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On Strongly Generalized w-closed Sets In w-spaces

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Abstract

The purpose of this note is to introduce the notions of strongly generalized w-closed set(simply, sgw-closed set) and strongly generalized w-open(simply, sgw-open) set in w-spaces, and study some basic properties of such the notions. It is investigated that every w-open set is sgw-open and every sgw-open set is gw-open but a gw-open set may not be sgw-open in a given w-space.

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1. Introduction

Siwiec [18] introduced the notions of weak neighborhoods and weak base in a topological space. We introduced the weak neighborhood systems defined by using the notion of weak neighborhoods in [13]. The weak neighborhood system induces a weak neighborhood space which is independent of neighborhood spaces [4] and general topological spaces [2]. The notions of weak structure and w-space were investigated in [14]. In fact, the set of all g-closed subsets [5] in a topological space is a kind of weak structure.

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Levine [5] introduced the notion of g-closed subsets in topological spaces. In the same way, in [16], we introduced the notion of generalized w-closed set (simply, gw-closed set) in weak spaces, and investigated some basic properties of such notions. The one purpose of our research is to introduce strongly generalize w-open sets (strongly generalize w-closed sets) in w-spaces which are gw-open but not conversely, and investigate some of basic properties of such notions.

2. Preliminaries

Definition 2.1. [14] Let X be a nonempty set. A subfamily w_X of the power set $P(X)$ is called a *weak structure* on X if it satisfies the following:

(1) $\emptyset \in w_X$ and $X \in w_X$.

(2) For $U_1, U_2 \in w_X$, $U_1 \cap U_2 \in w_X$.

Then the pair (X, w_X) is called a *w-space* on X. Then $V \in w_X$ is called a *w-open* set and the complement of a w-open set is a *w-closed* set.

The collection of all w-open sets (resp., w-closed sets) in a w-space X will be denoted by $WO(X)$ (resp., $WC(X)$). We set $W(x) = \{U \in WO(X) : x \in U\}$.

Let S be a subset of a topological space X. The closure (resp., interior) of S will be denoted by clS (resp., $intS$). A subset S of X is called a *preopen* set [11] (resp., α-open set [17], *semi-open* [6]) if $S \subset int(cl(S))$ (resp., $S \subset int(cl(int(S)))$, $S \subset cl(int(S))$). The complement of a preopen set (resp., α-open set, *semi-open*) is called a *preclosed* set (resp., α-closed set, *semi-closed*). The family of all preopen sets (resp., α-open sets, semi-open sets) in X will be denoted by $PO(X)$ (resp., $\alpha(X)$, $SO(X)$). We know the family $\alpha(X)$ is a topology finer than the given topology on X.

A subset A of a topological space (X, τ) is said to be:

(a) g-closed [5] if $cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(b) gp-closed [7] if $pCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(c) gs-closed [1, 3] if $sCl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(d) $g\alpha$-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is α-open in X where $\tau^\alpha = \alpha(X)$;

(e) $g\alpha^*$-closed [8] if $\tau^\alpha Cl(A) \subset int(U)$ whenever $A \subset U$ and U is α-open in X;

(f) $g\alpha^{**}$-closed [8] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is α-open in X;

(g) αg-closed [9] if $\tau^\alpha Cl(A) \subset U$ whenever $A \subset U$ and U is open in X;

(h) $\alpha^{**}g$-closed [9] if $\tau^\alpha Cl(A) \subset int(cl(U))$ whenever $A \subset U$ and U is open in X.

Then the family τ, $GO(X)$, $g\alpha O(X)$, $g\alpha^* O(X)$, $g\alpha^{**} O(X)$, $\alpha g O(X)$ and $\alpha^{**} g O(X)$ are all weak structures on X. But $PO(X)$, $GPO(X)$ and $SO(X)$ are not weak structures on X. A subfamily m_X of the power set $P(X)$ of a nonempty set X is called a *minimal structure* on X [10] if $\emptyset \in w_X$ and $X \in w_X$. Thus clearly every weak structure is a minimal structure.

Definition 2.2. [14] Let (X, w_X) be a w-space. For a subset A of X, the *w-closure* of A and the *w-interior* of A are defined as follows:

(1) $wC(A) = \cap\{F : A \subseteq F, X - F \in w_X\}$.

