New Proof of the Theorem of Existence and Uniqueness of Geometric Fractional Brownian Motion Equation

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Abstract

This paper provides a proof of existence and uniqueness theorem for the geometric fractional Brownian motion equation. The proof uses an approximation approach in addition to some inequalities.

Keywords: geometric fractional Brownian motion equation, approximation approach, stochastic differential equation.

INTRODUCTION

In general, the study of the stochastic differential equations (SDEs) heavily depends on the definition of stochastic integrals (Tien (2013)). In this work we use an approximation approach in $L^2(\Omega,F,P)$ to investigate existence and uniqueness of a geometric fractional Brownian motion (GFBM) equation of the form:

$$dX_t = \mu X_t dt + \sigma(t)X_t dB_t^H$$

(1)

which is a special case of fractional stochastic differential equations (FSDE) of the form:

$$dX_t = b(s,X_s)dt + \sigma(s,X_s)dB_t^H$$

(2)

where $b(t,x)$ and $\sigma(t,x)$ are two continuous functions and $X_0$ is a given random variable such that $E[X_0^2] < \infty$. 
Some necessary theorems and definitions are presented to show the existence and uniqueness of the solution for stochastic differential equation driven by fractional Brownian motion (FBM).

**Definition 1: (Lipschitz condition; Ricco (2004))** Function \( f: A \rightarrow \mathbb{R}^m, A \subset \mathbb{R}^n \) satisfies a Lipschitz condition on the closed interval \([a; b]\) if there is a constant \( K \) such that
\[
|f(x) - f(y)| \leq K|x - y|
\]
(3)
for every pair of points \( x \) and \( y \in A \). Further, function \( f \) is called locally Lipschitz if for each \( x_0 \in A \) there exists constant \( M < 0 \) and \( \delta < 0 \) such that \( |x - x_0| < 0 \) implies that \( |f(x) - f(x_0)| \leq M|x - x_0|\).

**Lemma 1: (Jensen’s Inequality; Lin and Bai (2011)):** Let \( X \) be a random variable with \( E[X] < \infty \). For any convex function \( f(X) \) such that \( E[|f(X)|] < \infty \) then
\[
E[f(X)] \leq f(E[X]).
\]
(4)

**Lemma 2: (Gronwall’s Inequality; Oguntuase (2001)):** Let \( f(t) \) and \( u(t) \) are nonnegative continuous functions on \([0, T]\) such that \( f(t) \leq C + \int_0^t f(s)u(s)ds \) for \( 0 \leq t \leq T \) and for some non-negative constant \( C \) then
\[
f(t) \leq C \ Exp[\int_0^t u(s) \ ds].
\]
(5)

**Note:** If \( u(t) = A \) in Equation (5), we get \( f(t) \leq C \ Exp[At] \)

**Lemma 3: (Ito Isometry; Oksendal (2003)):** If \( f(t,w) \) is bounded and elementary then
\[
E[(\int_s^t f(t,w) dW_s)^2] = E[\int_s^t f^2(s,t) dt],
\]
(6)
where \( W_s \) is a standard Brownian motion.

**Lemma 4 (Doob’s martingale inequality; Oksendal (2003)):** If \( M_t \) is a martingale such that \( t \mapsto M_t(w) \) is continuous almost surely, then for all \( p \geq 1, T \geq 0 \) and all \( \lambda > 0 \),
\[
P \left( \sup_{0 \leq t \leq T} |M_t| \geq \lambda \right) \leq \frac{1}{\lambda^p} E[|M_T|^p].
\]
(7)

**Lemma 5: (Borel – Cantelli Lemma; Chandra(2012)):** Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( \{A_k\}_{k=1}^\infty \) be a sequence of events in \( \mathcal{F} \). If \( \sum_{k=1}^\infty P(A_k) \) converges then \( P \left( \lim \sup_{k \to \infty} A_k \right) = 0 \). If the events \( A_k \) are independent and \( \sum_{k=1}^\infty P(A_k) = \infty \) then \( P \left( \lim \sup_{k \to \infty} A_k \right) = 1 \).

