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# On poly-tangent numbers and polynomials and distribution of their zeros

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**Abstract**

In this paper we introduce the poly-tangent polynomials and numbers. We also give some properties, explicit formulas, several identities, a connection with poly-tangent numbers and polynomials, and some integral formulas. Finally, we investigate the zeros of the poly-tangent polynomials by using computer.

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**Keywords:** Tangent numbers and polynomials, poly-tangent numbers and polynomials, Cauchy numbers, Stirling numbers.

## 1. Introduction

Many mathematicians have worked in the area of the Bernoulli numbers and polynomials, Euler numbers and polynomials, Genocchi numbers and polynomials, tangent numbers and polynomials, poly-Bernoulli numbers and polynomials, poly-Euler numbers and polynomials(see [1-11]). In this paper, we define poly-tangent polynomials and numbers and study some properties of the poly-tangent polynomials and numbers. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We remember that the classical Stirling numbers of the first kind $S_1(n, k)$ and $S_2(n, k)$ are defined by the relations(see [11])

$$(x)_n = \sum_{k=0}^{n} S_1(n, k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n, k)(x)_k, \tag{1.1}$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order $n$. The numbers $S_2(n, m)$ also admit a representation in terms of a generating function

$$(e^t - 1)^m = m! \sum_{n=m}^{\infty} S_2(n, m)\frac{t^n}{n!}. \tag{1.2}$$

We also have

$$m!\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}=(\log(1+t))^m. \tag{1.3}$$

We also need the binomial theorem: for a variable $x$,

$$\frac{1}{(1-t)^c}=\sum_{n=0}^{\infty}\binom{c+n-1}{n}t^n. \tag{1.4}$$

The poly-Bernoulli numbers $B_n^{(k)}$ were introduced by Kaneko [5] by using the following generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{1-e^{-t}}=\sum_{n=0}^{\infty}B_n^{(k)}\frac{t^n}{n!}, \quad (k\in\mathbb{Z}), \tag{1.5}$$

where

$$\mathrm{Li}_k(t)=\sum_{n=1}^{\infty}\frac{t^n}{n^k} \tag{1.6}$$

is the $k$th polylogarithm function.

The poly-Euler polynomials $E_n^{(k)}(x)$ are defined by generating function

$$\frac{\mathrm{Li}_k(1-e^{-t})}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}, \quad (k\in\mathbb{Z}). \tag{1.7}$$

The familiar tangent polynomials $\mathbf{T}_n(x)$ are defined by the generating function([7]):

$$\left(\frac{2}{e^{2t}+1}\right)e^{xt}=\sum_{n=0}^{\infty}\mathbf{T}_n(x)\frac{t^n}{n!}, \quad (|2t|<\pi). \tag{1.8}$$

When $x=0$, $\mathbf{T}_n(0)=\mathbf{T}_n$ are called the tangent numbers. The tangent polynomials $\mathbf{T}_n^{(r)}(x)$ of order $r$ are defined by

$$\left(\frac{2}{e^{2t}+1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{T}_n^{(r)}(x)\frac{t^n}{n!}, \quad (|2t|<\pi). \tag{1.9}$$

It is clear that $r=1$ we recover the tangent polynomials $\mathbf{T}_n(x)$.

The Bernoulli polynomials $\mathbf{B}_n^{(r)}(x)$ of order $r$ are defined by the following generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt}=\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}, \quad (|t|<2\pi). \tag{1.10}$$

The Frobenius-Euler polynomials of order $r$, denoted by $\mathbf{H}_n^{(r)}(u,x)$, are defined as

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u,x)\frac{t^n}{n!}. \tag{1.11}$$

The values at $x = 0$ are called Frobenius-Euler numbers of order $r$; when $r = 1$, the polynomials or numbers are called ordinary Frobenius-Euler polynomials or numbers.

Many kinds of of generalizations of these polynomials and numbers have been presented in the literature (see [1-11]). In the following section, we introduce the poly-tangent polynomials and numbers. After that we will investigate some their properties. We also give some relationships both between these polynomials and tangent polynomials and between these polynomials and cauchy numbers. Finally, we investigate the zeros of the poly-tangent polynomials by using computer.

## 2. Poly-tangent polynomials and numbers

In this section, we define poly-tangent numbers and polynomials and provide some of their relevant properties.