(2) $wI(A) = \cup\{U : U \subseteq A, U \in w_X\}$.

Theorem 2.3. [14] Let (X, w_X) be a w-space and $A \subseteq X$.

(1) $x \in wI(A)$ if and only if there exists an element $U \in W(x)$ such that $U \subseteq A$.

(2) $x \in wC(A)$ if and only if $A \cap V \neq \emptyset$ for all $V \in W(x)$.

(3) If $A \subset B$, then $wI(A) \subset wI(B)$; $wC(A) \subset wC(B)$.

(4) $wC(X - A) = X - wI(A)$; $wI(X - A) = X - wC(A)$.

(5) If A is w-closed (resp., w-open), then $wC(A) = A$ (resp., $wI(A) = A$).

Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a generalized w-closed set* (simply, gw-closed set) [16] if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is w-open. Then if the w_X-structure is a topology, the generalized w-closed set is exactly a generalized closed set in sense of Levine in [5]. And every w-closed set is generalized w-closed, but in general, the converse is not true. From now on, the family of all gw-open sets (resp., gw-closed sets) in X will be denoted by $GWO(X)$ (resp., $GWC(X)$).

3. Main results

Definition 3.1. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a strongly generalized w-closed set* (simply, sgw-closed set) if $wC(A) \subseteq U$, whenever $A \subseteq U$ and U is gw-open.

The family of all sgw-open sets (resp., sgw-closed sets) in X will be denoted by $SGWO(X)$ (resp., $SGWC(X)$).

Remark 3.2. If the w_X-structure is a topology, the strongly generalized w-closed set is exactly a strongly g-closed set introduced in [19].

Theorem 3.3.

(1) Every w-closed set is strongly generalized w-closed.

(2) Every strongly generalized w-closed set is generalized w-closed.

The converses of the above theorem are not true, in general, as seen in the next example.

Example 3.4. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{a, c\}, \{a, d\}, X\}$ be a w-structure in X. Note that:

$$\begin{aligned}
WC(X) &= \{\emptyset, \{b, c\}, \{b, d\}, \{b, c, d\}, \{a, c, d\}, X\};\\
GWC(X) &= \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\},\\
&\qquad \{a, c, d\}, \{b, c, d\}, X\};\\
SGWC(X) &= \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}.
\end{aligned}$$

From the facts, we can show that the converses of the above theorem are not always true.

Lemma 3.5. [14] Let (X, w_X) be a w-space and $A, B \subseteq X$. Then the following things hold:

(1) $wI(A) \cap wI(B) = wI(A \cap B)$.

(2) $wC(A) \cup wC(B) = wC(A \cup B)$.

Theorem 3.6. Let (X, w_X) be a w-space. Then the union of two sgw-closed sets is a sgw-closed set.

Proof. Let A and B be any two sgw-closed sets. Let G be any gw-open set such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since A and B are sgw-closed sets, $wC(A) \subseteq G$ and $wC(B) \subseteq G$. Now, by Lemma 3.5, $wC(A \cup B) = wC(A) \cup wC(B) \subseteq G$. So, $A \cup B$ is sgw-closed. ■

In general, the intersection of two sgw-closed sets is not sgw-closed:

Example 3.7. Let $X = \{a, b, c, d\}$ and $w_X = \{\emptyset, \{a\}, \{b\}, \{a, d\}, X\}$ be a w-structure in X. Note:

$$\begin{aligned}
WC(X) &= \{\emptyset, \{b, c\}, \{b, c, d\}, \{a, c, d\}, X\};\\
GWC(X) &= \{\emptyset, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\},\\
&\qquad \{a, c, d\}, \{b, c, d\}, X\};\\
GWO(X) &= \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\},\\
&\qquad \{a, b, d\}, X\};\\
SGWC(X) &= \{\emptyset, \{c\}, \{b, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}, \{b, c, d\}, X\}.
\end{aligned}$$

Now, consider two sgw-closed sets $A = \{a, b, c\}$ and $B = \{a, c, d\}$. Then $A \cap B$ is not sgw-closed.

