{ds}z(s) = \frac{1}{1-k_d}[a\tau(z(s-1)+u_*)e^{-b(z(s-1)+u_*)} - c\tau(z(s)+u_*)+k_p z(s)] \tag{3.20}$$

Comparing Equations (3.20) and (2.4), we find that the controlled model and uncontrolled model have the same equilibrium point. Expanding the right side of Equation (3.20) into Taylor series at $z(t) = 0$, we consider the linear part

$$\frac{d}{ds}z(s) = \frac{1}{1-k_d}[c\tau((1-bu_*)z(s-1) - z(s)) + k_p z(s)] \tag{3.21}$$

let $a_1 = \dfrac{\tau c(1-bu_*)}{1-k_d}$ and $a_2 = \dfrac{k_p - \tau c}{1-k_d}$ then

$$\frac{d}{ds}z(s) = a_1 z(s-1) + a_2 z(s) \tag{3.22}$$

Characteristic equation corresponding to Equation (3.22) is

$$\lambda^* - a_1 e^{-\lambda^*} - a_2 = 0 \tag{3.23}$$

Assume Equation (3.23) has a pair of pure imaginary roots $\lambda^* = \pm i\omega^*$ with $\omega^* > 0$. Inserting them into Equation (3.23). Then decompose the characteristice quationin to real and imaginary parts

$$a_2 + a_1 cos(\omega^*) = 0 \tag{3.24}$$

$$\omega^* + a_1 sin(\omega^*) = 0 \tag{3.25}$$

From Equations (3.24), (3.25) and combined with the formula: $sin^2(\omega^*)+cos^2(\omega^*) = 1$, we obtain

$$\omega^* = \sqrt{a_1^2 - a_2^2} \tag{3.26}$$

and

$$\omega^* = \arccos\left(-\frac{a_2}{a_1}\right) \tag{3.27}$$

Combine Equations (3.26), (3.27) and we can get the equationas follows:

$$\sqrt{a_1^2 - a_2^2} = \arccos\left(-\frac{a_2}{a_1}\right) \tag{3.28}$$

equation (3.28) is a transcendental equation, we can get its numerical solution by numerical simulation software. [2], We need to get the fixed point ${\tau^*}_k$ of the system such that $\lambda^*({\tau^*}_k) = i\omega^*$, the value of ω^* can be derived from equation (3.23) then.

We still need to check the following transversal condition in order to verify the onset of Hopf bifurcation:

Lemma 3.1. For equation (3.23) if

$$k_p > \frac{{\tau^*}_k^2 c^2}{k_d - 1 + {\tau^*}_k c}$$

then

$$\frac{dRe(\lambda)}{d\tau}|_{\tau={\tau^*}_k} > 0.$$

Proof. Let $r = \dfrac{c(1-bu_*)}{1-k_d}$, differentiating both sides of equation (3.23) with respect to τ, we obtain

$$d\lambda - re^{-\lambda}d\tau + \tau re^{-\lambda}d\lambda + \frac{c}{1-k_d}d\tau = 0$$

Therefore

$$\frac{d\lambda}{d\tau}|_{\tau=\tau_k^*} = \frac{rcos\omega^* - \dfrac{c}{1-k_d} - irsin\omega^*}{1 + {\tau^*}_k rcos\omega^* - i{\tau^*}_k rsin\omega^*}$$

this implies that

$$\frac{dRe(\lambda)}{d\tau}|_{\tau=\tau_k^*} = \frac{k_p(k_d-1) + {\tau^*}_k c(k_p - {\tau^*}_k c) + {{\tau^*}_k}^2 r^2 (1-k_d)^2}{{\tau^*}_k (1-k_d)^2[(1+{\tau^*}_k rcos\omega^*)^2 + ({\tau^*}_k rsin\omega^*)^2]}$$

so that completing proof. ■

Therefore, the transversal condition for Hopf bifurcation in the PD controlled system (3.20) is satisfied. We conclude above analyzes in following:

Theorem 3.2. For the controlled system (3.20), there exists a Hopf bifurcation emerging from its equilibrium $z^* = 0$ when the positive parameter τ passes through the critical value ${\tau^*}_k$.