The poly-tangent polynomials $T_n^{(k)}(x)$ are defined by the generating function:

$$\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}e^{xt} = \sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!}, \quad (k \in \mathbb{Z}). \tag{2.1}$$

When $x = 0$, $T_n^{(k)}(0) = T_n^{(k)}(x)$ are called the poly-tangent numbers. Upon setting $k = 1$ in (2.1), we have

$$T_n^{(1)}(x) = n\mathbf{T}_{n-1}(x) \text{ for } n \geq 1.$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\right)e^{xt}\\ &= \sum_{n=0}^{\infty} T_n^{(k)}\frac{t^n}{n!}\sum_{n=0}^{\infty} x^n\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}x^{n-l}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

By comparing the coefficients on both sides of (2.2), we have the following theorem.

**Theorem 2.1.** For $n \in \mathbb{Z}_+$, we have

$$T_n^{(k)}(x) = \sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}x^{n-l}.$$

The following elementary properties of the poly-tangent numbers $T_n^{(k)}$ and polynomials $T_n^{(k)}(x)$ are readily derived form (2.1). We, therefore, choose to omit details involved.

**Theorem 2.2.** For $k \in \mathbb{Z}$, we have

$$(1) \quad T_n^{(k)}(x+y) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)}(x) y^{n-l}.$$

$$(2) \quad T_n^{(k)}(2-x) = \sum_{l=0}^{n} (-1)^l \binom{n}{l} T_{n-l}^{(k)}(2) x^l$$

**Theorem 2.3.** For any positive integer $n$, we have

$$\begin{aligned}
&(1) \quad T_n^{(k)}(mx) = \sum_{l=0}^{n} \binom{n}{l} T_l^{(k)}(x)(m-1)^{n-l} x^{n-l}. \\
&(2) \quad T_n^{(k)}(x+1) - T_n^{(k)}(x) = \sum_{l=0}^{n-1} \binom{n}{l} T_l^{(k)}(x). \\
&(3) \quad \frac{d}{dx} T_n^{(k)}(x) = n T_{n-1}^{(k)}(x). \\
&(4) \quad T_n^{(k)}(x) = T_n^{(k)} + n \int_0^x T_{n-1}^{(k)}(t) dt.
\end{aligned} \tag{2.3}$$

From (1.6), (1.8), and (2.1), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x) \frac{t^n}{n!} &= \left( 2 \frac{\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1} \right) e^{xt} = \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{(l+1)^k} \frac{2e^{xt}}{e^{2t}+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \frac{2e^{(x-i)t}}{e^{2t}+1} \\
&= \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \sum_{n=0}^{\infty} \mathbf{T}_n(x-i) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{i=0}^{l+1} \binom{l+1}{i} (-1)^i \mathbf{T}_n(x-i) \right) \frac{t^n}{n!}.
\end{aligned} \tag{2.4}$$

By comparing the coefficients on both sides of (2.4), we have the following theorem.

**Theorem 2.4.** For $n \in \mathbb{Z}_+$, we have

$$T_n^{(k)}(x) = \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \mathbf{T}_n(x-j).$$

By using definition of tangent polynomials and Theorem 2.4, we have the following corollary.

**Corollary 2.5.** For any positive integer $n$, we have

$$T_n^{(k)}(2-x) = (-1)^n \sum_{l=0}^{\infty} \frac{1}{(l+1)^k} \sum_{j=0}^{l+1} \binom{l+1}{j} (-1)^j \mathbf{T}_n(x+j).$$

By (2.1), we note that

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} &= 2\sum_{l=0}^{\infty}(-1)^l e^{2lt} \sum_{l=0}^{\infty} \frac{(1-e^{-t})^{l+1}}{(l+1)^k} e^{xt} \\
&= 2\sum_{l=0}^{\infty}(-1)^l e^{2lt} \sum_{l=0}^{\infty} \frac{(1-e^{-t})}{(l+1)^k} e^{xt} \\
&= \sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1} \frac{2(-1)^{l+j-i}\binom{i+1}{j}}{(i+1)^k} e^{(2l-2i-j+x)t} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1} \frac{2(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x)^n}{(i+1)^k}\right)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients on both sides, we have the following theorem.

**Theorem 2.6.** For $n \in \mathbb{Z}_+$, we have

$$T_n^{(k)}(x) = 2\sum_{l=0}^{\infty}\sum_{i=0}^{l}\sum_{j=0}^{i+1} \frac{(-1)^{l+j-i}\binom{i+1}{j}(2l-2i-j+x)^n}{(i+1)^k}.$$

## 3. Some identities involving poly-tangent numbers and polynomials

In this section, we give several combinatorics identities involving poly-tangent numbers and polynomials in terms of Stirling numbers, falling factorial functions, raising factorial functions, Beta functions, Bernoulli polynomials of higher order, and Frobenius-Euler functions of higher order.