**Lemma 6: (Fatou’s Lemma; Knapp (2005)):** If \( S \) is a measurable set and if \( \{Y_n\} \) is sequence of non-negative measurable functions. Then,
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$$\int_S \lim_{n \to \infty} \inf Y_n \, dM \leq \lim_{n \to \infty} \int_S Y_n \, dM.$$ (8)

In particular, if $Y(s) = \lim_{n \to \infty} \inf f_n(s)$, for all $s \in S$. Then $Y$ is measurable and

$$\int_S Y \, dM \leq \lim_{n \to \infty} \inf \int_S f_n(s) \, dM.$$ (9)

**Approximation Approach**

In terms of a practical approach to the theory, Thao (2006; 2014), Thao and Christine (2003), Thao, Sattayatham and Plienpanich (2008), Plienpanich, Sattayatham and Thao (2009), Dung (2011), Dung and Thao (2010), Tein (2013 a; 2013b), and Intarasit and Sattayatham (2010) studied fractional stochastics driven by FBM of the Liouville form (LFBM) based on a crucial fact that any LFBM can be approximated in the space $L^2(\Omega, F, P)$ by semimartingales,

$$B_t^H = \int_0^t (t-s)^\alpha \, dW_s.$$ (10)

Also, Mozet and Nualart (2000) introduced the semimartingale

$$B_t^{H,\epsilon} = \int_0^t (t-s + \epsilon)^\alpha \, dW_s,$$ (11)

where $\alpha = H - 1/2$ and $W_t$ is a standard Brownian motion. Furthermore

$$dB_t^{H,\epsilon} = \alpha \Phi_t^\epsilon \, dt + \epsilon^\alpha \, dW_s,$$ (12)

where

$$\Phi_t^\epsilon = \int_0^t (t-s + \epsilon)^{\alpha-1} \, dW_s.$$ (13)

Thao (2006) proved that $B_t^{H,\epsilon}$ converges uniformly to $B_t^H$ in $t \in [0, T]$, further lead to

$$\int_0^t f(s, w)dB_s^{H,\epsilon}$$ converges to $\int_0^t f(s, w)dB_s^H$. Thao (2006) further compute the integral of $\Phi_t^\epsilon$ as:

$$\int_0^t \Phi_t^\epsilon \, ds = \int_0^t \int_0^s (s-u+\epsilon)^{\alpha-1} \, ds \, dW_s$$

$$= \int_0^t \left[ \int_0^s (s-u+\epsilon)^{\alpha-1} \, ds \right] \, dW_s$$

$$= \frac{1}{\alpha} \left[ \int_0^t (t-s+\epsilon)^{\alpha} \, dW_s - \epsilon^\alpha W_t \right]$$

$$= \frac{1}{\alpha} \left[ B_t^{H,\epsilon} - \epsilon^\alpha W_t \right].$$ (14)

We are now ready to prove the theorem of existence and uniqueness for GFBM.
THEOREM OF EXISTENCE AND UNIQUENESS FOR GEOMETRIC FRACTIONAL BROWNIAN MOTION

In this subsection we used approximation approach in $L^2(\Omega, F, P)$ suggested by Thao (2013) to construct the proving of existence and uniqueness of the equations of GFBM:

$$X_t = X_0 + \int_0^t \mu X_s \, ds + \int_0^t \sigma(s) X_s \, dB_s^H,$$

where $\mu X_t(x)$ and $\sigma(s) X_t(x)$ are two continuous functions and $X_0$ is a given random variable such that $E[X_0^2] < \infty$.

Let $(\Omega, F, P)$ be a probability space where $F$ is the $\sigma$– algebra of set $\Omega$, and $P$ is a probability measure. Let $\{X_t\}_{t \in [0,\infty]}$ be a stochastic process defined on $(\Omega, F, P)$ such that for all $t \in [0,\infty]$ we have a random variable $w \in \Omega$ where $w \mapsto X_t(w)$ is a continuous function $X_t(w)$ represents the result at time $t$ of the experiment $w$. We can write $X_t(w) = X(t,w)$ and define a function $T \times \Omega \rightarrow \mathbb{R}^n$ as $(t, w) \mapsto X(t,w)$. Let $x \in \mathbb{R}^n$ then $x \mapsto (\sum_{k=1}^n x_k^2)^{\frac{1}{2}} \equiv ||x||_{L^2}$ is the Euclidean norm.

