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Balanced Biclique Polynomial of Graphs

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Abstract

This paper introduces the notion of balanced biclique polynomial of graphs. We characterized the balanced biclique subgraphs of some special graphs. In addition, we obtained explicit forms of the balanced biclique polynomials of these graphs.

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1. Introduction

There are a number of graph polynomials that have been widely studied. Chromatic polynomials count the number of proper colourings of a graph [8, 11, 12, 24]. Matching polynomials enumerate matching [17]. Independence polynomials are generating polynomials for the number of independent sets of each cardinality [19]. One of the most general approaches to graph polynomials was proposed by Farrel in 1979 in his theory of F-polynomials of a graph. According to Farrel [14], any such polynomial corresponds to a strictly prescribed family of connected subgraphs of the respective graph. For the matching polynomial of a graph G, this family consist of all edges of G, for the independence polynomial of G, this family includes all the stable set of G.

J. I. Brown, et al. [9], examined the effects of various graph operations on neighborhood polynomials, which are generating functions for the number of faces of each cardinality in the neighborhood complex of a graph. They provide explicit polynomials for hypercubes, for graphs not containing four-cycle and for graphs resulting from joins and Cartesian products.

F.M. Dong, et al. [13], had determined the vertex-cover polynomial of the path,cycle, wheel, and complete bipartite graph. Moreover, they developed a method to calculate the vertex-cover polynomial of a graph. Motivated by a problem in biological systematics, they consider a mapping f from $\{1, 2, \dots, m\}$ into the vertex set of a graph, subject to $f^{-1}(u) \cup f^{-1}(v) \neq \varnothing$ for every edge xy in G. They showed that the number of such mappings can be determined from the vertex-cover polynomial.

Saieed Akbari, et al. [1] introduced the edge cover polynomial. They showed that if $E(G, x) = E(H, x)$, then the degree sequence of G and H are the same. They showed that cycles and complete bipartite graphs are determined by their edge cover polynomials. Also they determined all graphs G for which $E(G, x) = E(P_n, x)$.

A. Vijayan [21] introduced a total edge fixed geodominating sets and polynomials of graphs $G_t(G, x)$. They obtained some properties of $G_t(G, x)$ and its coefficients. They also compute polynomials for complete graph, bipartite graph and the corona of any graph G with complete graph K_1 of order 1.

Ali, et al. [2] obtained the Wiener polynomial $W_n(G, x)$ for some special graphs including paths and cycle graphs. Moreover, for vertex-disjoint connected graphs G_1 and G_2 formulas for Wiener polynomials of Steiner n-distance of compound graphs are also obtained in terms of those polynomials for G_1 and G_2.

Laja and Artes [18] draw some properties of convex subgraph polynomials and generated the explicit forms of this polynomial. Their paper suggest that the sum of the zeros of this polynomial is the negative of the coefficient of $x^{|V(G)|-1}$ and the product of the zeros is $(-1)^{|V(G)|}$.

Vijayan and Vijila Dafini [22] come out with geodetic polynomials of the centipedes which is $P_n^* = x^n(1 + x)^n$.

Alikhani and Peng [4] shows the relationship between the denomination polynomial of graphs containing an induced at least three path of length and the denomination polynomial of related graphs obtain by replacing the path by shorter path.

Askari and Alaeiyan [7] derive some properties of the coefficients of the edge denomination polynomial and state that the edge domination polynomial of G is equal to the vertex domination polynomias of the line graph $L(G)$ of G.

Arocha and Llano [6] stress that the mean value of the matching polynomial is computed in the family of all labeled graphs. Also, the mean value of dominating polynomial is determined in a special family of bipartite graphs.

2. Preliminaries

An induced subgraph H of a graph G is a balanced biclique of G if $H \equiv K_{i,i}$ for some $i \in \left\{1, 2, \dots, \frac{|V(G)|}{2}\right\}$. In this case, the order of H is exactly $2i$. The balanced biclique polynomial of G is given by

$$b(G, x) = \sum_{i=1}^{\frac{\beta(G)}{2}} b_i(G)x^{2i},$$

where $b_i(G)$ is the number of balanced bicliques of G of order $2i$ and $\beta(G)$ is the cardinality of a maximum balanced biclique of G.

Note that $b_1(G) = |E(G)|$ and $b_2(G)$ is just the number of induced C_4 of G.

If G is a tree, then $\beta(G) = 2$. In this case, $b(G, x) = (n-1)x^2$ since a tree has $n-1$ edges. Hence, it would be interesting to consider cyclic graphs.