Theorem 3.8. Let (X, w_X) be a w-space. Then if A is a sgw-closed set, then $wC(A)-A$ contains no non-empty gw-closed set.

Proof. Suppose that there is a gw-closed set F such that $F \subseteq wC(A) - A$. Then $A \subseteq X - F$, and since $X - F$ is gw-open and A is sgw-closed, $wC(A) \subseteq X - F$. It implies that $F \subseteq X - wC(A)$, and so $F \subseteq wC(A) \cap (X - wC(A)) = \emptyset$. Hence, $F = \emptyset$. ■

In general, the converse in Theorem 3.8 is not true as shown in the next example.

Example 3.9. In Example 3.7, consider $A = \{a, c\}$. Then $wC(A) = \{a, c, d\}$ and $wC(A) - A = \{d\}$. Since $\{d\}$ is not gw-closed, $wC(A) - A$ contains no any non-empty sw-closed set, but A is not sgw-closed.

Corollary 3.10. Let (X, w_X) be a w-space. Then if A is a sgw-closed set, then $wC(A)-A$ contains no non-empty w-closed set.

Proof. Since every w-closed set is gw-closed, it follows from the above theorem. ■

Theorem 3.11. Let (X, w_X) be a w-space. Then if A is a gw-closed set and $A \subseteq B \subseteq wC(A)$, then B is sgw-closed.

Proof. Let U be any gw-open set satisfying $B \subseteq U$. Then since A is a sgw-closed set, clearly it is obtained $wC(A) \subseteq U$. By hypothesis, $wC(B) = wC(A) \subseteq U$ and so B is sgw-closed set. ■

Definition 3.12. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is called *a strongly generalized w-open set* (simply, sgw-open set) if $X - A$ is gw-closed.

Theorem 3.13. Let (X, w_X) be a w-space and $A \subseteq X$. Then A is sgw-open if and only if $F \subseteq wI(A)$ whenever $F \subseteq A$ and F is gw-closed.

Proof. Obvious. ■

Theorem 3.14. Let (X, w_X) be a w-space. Then the intersection of two sgw-open sets is a sgw-open set.

Proof. It is obvious from Theorem 3.6. ■

In general, the union of two sgw-open sets is not sgw-open (See Example 3.7).

Theorem 3.15. Let (X, w_X) be a w-space and $A \subseteq X$. Then if A is sgw-open, then $U = X$, whenever $wI(A) \cup (X - A) \subseteq U$ and U is gw-open.

Proof. Let U be any gw-open set and $wI(A) \cup (X - A) \subseteq U$. Then $X - U \subseteq (X - wI(A)) \cap A = wC(X - A) \cap A = wC(X - A) - (X - A)$. Since $X - A$ is sgw-closed, by Theorem 3.8, the gw-closed set $X - U$ must be empty. Hence, $U = X$. ■

Theorem 3.16. Let (X, w_X) be a w-space. Then if A is a sgw-open set and $wI(A) \subseteq B \subseteq A$, then B is sgw-open.

Proof. It is similar to the proof of Theorem 3.11. ■

Theorem 3.17. Let (X, w_X) be a w-space. Then if A is a sgw-closed set, then $wC(A) - A$ is sgw-open.

Proof. Suppose that A is a sgw-closed set. Then by Theorem 3.8, the empty set is the only one gw-closed subset of $wC(A) - A$. So, for the only gw-closed subset $\emptyset$ of $wC(A) - A$, $\emptyset \subseteq wC(A) - A$ and $\emptyset \subseteq wI(wC(A) - A)$. Hence, $wC(A) - A$ is sgw-open. ■

Theorem 3.18. Let (X, w_X) be a w-space. Then if A is a sgw-open set, then $wI(A) \cup (X - A)$ is sgw-closed.

Proof. Suppose that A is a sgw-open set. Then by Theorem 3.15, the whole set X is the only one gw-open set containing $wI(A) \cup (X - A)$. So, $wI(A) \cup (X - A) \subseteq X$ and $wC(wI(A) \cup (X - A)) \subseteq X$. Hence, $wI(A) \cup (X - A)$ is sgw-closed. ■

Let (X, w_X) be a w-space. For a subset A of X, *sgw-closure* of A and *sgw-interior* of A are defined as the following:

$$sgwC(A) = \cap\{F : A \subseteq F, F \text{ is } sgw\text{-closed}\} :$$
$$sgwI(A) = \cup\{U : U \subseteq A, U \text{ is } sgw\text{-open}\}.$$

Theorem 3.19. Let (X, w_X) be a w-space and $A \subseteq X$.