Theorem:

Let $T > 0$ and $b(\cdot, \cdot) = \mu X_t : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma(\cdot, \cdot) = \sigma(t) X_t : [0, T] \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ be measurable functions satisfying

$$|\mu X_t(x) - \mu X_t(y)| + |\sigma(t) X_t(x) - \sigma(t) X_t(y)| \leq K |x - y| : x, y \in \mathbb{R}^n, t \in [0, T]$$

such that

$$|\mu X_t| + |\sigma(t) X_t| \leq C(1 + |x|), \text{ for some constant } C, x \in \mathbb{R}^n, t \in [0, T].$$

Then GFBM

$$dX_t = \mu X_t \, dt + \sigma(t) X_t \, dB_t^H$$

has unique solution in $t \in [0, T]$.

Proof:

FIRST: THE PROOF OF THE UNIQUENESS.

Let $X^1_t$ and $X^2_t$ are two solutions of Equation (17). Suppose that

$$a = a(s,w) = \mu(X^1_s - X^2_s)$$

and

$$\gamma = \gamma(s,w) = \sigma(s)(X^1_s - X^2_s).$$

Define $\tau^1_n = \inf\{ t \geq 0 \mid |X^1_t| \geq n \}$ and $\tau^2_n = \inf\{ t \geq 0 \mid |X^2_t| \geq n \}$, and let $S_n = \min\{\tau^1_n, \tau^2_n\}$. We need to prove that $E[|X^2_{t \wedge S_n} - X^1_{t \wedge S_n}|^2] \rightarrow 0$ for all $t \in [0, T]$. --end--
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Now for all \( t \in [0, T] \),
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] = E[| \int_0^{t\alpha S_n} \mu X_s^1 ds + \int_0^{t\alpha S_n} \sigma(s) X_s^1 dB_s^H - \int_0^{t\alpha S_n} \mu X_s^2 ds - \int_0^{t\alpha S_n} \sigma(s) X_s^2 dB_s^H |^2 ]
\]
\[
= E[(\int_0^{t\alpha S_n} a ds + \int_0^{t\alpha S_n} \gamma dB_s^H)^2]. \quad (18)
\]
However, \(|a + b|^2 \leq 2|a|^2 + 2|b|^2\), leading to the last expression will be
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] \leq E[2(\int_0^{t\alpha S_n} a ds)^2 + 2(\int_0^{t\alpha S_n} \gamma dB_s^H)^2]. \quad (19)
\]
By Using Lemma 1, Equation (19) will become
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] \leq 2E[(\int_0^{t\alpha S_n} a ds)^2] + 2E[(\int_0^{t\alpha S_n} \gamma dB_s^H, e)^2]. \quad (20)
\]
However, \( \int_0^t f(s, w)dB_s^H, e \rightarrow \int_0^t f(s, w)dB_s^H \), thus Equation (20) is approximately equal to
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] \cong 2E[(\int_0^{t\alpha S_n} a ds)^2] + 2E[(\int_0^{t\alpha S_n} \gamma dB_s^H, e)^2]. \quad (21)
\]
We substitute Equation (12) into Equation (21) and get
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] = 2E[(\int_0^{t\alpha S_n} a ds)^2] + 2E[(\int_0^{t\alpha S_n} \gamma (\alpha \phi_s ds + e^\alpha dW_s))^2]
\]
\[
= 2E[(\int_0^{t\alpha S_n} a ds)^2] + 2E[(\alpha \int_0^{t\alpha S_n} \gamma \phi_s ds + e^\alpha \int_0^{t\alpha S_n} \gamma dW_s)^2]
\]
\[
\leq 2E[(\int_0^{t\alpha S_n} a ds)^2] + 2E[2(\alpha \int_0^{t\alpha S_n} \gamma \phi_s ds)^2] + 2(e^\alpha \int_0^{t\alpha S_n} \gamma dW_s)^2]
\]
\[
\leq 2E[(\int_0^{t\alpha S_n} a ds)^2] + 4\alpha^2 E[(\int_0^{t\alpha S_n} \gamma \phi_s ds)^2] + 4e^{2\alpha} E[(\int_0^{t\alpha S_n} \gamma dW_s)^2]
\]
By Lemma 3,
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] \leq 2tE[\int_0^{t\alpha S_n} a^2 ds] + 4\alpha^2 E[(\int_0^{t\alpha S_n} \gamma \phi_s ds)^2] + 4e^{2\alpha} E[\int_0^{t\alpha S_n} \gamma^2 ds]. \quad (22)
\]
By applying Equation (15) to (22) we get;
\[
E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] \leq (2t + 4e^{2\alpha})K_n^2 \int_0^{t\alpha S_n} E[|X_{t\alpha S_n}^2 - X_{t\alpha S_n}^1|^2] ds + 4\alpha^2 E[(\int_0^{t\alpha S_n} \gamma \phi_s ds)^2]. \quad (23)
\]
Now, we aim to show that \( \mathbb{E}\left[(\int_0^{t_{\Delta N}} \mathcal{G}_s ds)^2\right] = 0 \) is the last expression in Equation (23).