We note that the equilibrium point remains unchanged and the critical value ${\tau^*}_k$ varies with control parameters k_p and k_d. Theorem (3.19) shows that the PD controller can be applied to the ecological model for the purpose of bifurcation control.

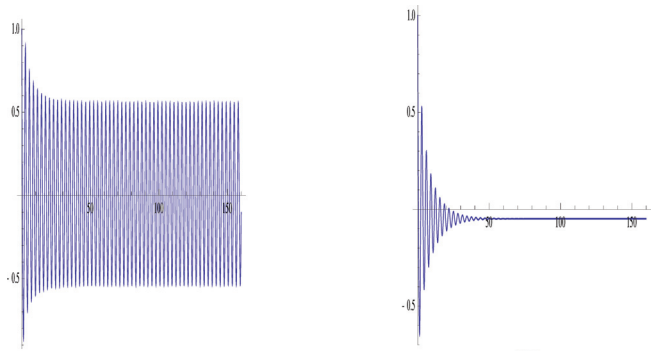

Figure 2: the numerical solution of control ecological modle with corresponding to (A) $a = 30, b = 2, c = 2, \tau = 0.4, k_d = 0.5, k_p = 0.1$ and (B) $a = 30, b = 2, c = 2, \tau = 0.4, k_d = 0.3, k_p = 0.1$.

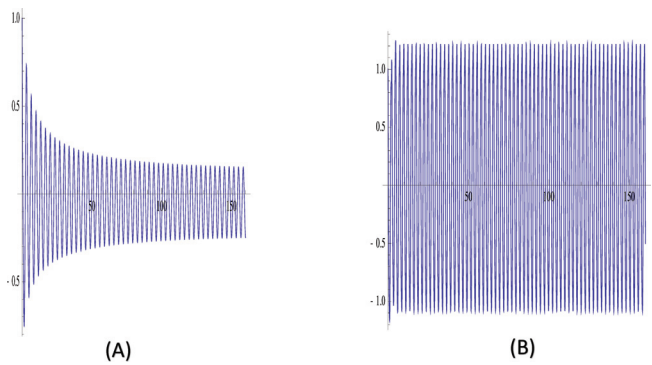

Figure 3: the numerical solution of control ecological modle with corresponding to (A) $a = 30, b = 2, c = 2, \tau = 0.4, k_d = 0.3, k_p = 0.4$ and (B) $a = 30, b = 2, c = 2, \tau = 0.8, k_d = 0.5, k_p = 0.1$.

4. Stability and direction of bifurcating periodic solutions

In this section we study the direction of the Hopf bifurcation and the stability of the bifurcation periodic solutions when $\tau = \tau^*{}_k$ under the condition $k_p > \dfrac{\tau^{*2}_k c^2}{k_d - 1 + \tau^*{}_k c}$ and using techniques from normal form and center manifold theory by Hassard et al. [7].

For convenience, let $\mu = \tau - \tau^*{}_k$, therefore $\mu = 0$ is the value of hopf bifurcation for equation (3.20). In this section we will be discussed in multiple parts. In the frist part we will formulate the operator form for (3.20), describe a process of determining the critical eigenvalues for the problem, formulate the adjoint operator equation and Apply Orthogonal Decomposition. then Compute the Center Manifold Form, Develop the Normal Form on the Center Manifold and at the end examine periodic solutions conditions on center manifold.

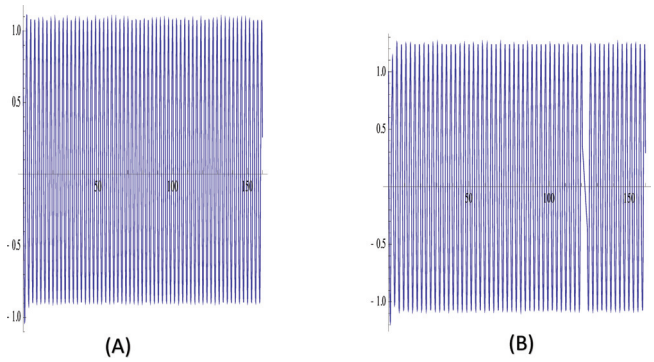

Figure 4: the numerical solution of control ecological modle with corresponding to (A) $a = 30, b = 2, c = 2, \tau = 0.8, k_d = 0.3, k_p = 0.1$ and (B) $a = 30, b = 2, c = 2, \tau = 0.8, k_d = 0.3, k_p = 0.4$.