(1) If A is sgw-open, then $sgwI(A) = A$.

(2) If A is sgw-closed, then $sgwC(A) = A$.

Proof. Obvious. ■

But the converses in the above theorem are not always true as shown in the next example.

Example 3.20. In Example 3.7, let $F = \{a, c\}$. Then $gwC(F) = \{a, c\}$. But F is not sgw-closed. Consider $A = \{a, c, d\}$. Then $gwI(A) = A$. But A is not sgw-open.

Theorem 3.21. Let (X, w_X) be a w-space and $A, B \subseteq X$. Then the following things hold:

(1) If $A \subseteq B$, then $sgwI(A) \subseteq sgwI(B)$ and $sgwC(A) \subseteq sgwC(B)$.

(2) $sgwC(X - A) = X - sgwI(A)$; $sgwI(X - A) = X - sgwC(A)$.

(3) $x \in sgwI(A)$ if and only if there exists a sgw-open set U containing x such that $U \subseteq A$.

(4) $x \in sgwC(A)$ if and only if $A \cap V \neq \emptyset$ for all sgw-open set V containing x.

Proof. From definitions of the sgw-closure and sgw-interior, it is proved directly. ∎

Theorem 3.22. Let (X, w_X) be a w-space and $A, B \subset X$. Then the following things hold:

(1) $\emptyset = sgwC(\emptyset)$.

(2) $A \subseteq sgwC(A)$.

(3) $sgwC(A \cup B) = sgwC(A) \cup sgwC(B)$.

(4) $sgwC(sgwC(A)) = sgwC(A)$.

Proof. (1) and (2) are obvious.

(3) From Theorem 3.21, $sgwC(A \cup B) \supseteq sgwC(A) \cup sgwC(B)$. Now, for the proof of the other inclusion, let $x \notin sgwC(A) \cup sgwC(B)$. Then there exist sgw-closed sets F_1 and F_2 such that $x \notin F_1$ and $A \subseteq F_1$; $x \notin F_2$ and $B \subseteq F_2$. So $x \notin F_1 \cup F_2$ and $A \cup B \subseteq F_1 \cup F_2$. Since $F_1 \cup F_2$ is sgw-closed, we have that $x \notin sgwC(A \cup B)$. Consequently, $sgwC(A \cup B) = sgwC(A) \cup sgwC(B)$.

(4) We only show that $sgwC(sgwC(A)) \subseteq sgwC(A)$. For any sgw-closed set F satisfying $A \subseteq F, sgwC(A) \subseteq sgwC(F) = F$, and so $\{F : A \subseteq F, F \text{ is } sgw\text{-closed}\} \subseteq \{K : sgwC(A) \subseteq K, K \text{ is } sgw\text{-closed}\}$. This implies that $sgwC(sgwC(A)) = \cap\{K : sgwC(A) \subseteq K, K \text{ is } sgw\text{-closed}\} \subseteq \cap\{F : A \subseteq F, F \text{ is } sgw\text{-closed}\} = sgwC(A)$. ∎

Theorem 3.23. Let (X, w_X) be a w-space and $A, B \subset X$. Then the following things hold:

(1) $X = sgwI(X)$.

(2) $sgwI(A) \subseteq A$.

(3) $sgwI(A \cap B) = sgwI(A) \cap sgwI(B)$.

(4) $sgwI(sgwI(A)) = sgwI(A)$.

Proof. These are easily obtained by Theorem 3.21 and Theorem 3.22. ∎

Finally, we have the following theorem:

Theorem 3.24. Let (X, w_X) be a w-space. Then

(1) the family $SGWO(X)$ of all sgw-open sets is a w-space finer than w_X;

(2) the family $\tau^* = \{U \subseteq X : U = sgwI(U)\}$ is a finer topology than $SGWO(X)$.

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