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Differential equations associated with Genocchi polynomials

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Abstract

Recently, several authors have studied nonlinear differential equations arising from the generating functions of various special functions (see [3, 8–15]). We derive a family of non-linear differential equations from the generating functions of the Genocchi polynomials and study the solutions of these differential equations.

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1. Introduction

The Euler polynomials are defined by the generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}\quad \text{(see [4 − 7])}. \tag{1}$$

When $x=0$, $E_n=E_n(0)$, $(n\geq 0)$, are called Euler numbers.

The Genocchi polynomials $G_n(x)$ are defined by generating functions as follows:

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!},\quad \text{(see [1, 2, 5, 7])}. \tag{2}$$

In the special case $x=0$, $G_n(0)=G_n$ for $n=0,1,\dots$ are called the n-th Genocchi numbers.

For $r\in\mathbb{N}$, we consider that the higher order Genocchi polynomials $G_n^{(r)}(x)$ are defined by generating functions as follows:

$$\left(\frac{2t}{e^t+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}G_n^{(r)}(x)\frac{t^n}{n!},\quad \text{(see [1, 2])}. \tag{3}$$

In the special case $x=0$, $G_n^{(r)}(0)=G_n^{(r)}$ for $n=0,1,\dots$ are called the higher order n-th Genocchi numbers.

Recently, T. Kim et al. have studied nonlinear differential equations arising from Bernoulli, Euler, Frobenius-Euler, Changhee, Mittag-Leffer, degenerate Bell, λ-Changhee, degenerate Bernoulli polynomials. (see [3, 8-15]). In this paper, we use the idea recently developed by Kim et al. (see [14]). We derive a family of non-linear differential equations from the generating function of Genocchi polynomials and study the solutions of these differential equations.

2. The non-linear differential equations for Genocchi polynomials

Let

$$F=F(t)=\frac{2}{e^t+1}. \tag{4}$$

From (4), we get

$$F'=F'(t)=\frac{-2e^t}{(e^t+1)^2}=-F+\frac{1}{2}F^2. \tag{5}$$

It is arranged as following.

$$\frac{1}{2}F^2=F+F'. \tag{6}$$

Let us take derivative in (6) with respect to t. Then we obtain

$$\frac{1}{2}2!FF' = F' + F''. \tag{7}$$

From (5) and (7), we can derive the following equation.

$$\frac{1}{2^2}2!F^3 = 2F + 3F' + F''. \tag{8}$$

and furthermore

$$\frac{1}{2^3}3!F^4 = 6F + 11F' + 6F'' + F^{(3)}, \tag{9}$$

$$\frac{1}{2^4}4!F^5 = 24F + 50F' + 35F'' + 10F^{(3)} + F^{(4)}. \tag{10}$$

Continuing this process, we set

$$\frac{1}{2^N}N!F^{N+1} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)}, N = 1, 2, \ldots, \tag{11}$$

where $\left(\frac{d}{dt}\right)^k F(t) = F^{(k)}$.

Let us take derivative in (11) with respect to t. From (5) and (11), we obtain

$$\frac{1}{2^N}(N+1)!F^N\Big(-F + \frac{1}{2}F^2\Big) = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k+1)}$$

so that

$$\frac{1}{2^{N+1}}(N+1)!F^{N+2} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k+1)} + (N+1) \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)}. \tag{12}$$

On the other hand, by replacing N by $N+1$ in (11), we have

$$\frac{1}{2^{N+1}}(N+1)!F^{N+2} = \sum_{k=0}^{N+1} \alpha_k^{(N+1)} F^{(k)} \tag{13}$$

so that, by (12) and (13), we get the followings.

$$\begin{aligned} \alpha_0^{(N+1)} &= (N+1)\alpha_0^{(N)}, \\ \alpha_{N+1}^{(N+1)} &= \alpha_N^{(N)} \end{aligned} \tag{14}$$

and for $k = 1, 2, \ldots, N$,

$$\alpha_k^{(N+1)} = (N+1)\alpha_k^{(N)} + \alpha_{k-1}^{(N)}. \tag{15}$$

We see $\alpha_0^{(1)} = \alpha_1^{(1)} = 1$ from (6), so that, from (14) we have the followings.