Step 1: Form the Operator Equation
With expanding the right side of Equation (3.20) into Taylor series at $z(s) = 0$, then

$$\frac{d}{ds}x(s) = U(\mu)x(s) + V(\mu)x(s-1) + f(x(s), x(s-1), \mu) \tag{4.29}$$

that $U(\mu) = a_2, V(\mu) = a_1$ and $f(x(s), x(s-1), \mu) = O(x^2(s), x^2(s-1))$ we consider The linear portion of equation (4.29) that is given by

$$\frac{d}{ds}x(s) = a_2x(s) + a_1x(s-1) \tag{4.30}$$

the equations (4.29) and (4.30) can be thought of as maps of entire functions. we will start with the class of continuous functions defined on the interval [-1, 0] with values in R^n and refer to this as class C_0. The maps are constructed by defining a family of solution operators for the linear portion of equation (4.29) by

$$(\Upsilon(s)\phi)(\Theta) = (x_s(\phi))(\Theta) = x(s + \Theta) \tag{4.31}$$

for $\phi \in C_0, \Theta \in [-1, 0], s \geq 0$. This is a mapping of a function in C_0 to another function in C_0. so equations (4.29), (4.30) can be thought of as maps from C_0 to C_0 and $\Upsilon(s), s \geq 0$ is a semigroup.

An infinitesimal generator of a semigroup $\Upsilon(s)$ is defined by

$$\Lambda\phi = \lim_{s\to 0^+} \frac{1}{s}[\Upsilon(s)\phi - \phi]$$

for $\phi \in C_0$. in the linear equation (4.30) the infinitesimal generator can be constructed as

$$\Lambda(\mu)\phi = \begin{cases} \dfrac{d\phi}{d\Theta}(\Theta) & -1 \leq \Theta < 0 \\ a_2\phi(0) + a_1\phi(-1) & \Theta = 0 \end{cases} \tag{4.32}$$

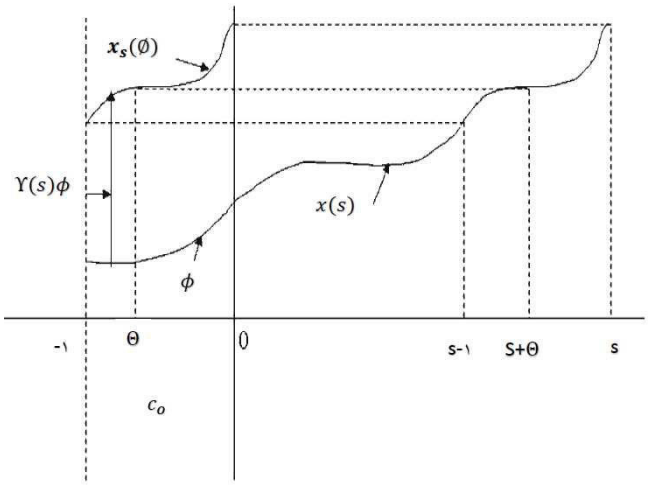

Figure 5: The solution operator maps functions in C_0 to functions in C_0

Then $\Upsilon(s)\phi$ satisfies $\frac{d}{ds}\Upsilon(s)\phi = \Lambda(\mu)\Upsilon(s)\phi$, the operator form for the nonlinear equation (4.29) is

$$\frac{d}{ds}x_s(\phi) = \Upsilon(\mu)x_s(\phi)+F(x_s(\phi),\mu), that \quad F(\phi,\mu)(\Theta) = \begin{cases} 0 & -1 \leq \Theta < 0 \\ f(\phi,\mu) & \Theta = 0 \end{cases} \tag{4.33}$$