Since \( \sigma(t)X_t \) is bounded, for some constant \( M \) will provide

\[
\mathbb{E}\left[(\int_0^{t_{\Delta N}} \mathcal{G}_s ds)^2\right] \leq M^2 \mathbb{E}\left[(\int_0^{t_{\Delta N}} \mathcal{G}_s ds)^2\right].
\]

By Equation (14) now become

\[
\left(\int_0^t \mathcal{G}_s ds\right)^2 = \frac{1}{a^2} [B_{t}^{H,e} - \epsilon^a W_t]^2 \leq \frac{1}{a^2} \left[ (B_{t}^{H,e})^2 + (\epsilon^a W_t)^2 \right].
\]

Then

\[
\mathbb{E}\left[(\int_0^{t_{\Delta N}} \mathcal{G}_s ds)^2\right] \leq \frac{1}{a^2} \left[ \mathbb{E}\left[B_{t}^{H,e} \right]^2 + 2\alpha \mathbb{E}[W_t^2] \right] \leq \frac{1}{a^2} \left[ \left( \mathbb{E}[B_{t}^{H,e}] \right)^2 + 2\alpha (\mathbb{E}[W_t])^2 \right].
\]

As \( \mathbb{E}[B_{t}^{H,e}] = 0 \) and \( \mathbb{E}[W_t] = 0 \),

\[
\mathbb{E}\left[(\int_0^{t_{\Delta N}} \mathcal{G}_s ds)^2\right] = 0.
\]

Therefore Equation (19) becomes

\[
\mathbb{E}\left[|X_{t_{\Delta N}}^2 - X_{t_{\Delta N}}^1|^2\right] \leq (2t + 4\epsilon^2\alpha)K_n^2 \int_0^{t_{\Delta N}} \mathbb{E}\left[|X_{s_{\Delta N}}^2 - X_{s_{\Delta N}}^1|^2\right] ds.
\]

We define \( \psi(t) = \mathbb{E}\left[|X_{t_{\Delta N}}^1 - X_{t_{\Delta N}}^2|^2\right] \), so \( \forall t \in [0,T] \) we have

\[
\psi(t) \leq (2t + 4\epsilon^2\alpha)K_n^2 \int_0^t \psi(s)ds.
\]

By applying Lemma 2 \( C = 0 \) and \( u(s) = (2t + 4\epsilon^2\alpha)K_n^2 \), then we get \( \psi(t) = 0 \) or

\[
\mathbb{E}\left[|X_{t_{\Delta N}}^1 - X_{t_{\Delta N}}^2|^2\right] = 0
\]

which implies that \( X_{t_{\Delta N}}^1 = X_{t_{\Delta N}}^2 \) \( \forall t \in [0,S_n] \). Since \( t \mapsto X_t^1 \) and \( t \mapsto X_t^2 \) are continuous, this implies the result for \( t \in [0,S_n] \) as \( n \to \infty \), so we obtained the uniqueness of solution on \( [0,T] \).