$$\begin{aligned} \alpha_0^{(N+1)} &= (N+1)\alpha_0^{(N)} = (N+1)N\alpha_0^{(N-1)} \\ &= \cdots = (N+1)\cdots 2\alpha_0^{(2)} = (N+1)!, \\ \alpha_{N+1}^{(N+1)} &= \alpha_N^{(N)} = \cdots = \alpha_1^{(1)} = 1 \end{aligned} \tag{16}$$

For $k = 1, 2, \ldots, N$, we get from (15),

$$\begin{aligned} \alpha_k^{(N+1)} &= (N+1)\alpha_k^{(N)} + \alpha_{k-1}^{(N)} \\ &= (N+1)\big(N\alpha_k^{(N-1)} + \alpha_{k-1}^{(N-1)}\big) + \alpha_{k-1}^{(N)} \\ &= (N+1)^{<2>}\alpha_k^{(N-1)} + (N+1)\alpha_{k-1}^{(N-1)} + \alpha_{k-1}^{(N)} \\ &= (N+1)^{<3>}\alpha_k^{(N-2)} + (N+1)^{<2>}\alpha_{k-1}^{(N-2)} + (N+1)\alpha_{k-1}^{(N-1)} \\ &\quad + \alpha_{k-1}^{(N)} \\ &= \cdots \\ &= (N+1)^{<N+1-k>}\alpha_k^{(k)} + \sum_{n_1=k}^{N} (N+1)^{<N-n_1>}\alpha_{k-1}^{(n_1)} \\ &= \sum_{n_1=k-1}^{N} (N+1)^{<N-n_1>}\alpha_{k-1}^{(n_1)} \end{aligned} \tag{17}$$

where $N^{<i>} = N(N-1)\cdots(N-i+1)$.

Continuing the work,

$$\begin{aligned} \alpha_k^{(N+1)} &= \sum_{n_1=k-1}^{N} (N+1)^{<N-n_1>} \sum_{n_2=k-2}^{n_1-1} {n_1}^{<n_1-1-n_2>}\alpha_{k-2}^{(n_2)} \\ &= \cdots \\ &= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_m=k-m}^{n_{m-1}-1} (N+1)^{<N-n_1>}{n_1}^{<n_1-1-n_2>} \cdots \\ &\quad \times {n_{m-1}}^{<n_{m-1}-1-n_m>}\alpha_{k-m}^{(n_m)} \\ &= \cdots \\ &= \sum_{n_1=k-1}^{N} \sum_{n_2=k-2}^{n_1-1} \cdots \sum_{n_k=0}^{n_{k-1}-1} (N+1)^{<N-n_1>}{n_1}^{<n_1-1-n_2>} \cdots \\ &\quad \times {n_{k-1}}^{<n_{k-1}-n_k>}\alpha_0^{(n_k)} \end{aligned}$$

so that it gives

$$\begin{aligned}\alpha_k^{(N+1)} &= \sum_{n_1=k-1}^{N} (N+1)^{<N-n_1>} \sum_{n_2=k-2}^{n_1-1} {n_1}^{<n_1-1-n_2>} \times \\ &\quad \cdots \times \sum_{n_k=0}^{n_{k-1}-1} {n_{k-1}}^{<n_{k-1}-n_k>} {n_k}^{<n_k>} \\ &= \left(\prod_{m=1}^{k} \sum_{n_m=k-m}^{n_{m-1}-1} {n_{m-1}}^{<n_{m-1}-1-n_m>} \right) (n_k)! \end{aligned} \tag{18}$$

where $n_0 = N + 1$.

Therefore, we obtain the following theorem.