Remark 2.1. The degree of the balanced biclique polynomial is at most $2\Delta(G)$.

3. Bicliques of Cycles, Complete Graphs and Planar Grids

The following result characterizes the balanced bicliques of cycles of order greater than 4. For $n = 4$, $b(C_4, x) = 4x^2 + x^4$.

Lemma 3.1. Let $n \geq 5$. A subset S of $V(C_n)$ induces a balanced biclique in C_n if and only if $\langle S \rangle = K_{1,1}$.

The above result is clear since for $n \geq 5$, C_n has no $K_{2,2}$. Also, ff G has $K_{r,r}$, then it has $K_{i,i}$ for every $i \in \{1, 2, \ldots, r-1\}$.

It is interestng to note that the maximum balanced biclique of a graph G has cardinality atmost twice the independence number of G. This follows from the fact that each vertex partition of $K_{i,i}$ is an independent set. Hence, if $I(G)$ is the independence number of G, then $\deg b(G, x) \leq 2I(G)$. In addition, this bound is sharp. Indeed, $\deg b(K_{m,m}, x) = 2m = |K_{m,m}|$, where m here is also the independence number of $K_{m,m}$.

Theorem 3.2. Let $n \geq 5$. Then $b(C_n, x) = nx^2$.

The following result characterises the balanced bicliques of complete graphs K_n.

Lemma 3.3. A subset S of $V(K_n)$ induces a balanced biclique in K_n if and only if $\langle S \rangle = K_{1,1}$.

Proof. The maximum independent set of K_n has cardinality 1. The result follows immediately. ■

Hence, the following is immediate.

Theorem 3.4. For $n \geq 2$,

$$b(K_n, x) = \binom{n}{2} x^2.$$

We consider now a planer grid $P_m \times P_n$. In the following result, we characterized the balanced bicliques of $P_m \times P_n$ for $m, n \geq 2$.

Lemma 3.5. A subset S of $V(P_m \times P_n)$ with $|S_2| > 2$ induces a balanced biclique of $P_m \times P_n$ if and only if $\langle S \rangle = C_4$.

Proof. Note that for $m, n \geq 3$, $\Delta(P_m \times P_n) = 4$. Hence, a balanced biclique has at most 8 vertices. Clearly, $P_m \times P_n$ has no $K_{4,4}$. Moreover, it has no $K_{3,3}$. Consequently, $\langle S \rangle = K_{2,2} = C_4$. The converse follows from the definition of a balanced biclique of a graph. ■

The next result establishes the balanced biclique polynomial of the planar grid $P_m \times P_n$.

Theorem 3.6. Let $m, n \geq 2$. Then

$$b(P_m \times P_n, x) = ((n-1)m + (m-1)n)x^2 + (n-1)(m-1)x^4.$$

Proof. The size of $P_m \times P_n$ is $(n-1)m + (m-1)n$, which is the coefficient of x^2. Moreover, $P_m \times P_n$ has exactly $(n-1)(m-1)$ induced C_4. The assertion follows from Lemma 3.5. ■

Theorem 3.7. If G has no induced C_4 and $\Delta(G) = 2$, then $\deg(G, x) = 2$.

Proof. If G has no induced C_4 and $\Delta(G) = 2$, then $G = P_n$ or $G = C_n$ for some $n \geq 2$. Note that $b_i(G) = 0$ for $i > \Delta(G)$. ■

4. Bicliques of Complete q-partite Graphs

Next, we present a result on the balanced bicliques of the complete bipartite graph $K_{m,n}$. We have the following characterization.

Lemma 4.1. A subset S of $V(K_{m,n})$ induces a balanced biclique in $K_{m,n}$ if and only if $S = S_1 \cup S_2$ where $S_1 \subseteq V(\overline{K_m})$ and $S_2 \subseteq V(\overline{K_n})$ with $|S_1| = |S_2|$.

Proof. The sets $V(\overline{K_m})$ and $V(\overline{K_n})$ are totally independent. Moreover, every vertex of $V(\overline{K_m})$ is adjacent with every vertex of $V(\overline{K_n})$. Hence, any balanced biclique of $K_{m,n}$ is of the form $\langle S \rangle = \langle S_1 \cup S_2 \rangle$ where $S_1 \subseteq V(\overline{K_m})$ and $S_2 \subseteq V(\overline{K_n})$ with $|S_1| = |S_2|$. The converse follows from the definition of a complete bipartite graph. ■

From the above result we have the balanced biclique polynomial of the complete bipartite graph $K_{m,n}$.