**SECOND: THE PROOF OF THE EXISTENCE.**

Consider a stochastic differential equation (SDE),

\[
dX_t = \mu X_t dt + \sigma(t)X_t dB_t^H.
\]

When \( X_0^\epsilon = X_0 \), the corresponding approximation equation of Equation (26) will become

\[
dX_t^\epsilon = \mu X_t^\epsilon dt + \sigma(t)X_t^\epsilon dB_t^{H,e}
\]

By using Equation (12), we can write Equation (27) as

\[
dX_t^\epsilon = \mu X_t^\epsilon dt + \sigma(t)X_t^\epsilon \{\alpha \Phi_t^\epsilon dt + \epsilon^a dW_s\}
\]

\[
= (\mu X_t^\epsilon + \sigma(t)X_t^\epsilon \alpha \Phi_t^\epsilon) dt + \epsilon^a \sigma(t)X_t^\epsilon dW_s.
\]
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Replace $b_1(t, X^e_t) = \mu X^e_t + \sigma(t)X^e_t \alpha \Phi^e_t$ and $\sigma_1(t, X^e_t) = e^a(\sigma(t)X^e_t)$ into Equation (28) leads to

$$dX^e_t = b_1(t, X^e_t) dt + \sigma_1(t, X^e_t) dW_s.$$ (29)

Equation (29) can also be written as

$$X^e_t = X_0 + \int_0^t b_1(t, X^e_s) ds + \int_0^t \sigma_1(t, X^e_s) dW_s.$$ (30)

Equation (29) and Equation (30) represent stochastic differential equation with standard Brownian motion $W_s$ where $b_1(t, X^e_t)$ and $\sigma_1(t, X^e_t)$ satisfy Definition 2, Equation (12) and Equation (17) i.e.

$$|b_1(t, x) - b_1(t, y)| + |\sigma_1(t, x) - \sigma_1(t, y)| \leq D |x - y|$$ (31)

and

$$|b_1(t, x)| + |\sigma_1(t, x)| \leq L(1 + |x|).$$ (32)

If solution for Equation (30) exists, this implies that the solution for Equation (27) also exists. Therefore, it exists for Equation (26).

To prove the existence of Equation (30), we will follow the approach of Oksendal (2003).

We define $Y^0_t = X_0$ and $Y^k_t = Y^k_t(w)$ such that

$$Y^{k+1}_t = X_0 + \int_0^t b_1(t, Y^k_s) ds + \int_0^t \sigma_1(t, Y^k_s) dW_s.$$ (33)

By similar computations as in the case of uniqueness, we have

$$E \left[ |Y^{k+1}_t - Y^k_t|^2 \right] \leq (2t + 4e^{2a}) D^2 \int_0^{T \wedge S_n} E \left[ |Y^k_t - Y^{k-1}_t|^2 \right] ds.$$ (34)

Now mathematical induction is applied to Equation (34),

Let $k \geq 1$ and $t \leq T$, then we have

$$E[|Y^1_t - Y^0_t|^2] \leq 2E \left[ \int_0^t b^2_1(t, X^e_0) ds \right] + 2E \left[ \int_0^t \sigma^2_1(t, X^e_0) dW_s \right].$$ (35)

By Lemma 3

$$E[|Y^1_t - Y^0_t|^2] \leq 2tE \left[ \int_0^t b^2_1(t, X^e_0) ds \right] + 2E \left[ \int_0^t \sigma^2_1(t, X^e_0) dW_s \right].$$ (36)

We adopt Equation (32) to Equation (35) implies

$$E[|Y^1_t - Y^0_t|^2] \leq 2tE \left[ \int_0^t L^2 (1 + |X^e_0|^2) ds \right] + 2E \left[ \int_0^t L^2 (1 + |X^e_0|^2) dW_s \right].$$

$$\leq 2tL^2 \int_0^t E(1 + 2|X^e_0| + |X^e_0|^2) ds + 2L^2 \int_0^t E(1 + 2|X^e_0| + |X^e_0|^2) ds.$$ (37)

$$\leq 2tL^2 (1 + E(|X^e_0|^2)) + 2tL^2 (1 + E(|X^e_0|^2)).$$
\[
\begin{align*}
\leq 2tL^2(1 + E[|X_0|^2])(t + 1) \\
\leq 2T L^2 (1 + E[|X_0|^2]) t = A_1 t,
\end{align*}
\]

where \( A_1 \) is a constant that depends on \( T, L \) and \( E[|X_0|^2] \). Now by induction on \( k \geq 0 \) and \( t \leq T \) we get

\[
E \left[|Y_{t+k+1}^k - Y_t^k|^2\right] \leq \frac{A_2^{k+1} t^{k+1}}{(k+1)!},
\]

where \( A_2 \) is a constant that depends on \( T, L, D \) and \( E[|X_0|^2] \).