Step 2: Define an Adjoint Operator

We define a form of adjoint Operator associated with (4.33). we consider $C_0{}^{\circledast} = C([0,1], R^n)$ that is the space of continuous functions from[0,1] to R^n. the adjoint equation associated with the equation (4.30) is

$$\frac{d}{ds}y(s,\mu) = -a_2 y(s,\mu) - a_1 y(s+1,\mu) \tag{4.34}$$

we define

$$(\Upsilon^{\circledast}(s)\psi)(\Theta) = (y_s(\psi))(\Theta) = y(s+\Theta) \tag{4.35}$$

for $\Theta \in [o,1], s \leqslant 0, y_s \in C_0{}^{\circledast}$ and $y_s(\psi)$ be the image of $\Upsilon^{\circledast}(s)\psi$, so relationship (4.35) defines semigroup with infinitesimal generator

$$\Lambda^{\circledast}(\mu)\psi = \begin{cases} -\frac{d\psi}{d\Theta}(\Theta) & 0 < \Theta \leq 1 \\ -\frac{d\psi}{d\Theta}(0) = a_2\psi(0) + a_1\psi(1) & \Theta = 0 \end{cases} \tag{4.36}$$

Note that, although the formal infinitesimal generator for (4.35) is defined as

$$\Lambda_0{}^{\circledast}\psi = \lim_{s\to 0^-}\frac{1}{s}[\Upsilon^{\circledast}(s)\psi - \phi]$$

for $\psi \in C_0$ Hale[5], that

$$\Lambda_0{}^{\circledast} = -\Lambda^{\circledast}$$

and family of operators (4.35) satisfies

$$\frac{d}{ds}\Upsilon^{\circledast}(s)\psi = -\Lambda^{\circledast}(\mu)\Upsilon^{\circledast}(s)\psi.$$

Step 3: Define a natural Inner product by way of an adjoint operator
Let $x(s) \in C_0$, $y(s) \in C_0{}^{\circledast}$ and define an inner product

$$\langle y(s), x(s)\rangle = \overline{y}^T(0)x(0) + \int_{-1}^{0} \overline{y}^T(s+1)a_1 x(s)\mathrm{d}s \tag{4.37}$$

Step 4: Get the Critical Eigenvalues and Apply Orthogonal Decomposition
As for $\phi \in C_0$, $\psi \in C_0{}^{\circledast}$ we have $\langle \psi, \Lambda(\mu)\phi\rangle = \langle \Lambda^{\circledast}(\mu)\psi, \phi\rangle$, obviously $\Lambda(0)$ and $\Lambda^{\circledast}(0)$ are adjoint operators and $\pm i\omega^*$ are eigenvalues of $\Lambda(0)$ and $\Lambda^{\circledast}(0)$. to gain Orthogonal Decomposition we need to calculate the eigenvector $\xi(\theta)$ of $\Lambda(0)$ and $\xi^{\circledast}(\theta)$ of $\Lambda^{\circledast}(0)$ associated with the eigenvalues $i\omega^*$ and $-i\omega^*$ respectively. let $\xi(\theta) = e^{i\omega^*\theta}A$, $\bar{\xi}(\theta) = e^{-i\omega^*\theta}\bar{A}$ that $\theta \in [-1, 0)$ and $\xi^{\circledast}(\theta) = e^{i\omega^*\theta}B$, $\bar{\xi}^{\circledast}(\theta) = e^{-i\omega^*\theta}\bar{B}$ that $\theta \in (0, 1]$. let $A = 1$ we choose B so that $\langle \xi^{\circledast}, \xi\rangle = 1$ and $\langle \xi^{\circledast}, \bar{\xi}\rangle = 0$. from (4.37) we have:

$$\langle \xi^{\circledast}, \xi\rangle = \overline{\xi^{\circledast}}^T(0)\xi(0) + \int_{-1}^{0} \overline{\xi^{\circledast}}^T(s+1)a_1\xi(s)\mathrm{d}s = \bar{B} + \int_{-1}^{0} e^{-i\omega^*(s+1)}\bar{B}a_1 e^{i\omega^* s}\mathrm{d}s \tag{4.38}$$

then

$$\bar{B} = \frac{1}{1 + a_1 e^{-i\omega^*}}.$$

Similarly

$$\langle \xi^{\circledast}, \bar{\xi}\rangle = \overline{\xi^{\circledast}}^T(0)\bar{\xi}(0) + \int_{-1}^{0} \overline{\xi^{\circledast}}^T(s+1)a_1\bar{\xi}(s)\mathrm{d}s = \bar{B} + \int_{-1}^{0} e^{-i\omega^*(s+1)}\bar{B}a_1 e^{-i\omega^* s}\mathrm{d}s \tag{4.39}$$

so

$$\bar{B}\left(1 - \frac{a_1 e^{-i\omega^*} - a_1 e^{i\omega^*}}{2i\omega^*}\right) = 0.$$

Let $x_s \in C_0$ the unique family of solutions of (4.33), where the μ notation has been dropped since we are only working with $\mu = 0$.

Define $\begin{pmatrix}\eta \\ \bar{\eta}\end{pmatrix} = \begin{pmatrix}\langle \xi^{\circledast}, x_s\rangle \\ \langle \bar{\xi}^{\circledast}, x_s\rangle\end{pmatrix}$ and the orthogonal family of functions $\omega_s = x_s -$

$(\xi, \bar{\xi})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}$ so that the equation (4,5) has now been decomposed as

$$\begin{cases} \frac{d}{ds}\eta(s) = i\omega^*\eta(s) + \bar{B}^T f\left(\omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta(s)\\ \bar{\eta}(s)\end{pmatrix}\right) \\ \frac{d}{ds}\bar{\eta}(s) = -i\omega^*\bar{\eta}(s) + B^T f\left(\omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta(s)\\ \bar{\eta}(s)\end{pmatrix}\right) \\ \frac{d}{ds}\omega_s(\Theta) = \begin{cases} (\Lambda\omega_s)(\Theta) - 2Re\left\{\left\langle \xi^{\circledast}, \omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}\right\rangle \xi(\Theta)\right\} & -1 \le \Theta < 0 \\ (\Lambda\omega_s)(0) - 2Re\left\{\left\langle \xi^{\circledast}, \omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}\right\rangle \xi(0)\right\} + f\left(\omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta(s)\\ \bar{\eta}(s)\end{pmatrix}\right) & \Theta = 0 \end{cases} \end{cases} \tag{4.40}$$

Step 5: Compute the Center Manifold Form

For Compute the center manifold form we start with the equations (4.40) and let

$$\chi_1(\eta, \bar{\eta}, \omega_s) = \bar{B}^T f\left(\omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta(s)\\ \bar{\eta}(s)\end{pmatrix}\right) \tag{4.41}$$

$$\chi_2(\eta, \bar{\eta}, \omega_s) = \begin{cases} -2Re\left\{\left\langle \xi^{\circledast}, \omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}\right\rangle \xi(\Theta)\right\} & -1 \le \Theta < 0 \\ -2Re\left\{\left\langle \xi^{\circledast}, \omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta\\ \bar{\eta}\end{pmatrix}\right\rangle \xi(0)\right\} + f\left(\omega_s + (\xi, \bar{\xi})\begin{pmatrix}\eta(s)\\ \bar{\eta}(s)\end{pmatrix}\right) & \Theta = 0 \end{cases} \tag{4.42}$$

and from equation (4.29), (3.20) we have

$$f(x(s), x(s-1), \mu) = a_3 x^2(s-1) + O(x^3(s-1)) \tag{4.43}$$

that $a_3 = (u_* - 2)\tau cb$ for to gain an approximate center manifold, can be given as a quadratic form in η and $\bar{\eta}$ with coefficients as functions of Θ