Theorem 4.2. For $m \leq n$,

$$b(K_{m,n}, x) = \sum_{i=1}^{m} \binom{m}{i}\binom{n}{i} x^{2i}.$$

Proof. Let $K_{m,n} = \overline{K_m} \oplus \overline{K_n}$ where $V(\overline{K_m}) = \{u_1, u_2, \ldots, u_m\}$ and $V(\overline{K_n}) = \{v_1, v_2, \ldots, v_n\}$. By Lemma 4.1, any balanced biclique of $K_{m,n}$ is of the form $\overline{K_p} \oplus \overline{K_p}$ where $p \leq m$. Note that for each p with $1 \leq p \leq m$, there are $\binom{m}{p}$ combinations of p-

sets of $V(\overline{K_m})$. Similarly, there are $\binom{n}{p}$ combinations of p-sets of $V(\overline{K_n})$. Consequently there are $\binom{m}{p}\binom{n}{p}$ combinations of $2p$-sets with p-sets coming from $V(K_m)$ and p-sets coming from $V(K_n)$. Hence, the coefficient of x^{2p} in $b(K_{m,n}, x)$ is exactly equal to $\binom{m}{p}\binom{n}{p}$. ■

Lastly, we generalized the above result to q-partite graph. The following result characterizes the balanced bicliques of the complete q-bipartite graph $K_{r_1,r_2,\ldots r_q}$ for $q > 2$.

Lemma 4.3. A subset S of $V(K_{r_1,r_2,\ldots,r_q})$ induces a balanced biclique in $K_{r_1,r_2,\ldots,r_q}$ if and only if $S = S_i \cup S_j$ where $S_i \subseteq V(\overline{K_i})$ and $S_j \subseteq V(\overline{K_j})$ with $|S_i| = |S_j|$.

Proof. Note that

$$K_{r_1,r_2,\ldots r_q} = \overline{K_{r_1}} \oplus \overline{K_{r_2}} \oplus \cdots \oplus \overline{K_{r_q}}$$

where $\langle \overline{K_{r_i}} \rangle_{i=1}^{q}$ is totally independent. Hence, any biclique of $K_{r_1,r_2,\ldots,r_q}$ must of the form $\langle S \rangle = \langle S_i \cup S_j \rangle$ for some $S_i \subseteq V(\overline{K_{r_i}})$ and $S_j \subseteq V(\overline{K_{r_j}})$, with $|S_i| = |S_j|$.

The converse follows from the definition of the balanced biclique of a graph. ■

Finally, we have the following result on the balanced biclique polynomial of the complete q- partite graph.

Theorem 4.4. Let $\langle r_i \rangle_{i=1}^{q}$ be an increasing sequence of positive integers. Then

$$b(K_{r_1,r_2,\ldots,r_q}, x) = \sum_{k=1}^{r_{q-1}} \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{k}\binom{r_j}{k} x^{2k}.$$

Proof. The total number of edges of $K_{r_1,r_2,\ldots,r_q}$ is $\sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{1}\binom{r_j}{1}$. Hence,

$$b_1(K_{r_1,r_2,\ldots,r_q}) = \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{1}\binom{r_j}{1}$$

Now, $K_{r_1,r_2,\ldots,r_q} = \overline{K_{r_1}} \oplus \overline{K_{r_2}} \oplus \cdots \oplus \overline{K_{r_q}}$. Fix $i \in \{1, 2, \ldots, q\}$. Taking two vertices of $V(\overline{K_{r_i}})$ and another two vertices in $V(\overline{K_{r_j}})$ with $i < j$ will induce a $K_{2,2}$. Thus,

$$b_2(K_{r_1,r_2,\ldots,r_q}) = \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{2}\binom{r_j}{2}$$

Similarly,

$$b_3(K_{r_1,r_2,\ldots,r_q}) = \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{3} \binom{r_j}{3}$$

Continuing the process gives for $r \leq r_{q-1}$,

$$b_r(K_{r_1,r_2,\ldots,r_q}) = \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{r} \binom{r_j}{r}$$

Combining all the cases gives the balanced biclique polynomial of the complete q-partite graph $K_{r_1,r_2,\ldots,r_q}$ which is given by

$$b(K_{r_1,r_2,\ldots,r_q}, x) = \sum_{k=1}^{r_{q-1}} \sum_{j=2}^{q} \sum_{i<j} \binom{r_i}{k} \binom{r_j}{k} x^{2k}.$$

■

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