Now,
\[
\sup_{0 \leq s \leq T} |Y_{t}^{k+1} - Y_t^k| \leq \int_0^T |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})| ds + \sup_{0 \leq s \leq T} \left| \int_0^t (\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1})) dW_s \right|.
\]  

We know that \( P(G > 2^{-k}) \leq P(G > 2^{-k-1}) \), imply Equation (37)
\[
P \left[ \sup_{0 \leq s \leq T} |Y_{t}^{k+1} - Y_t^k| > 2^{-k} \right]
\leq P \left[ \left( \int_0^T |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})| ds \right)^2 > 2^{-2k-2} \right] + P \left[ \sup_{0 \leq s \leq T} \left| \int_0^t (\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1})) dW_s \right| > 2^{-k-1} \right].
\]

Lemma (4) and Lemma (3) are applied to right hand side of Equation (38), then we get
\[
P \left[ \left( \int_0^T |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})| ds \right)^2 > 2^{-2k-2} \right] \leq 2^{2k+2} T \int_0^T E \left( |b_1(s, Y_s^k) - b_1(s, Y_s^{k-1})|^2 \right) ds
\]

and
\[
P \left[ \sup_{0 \leq s \leq T} \left| \int_0^t (\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1})) dW_s \right| > 2^{-k-1} \right] \leq 2^{2k+2} \int_0^T E \left( |\sigma_1(s, Y_s^k) - \sigma_1(s, Y_s^{k-1})|^2 \right) ds.
\]

By substituting Equation (39) and Equation (40) into Equation (38) and Equation (32), we obtain
\[
P \left[ \sup_{0 \leq s \leq T} |Y_{t}^{k+1} - Y_t^k| \right] \leq 2^{2k+2} D^2 (T + 1) \int_0^T E \left( |Y_t^k - Y_t^{k-1}|^2 \right) dt.
\]
By using Equation (36), Equation (41) can be written as
\[
P \left[ \sup_{0 \leq t \leq T} |Y_t^{k+1} - Y_t^k| > 2^{-k} \right] \leq 2^{2k+2} D^2 (T + 1) \int_0^T \frac{A_2^{k+1}}{(k+1)!} dt
\]
\[
= 2^{2k+2} D^2 (T + 1) \frac{A_2^{2k+2}}{(k+1)!}.
\]
However, \(A_2 > D^2 (T + 1)\), thus
\[
P \left[ \sup_{0 \leq t \leq T} |Y_t^{k+1} - Y_t^k| > 2^{-k} \right] \leq 2^{-k} a_{2k+2} \frac{A_2^{2k+2}}{(k+1)!}.
\]
This means that \(P \left[ \sup_{0 \leq t \leq T} |Y_t^{k+1} - Y_t^k| > 2^{-k} \right]\) is bounded, so Lemma (5) is used as follows
\[
P \left[ \lim_{n \to \infty} \sup_{0 \leq t \leq T} |Y_t^{k+1} - Y_t^k| > 2^{-k} \right] = 0.
\]
This follows that for almost all \(w\), there exists \(k_0\) such that
\[
\sup_{0 \leq t \leq T} |Y_t^{k+1} - Y_t^k| \leq 2^{-k} \text{ for } k \geq k_0.
\]
Therefore, the sequence
\[
Y_t^n (w) = Y_t^0 (w) + \sum_{k=0}^{n-1} (Y_t^{k+1} (w) - Y_t^k (w))
\]
is uniformly convergent in \([0, T]\), for almost all \(w\).

If we denote \(X_t^e = X_t (w) = \lim_{n \to \infty} Y_t^n (w)\), then \(X_t^e\) is continuous on \(t\) almost all \(w\) since \(Y_t^n (w)\) has the same property for all \(n\).