$$\omega_s(\eta, \bar{\eta})(\Theta) = \omega_{20}(\Theta)\frac{\eta^2}{2} + \omega_{11}(\Theta)\eta\bar{\eta} + \omega_{02}(\Theta)\frac{\bar{\eta}^2}{2} \tag{4.44}$$

then

$$\begin{aligned} f(\omega_s(\eta, \bar{\eta})(\Theta) + \xi(\Theta)\eta(s) + \bar{\xi}(\Theta)\bar{\eta}(s)) = a_3(\omega_s(\eta, \bar{\eta})(-1) + \xi(-1)\eta(s) \\ + \bar{\xi}(1)\bar{\eta}(s))^2 + O(3) \end{aligned} \tag{4.45}$$

$$\begin{aligned} f(\omega_s(\eta, \bar{\eta})(\Theta) + \xi(\Theta)\eta(s) + \bar{\xi}(\Theta)\bar{\eta}(s)) = a_3\gamma^2\eta^2 + 2a_3\gamma\bar{\gamma}\eta\bar{\eta} + a_3\bar{\gamma}^2\bar{\eta}^2 \\ + a_3[\omega_{20}(-1)\bar{\gamma} + 2\omega_{11}(-1)\gamma]\eta^2\bar{\eta} + O(3) \end{aligned} \tag{4.46}$$

that $\gamma = e^{-i\omega} - 1$. If we define

$$g_{20} = 2a_3\gamma^2\bar{B}, g_{11} = 2a_3\gamma\bar{\gamma}\bar{B}, g_{02} = 2a_3\bar{\gamma}^2\bar{B}$$

and

$$g_{21} = 2a_3[\omega_{20}(-1)\bar{\gamma} + 2\omega_{11}(-1)\gamma]\bar{B}$$

then we can to write

$$f(\omega_s(\eta,\bar{\eta})(\Theta) + \xi(\Theta)\eta(s) + \bar{\xi}(\Theta)\bar{\eta}(s)) = \frac{g_{20}\eta^2}{2\bar{B}} + \frac{g_{11}\eta\bar{\eta}}{\bar{B}} + \frac{g_{02}\bar{\eta}^2}{2\bar{B}} + \frac{g_{21}\eta^2\bar{\eta}}{2\bar{B}} \quad (4.47)$$

In order to compute g_{21} we needs to compute the center manifold coefficients ω_{20}, ω_{11} From the definition (4.42) for $-1 \leq \Theta < 0$

$$\begin{aligned}\chi_2(\eta,\bar{\eta})(\Theta) = &-[g_{20}\xi(\Theta) + g_{\bar{0}2}\bar{\xi}(\Theta)]\frac{\eta^2}{2} \\ &- [g_{11}\xi(\Theta) + g_{\bar{1}1}\bar{\xi}(\Theta)]\eta\bar{\eta} - [g_{02}\xi(\Theta) + g_{\bar{2}0}\bar{\xi}(\Theta)]\frac{\bar{\eta}^2}{2}\end{aligned} \quad (4.48)$$

and for $\Theta = 0$

$$\begin{aligned}\chi_2(\eta,\bar{\eta})(0) = &-\left[g_{20}\xi(0) + g_{\bar{0}2}\bar{\xi}(0) - \frac{g_{20}}{\bar{B}}\right]\frac{\eta^2}{2} \\ &-\left[g_{11}\xi(0) + g_{\bar{1}1}\bar{\xi}(0) - \frac{g_{11}}{\bar{B}}\right]\eta\bar{\eta} - \left[g_{02}\xi(0) + g_{\bar{2}0}\bar{\xi}(0) - \frac{g_{02}}{\bar{B}}\right]\frac{\bar{\eta}^2}{2}\end{aligned} \quad (4.49)$$

since

$$\frac{g_{02}}{\bar{B}} = \bar{g}_{20}B$$

then coefficients

$$\frac{\eta^2}{2}, \eta\bar{\eta}, \frac{\bar{\eta}^2}{2}$$

in $\chi_2(\eta,\bar{\eta})(\Theta)$ respectively as

$$\chi_2{}^{20}(\Theta) = \begin{cases} -[g_{20}\xi(\Theta) + g_{\bar{0}2}\bar{\xi}(\Theta)] & -1 \leq \Theta < 0 \\ -\left[g_{20}\xi(0) + g_{\bar{0}2}\bar{\xi}(0) - \frac{g_{20}}{\bar{B}}\right] & \Theta = 0 \end{cases} \quad (4.50)$$