Theorem 2.1. Let $F^{(k)}(t) = \dfrac{\mathrm{d}^k F(t)}{\mathrm{d}t^k}$ and $F^N(t) = \underbrace{F(t) \times \cdots \times F(t)}_{N\text{-times}}$. For $N = 2, 3, \ldots$, the following non-linear differential equation with respect to t:

$$\frac{1}{2^N} N! F^{N+1} = \sum_{k=0}^{N} \alpha_k^{(N)} F^{(k)} \tag{19}$$

has a solution $F = F(t) = \dfrac{2t}{e^t + 1}$. Where,

$$\begin{aligned}\alpha_k^{(N+1)} &= \sum_{n_1=k-1}^{N} (N+1)^{<N-n_1>} \sum_{n_2=k-2}^{n_1-1} {n_1}^{<n_1-1-n_2>} \times \\ &\quad \cdots \times \sum_{n_k=0}^{n_{k-1}-1} {n_{k-1}}^{<n_{k-1}-n_k>} {n_k}^{<n_k>} \\ &= \left(\prod_{m=1}^{k} \sum_{n_m=k-m}^{n_{m-1}-1} {n_{m-1}}^{<n_{m-1}-1-n_m>} \right) (n_k)!, \end{aligned} \tag{20}$$

where $n_0 = N + 1$.

It is not difficult to show that

$$\begin{aligned} F^{(k)} &= \left(\frac{d}{dt}\right)^k F(t) = \left(\frac{d}{dt}\right)^k \left(\frac{1}{t}\cdot\frac{2t}{e^t+1}\right) \\ &= \left(\frac{d}{dt}\right)^k \left(\frac{1}{t}\sum_{n=0}^{\infty} G_n \frac{t^n}{n!}\right) = \left(\frac{d}{dt}\right)^k \left(\frac{1}{t} + \sum_{n=1}^{\infty} G_n \frac{t^{n-1}}{n!}\right) \\ &= \left(\frac{d}{dt}\right)^k \left(\frac{1}{t} + \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1}\cdot\frac{t^n}{n!}\right) \\ &= (-1)^k k! t^{-k-1} + \sum_{n=0}^{\infty} \frac{G_{n+k+1}}{n+k+1}\cdot\frac{t^n}{n!}. \end{aligned} \tag{21}$$

From (21), we get

$$\begin{aligned} t^{N+1}\sum_{k=0}^{N}\alpha_k^{(N)}F^{(k)} &= \sum_{k=0}^{N}\alpha_k^{(N)}(-1)^k k! t^{N-k} \\ &\quad + \sum_{k=0}^{N}\sum_{n=0}^{\infty}\alpha_k^{(N)}\frac{G_{n+k+1}}{n+k+1}\cdot\frac{t^{N+1+n}}{n!} \\ &= \sum_{n=0}^{N}\left(\alpha_{N-n}^{(N)}(-1)^{N-n}(N-n)!n!\right)\frac{t^n}{n!} \\ &\quad + \sum_{n=N+1}^{\infty}\left(\sum_{k=0}^{N}\alpha_k^{(N)} n^{<N+1>}\frac{G_{n-N+k}}{n-N+k}\right)\frac{t^n}{n!}. \end{aligned} \tag{22}$$

On the other hand,

$$\begin{aligned} t^{N+1}\frac{1}{2^N}N!F^{N+1} &= \frac{1}{2^N}N!\left(\frac{2t}{e^t+1}\right)^{N+1} \\ &= \frac{1}{2^N}N!\sum_{n=0}^{\infty}G_n^{(N+1)}\frac{t^n}{n!}. \end{aligned} \tag{23}$$

Therefore, by Theorem 2.1, (22) and (23), we obtain the following.

Theorem 2.2. For $n \in \mathbb{N}\cup\{0\}$ and $N \in \mathbb{N}$, we have the followings.
For $n \geq N+1$,

$$G_n^{(N+1)} = \sum_{k=0}^{N} 2^N \alpha_k^{(N)} (N+1)\binom{n}{N+1}\frac{G_{n-N+k}}{n-N+k},$$

for $0 \leq n \leq N$,

$$G_n^{(N+1)} = 2^N \alpha_{N-n}^{(N)} (-1)^{N-n}\frac{1}{\binom{N}{n}}.$$

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