As we know that every Cauchy sequence is convergent, by using Equation (36) we have
\[
E[|Y_t|^2]^{1/2} = \|Y_t - Y_t^n\|_{L^2 (p)} = \left\| \sum_{k=0}^{n-1} (Y_t^{k+1} (w) - Y_t^k (w)) \right\|_{L^2 (p)}
\]
\[
\leq \sum_{k=0}^{n-1} \left\| Y_t^{k+1} (w) - Y_t^k (w) \right\|_{L^2 (p)} \leq \sum_{k=0}^{n-1} \left\| \frac{A_2^{k+1} t^{k+1}}{(k+1)!} \right\|_{L^2 (p)} \to 0 \text{ as } n \to \infty
\]
for \(m > n \geq 0\).

Equation (42) proved that sequence \(\{Y_t^n\}\) converges in \(L^2 (p)\) to certain limits say \(Y_t\).
Since subsequence \(Y_t^n (w)\) converges to \(Y_t (w)\) for all \(w\), we must have \(Y_t = X_t^e\) almost surely (i.e. \(X_t^e (w) = \lim_{n \to \infty} Y_t^n (w) = Y_t (w)\)).

Now, we will prove that \(X_t^e\) satisfies Equation (26) and Equation (29). For all \(n\),
\[
Y_t^{n+1} = X_0^e + \int_0^t b(s, Y_s^n) ds + \int_0^t \sigma_1 (s, Y_s^n) dW_s.
\]
Now, for all \(t \in [0, T]\), we have \(Y_t^{n+1} \to X_t^e\) as \(n \to \infty\) uniformly for almost all \(w\). By equation (44) and the Lemma (6), we have
\[
E \left[ \int_0^T |X_t^e - Y_t^m|^2 dt \right] \leq \lim_{m \to \infty} \sup \ E \left[ \int_0^T |Y_t^m - Y_t^n|^2 dt \right] \to 0 \text{ as } n \to \infty.
\]
By Lemma (3),
\[ E \left[ \int_0^T |X_t^\varepsilon - Y_t^n|^2 dt \right] = E \left[ \left( \int_0^T |X_t^\varepsilon - Y_t^n| dW_s \right)^2 \right] \rightarrow 0. \]
This implies that \( X_t^\varepsilon - Y_t^n \rightarrow 0 \), consequently
\[ \int_0^t b_1(s, Y_s^n)ds \rightarrow \int_0^t b_1(s, X_s^\varepsilon)ds \quad \text{and} \quad \int_0^t \sigma_1(s, Y_s^n)dW_s \rightarrow \int_0^t \sigma_1(s, X_s^\varepsilon)dW_s. \]
By taking the limit for Equation (45) as \( n \rightarrow \infty \),
\[ X_t^\varepsilon = \lim_{n \rightarrow \infty} Y_t^{n+1} = X_0^\varepsilon + \lim_{n \rightarrow \infty} \int_0^t b_1(s, Y_s^n)ds + \lim_{n \rightarrow \infty} \int_0^t \sigma_1(s, Y_s^n)dW_s \]
\[ = X_0^\varepsilon + \int_0^t \lim_{n \rightarrow \infty} b_1(s, Y_s^n)ds + \int_0^t \lim_{n \rightarrow \infty} \sigma_1(s, Y_s^n)dW_s \]
\[ = X_0^\varepsilon + \int_0^t b_1(s, X_s^\varepsilon)ds + \int_0^t \sigma_1(s, X_s^\varepsilon)dW_s. \]
This is the end of the proof.

**DISCUSSION**

In this work, we presented a new proof of the theorem of existence and uniqueness for the solution of geometric fractional Brownian motion equation. The proof of the uniqueness rests on approximation approach besides some identities and inequalities such as Ito isometry theorem, Jensen’s inequality and Gronwall’s inequality. Meanwhile, the proof of the existence uses the approximation approach that can convert fractional stochastic equation driven by fraction Brownian motion to an equivalent stochastic equation driven by standard Brownian motion. Then we prove the existence of solution for the converted equation by adopting Oskandal(2003). The proof depends on some inequalities and Lemmas, namely Ito isometry, Doob’s martingale inequality, Borel - Cantelli lemma and Fatou’s lemma.

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**REFERENCES**