$$\chi_2{}^{11}(\Theta) = \begin{cases} -[g_{11}\xi(\Theta) + g_{\bar{1}1}\bar{\xi}(\Theta)] & -1 \leq \Theta < 0 \\ -\left[g_{11}\xi(0) + g_{\bar{1}1}\bar{\xi}(0) - \frac{g_{11}}{\bar{B}}\right] & \Theta = 0 \end{cases} \quad (4.51)$$

and $\chi_2{}^{02}(\Theta) = \bar{\chi}_2^{20}(\Theta)$. By taking derivatives, the equation for the manifold becomes

$$\begin{aligned}&\left(\frac{d\omega_s(\eta,\bar{\eta})(\Theta)}{d\eta}\right)\left(\frac{d\eta(s)}{ds}\right) + \left(\frac{d\omega_s(\eta,\bar{\eta})(\Theta)}{d\bar{\eta}}\right)\left(\frac{d\bar{\eta}(s)}{ds}\right) \\ &= \Lambda\omega_s(\eta,\bar{\eta})(\Theta) + \chi_2(\eta,\bar{\eta})(\Theta)\end{aligned} \quad (4.52)$$

so

$$
\begin{aligned}
&i\omega^*\omega_{20}(\Theta)\eta^2(s) + i\omega^*\omega_{11}(\Theta)\bar{\eta}(s)\eta(s) - i\omega^*\omega_{11}(\Theta)\eta(s)\bar{\eta}(s) - i\omega^*\omega_{02}(\Theta)\bar{\eta}^2(s)\cdots \\
&\quad = (\Lambda\omega_{20})(\Theta)\frac{\eta^2}{2} + (\Lambda\omega_{11})(\Theta)\eta\bar{\eta} + (\Lambda\omega_{02})(\Theta)\frac{\bar{\eta}^2}{2} + {\chi_2}^{20}(\Theta)\frac{\eta^2}{2} \\
&\quad + {\chi_2}^{11}(\Theta)\eta\bar{\eta} + {\chi_2}^{02}(\Theta)\frac{\bar{\eta}^2}{2}\cdots
\end{aligned} \tag{4.53}
$$

we have

$$
\begin{cases}
2i\omega_{20}(\Theta) - (\Lambda\omega_{20})(\Theta) = {\chi_2}^{20}(\Theta) \\
-(\Lambda\omega_{11})(\Theta) = {\chi_2}^{11}(\Theta) \\
-2i\omega_{02}(\Theta) - (\Lambda\omega_{02})(\Theta) = {\chi_2}^{02}(\Theta)
\end{cases} \tag{4.54}
$$

then for $-1 \leq \Theta < 0$,

$$
\frac{d\omega_{11}(\Theta)}{\Theta} = g_{11}\xi(\Theta) + \bar{g_{11}}\bar{\xi}(\Theta)
$$

so that

$$
\omega_{11}(\Theta) = \frac{g_{11}}{i\omega}\xi(\Theta - \frac{\bar{g_{11}}}{i\omega}\bar{\xi}(\Theta) + Q
$$

and from $\Theta = 0$, $\mu = 0$, (4.32), (4.51) Q is will be calculated that $Q = -\frac{g_{11}}{\bar{B}(a_1 + a_2)}$.
then

$$
\omega_{11}(\Theta) = \frac{g_{11}}{i\omega^*}\xi(\Theta) - \frac{\bar{g_{11}}}{i\omega^*}\bar{\xi}(\Theta) - \frac{g_{11}}{\bar{B}(a_1 + a_2)} \tag{4.55}
$$

for Calculation $\omega_{20}(\Theta)$ from (4.54), (4.32), (4.50) we have

$$
\omega_{20}(\Theta) = -\frac{g_{20}}{i\omega^*}\xi(\Theta) - \frac{\bar{g_{02}}}{3i\omega^*}\bar{\xi}(\Theta) - \left[\frac{g_{20}}{\bar{B}(2i\omega^* - a_2 - a_1 e^{-2i\omega^*})}\right]e^{2i\omega^*} \tag{4.56}
$$

so that we can define g_{21}.

Step 6: Develop the Normal Form on the Center Manifold
With normal form theory, following the argument of Wiggins [6], can be used to reduce (4.40) to the form

$$
\begin{cases}
\dfrac{d\vartheta}{ds} = i\omega^*\vartheta + \Delta\vartheta^2\bar{\vartheta} \\[2ex]
\dfrac{d\bar{\vartheta}}{ds} = -i\omega^*\bar{\vartheta} + \Delta\bar{\vartheta}^2\vartheta
\end{cases} \tag{4.57}
$$

that

$$
\Delta = \frac{i}{2\omega}\left\{g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right\} + \frac{g_{21}}{2}
$$

We define

$$\begin{cases} \nu = -\dfrac{Re\{\Delta(0)\}}{Re\dot{\lambda}(0)} \\ \delta = -\dfrac{Im\{\Delta(0)\} + \nu Im\dot{\lambda}(0)}{\omega^*} \\ \varrho = 2Re\{\Delta(0)\} \end{cases} \tag{4.58}$$

According to the case described above, We can summarize the results in the following theorem:

Theorem 4.1. For the controlled system (3.20), the Hopf bifurcationis determined by the parameters ν, δ and β, the conclusions are summarized asfollows:

(i) Parameter ν determines the direction of the Hopf bifurcation. if $\nu > 0$, the Hopf bifurcation is supercritical,the bifurcating periodic solutions exist for $\tau > \tau^*{}_k$, if $\nu < 0$ the Hopf bifurcation is subcritical, the bifurcating periodic solutions exist for $\tau < \tau^*{}_k$.

(ii) Parameter ϱ determines the stability of the bifurcating periodic solutions. If $\varrho < 0$, the bifurcating periodic solutions is stable; if $\varrho > 0$, the bifurcating periodic solutions is unstable.

(iii) Parameter δ determines the period of the bifurcating periodic solution. If $\delta > 0$, the period increases; If $\delta < 0$, the period decreases.

5. Conclusions

In this paper, the problem of Hopf bifurcation control for an ecological model with time delay was studied. In order to control the Hopf bifurcation, a PD controller is applied to the model. This PD controller can successfully delay or advance the onset of an inherent bifurcation. The end theoren helped to improve model.

References

[1] Yuanyuan Wang, Lisha Wang, *Hybrid control of the Neimark-Sacker bifurcation in a delayed Nicholson's blowflies equation*, Wang and Wang Advances in Difference Equations (2015) 2015:306.

[2] DaweiDing, XiaoyunZhang, JindeCao, NianWang, DongLiang, *Bifurcation control of complex networks model via PD controller*, Neurocomputing (2015).

[3] Zunshui Cheng, JindeCao, *Hybrid control of Hopf bifurcation in complex networks with delays*, Neurocomputing 131(2014), 164–170.

[4] Lina MA, Dawei DING, Tianren CAO, Mengxi WANG, *Bifurcation control in a small-world network model via TDFC*, International Conference on Advances in Mechanical Engineering and Industrial Informatics (AMEII 2015).

[5] Jack K. Hale Sjoerd M. Verduyn Lunel, *Introduction to Functional Differential Equations*, springer, Mathematics Subject Classification (1991): 34K20, 34A30, 39A10.

[6] Stephen Wiggins, *Introduction to Applied Nonlinear Dynamical Systems and Chaos*, Springer, Mathematics Subject Classification (1991): 58 Fxx, 34Cxx, 70Kxx

[7] Balakumar Balachandran . Tam's Kalmr-Nagy David E. Gilsinn, *Delay Differential Equations*, Springer Science+Business Media, LLC 2009.