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\right)(1-(1-e^{-t}))^{-x} \\
&= \frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\sum_{l=0}^{\infty}\binom{x+l-1}{l}(1-e^{-t})^l
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{\infty} < x >_l \frac{(e^t-1)^l}{l!}\left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}e^{-lt}\right)\\
&= \sum_{l=0}^{\infty} < x >_l \sum_{n=0}^{\infty} S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty} T_n^{(k)}(-l)\frac{t^n}{n!} \qquad (3.1)\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k)}(-l) < x >_l\right)\frac{t^n}{n!},
\end{aligned}$$

where $< x >_l = x(x+1)\cdots(x+l-1)(l \geq 1)$ with $< x >_0 = 1$.

By comparing the coefficients on both sides of (3.1), we have the following theorem.

**Theorem 3.1.** For $n \in \mathbb{Z}_+$, we have

$$T_n^{(k)}(x) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k)}(-l) < x >_l .$$

By Theorem 2.1, get

$$\begin{aligned}
\int_0^1 T_n^{(k)}(x)dx &= \int_0^1 \sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}x^{n-l}dx\\
&= \sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}\frac{1}{n-l+1}.
\end{aligned} \qquad (3.2)$$

By (3.2) and Theorem 3.1, we have the following theorem.

**Theorem 3.2.** For any positive integer $n$, we have

$$\sum_{l=0}^{n}\binom{n}{l}T_l^{(k)}\frac{1}{n-l+1} = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k)}(-l)(-1)^l\widehat{c_l},$$

where $\widehat{c_l}$ are Cauchy numbers of the second kind (see [5]).

By (2.1) and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\right)((e^t-1)+1)^x\\
&= \frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\sum_{l=0}^{\infty}\binom{x}{l}(e^t-1)^l\\
&= \sum_{l=0}^{\infty}(x)_l\frac{(e^t-1)^l}{l!}\left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{l=0}^{\infty}(x)_l \sum_{n=0}^{\infty} S_2(n,l)\frac{t^n}{n!}\sum_{n=0}^{\infty} T_n^{(k)}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)T_{n-i}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.3}$$

By comparing the coefficients on both sides of (3.3), we have the following theorem.

**Theorem 3.3.** For $n \in \mathbb{Z}_+$, we have

$$T_n^{(k)}(x) = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}(x)_l S_2(i,l)T_{n-i}^{(k)}.$$

By (3.2) and Theorem 3.3, we have the following theorem.

**Theorem 3.4.** For any positive integer $n$, we have

$$\sum_{l=0}^{n}\binom{n}{l}\frac{T_{n-l}^{(k)}}{l+1} = \sum_{l=0}^{\infty}\sum_{i=l}^{n}\binom{n}{i}S_2(i,l)T_{n-i}^{(k)}c_l,$$

where $c_l$ are Cauchy numbers of the first kind (see [5]).

By Theorem 2.2, we note that

$$\begin{aligned}
\int_0^1 y^n T_n^{(k)}(x+y)dy &= \int_0^1 y^n \sum_{l=0}^{n}\binom{n}{l}T_{n-l}^{(k)}(x)y^l dy \\
&= \sum_{l=0}^{n}\binom{n}{l}T_{n-l}^{(k)}(x)\int_0^1 y^{n+l}dy \\
&= \sum_{l=0}^{n}\binom{n}{l}T_{n-l}^{(k)}(x)\frac{1}{n+l+1}.
\end{aligned} \tag{3.4}$$

From (2.1) and Theorem 2.2, we note that

$$\begin{aligned}
&\int_0^1 y^n T_n^{(k)}(x+y)dy = \left.\frac{y^n T_{n+1}^{(k)}(x+y)}{n+1}\right|_0^1 - \int_0^1 ny^{n-1}\frac{T_{n+1}^{(k)}(x+y)}{n+1}dy \\
&= \frac{T_{n+1}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}T_{n+1}^{(k)}(x+y)dy \\
&= \frac{T_{n+1}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_l^{(k)}(x)y^{n+1-l}dy \\
&= \frac{T_{n+1}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_l^{(k)}(x)\frac{1}{2n-l+1}.
\end{aligned} \tag{3.5}$$

Therefore, by (3.4) and (3.5), we obtain the following theorem.

**Theorem 3.5.** For $n \in \mathbb{Z}_+$, we have

$$T_{n+1}^{(k)}(x+1) = \sum_{l=0}^{n+1} \binom{n+1}{l} T_l^{(k)}(x) \frac{n}{2n-l+1} + \sum_{l=0}^{n} \binom{n}{l} T_{n-l}^{(k)}(x) \frac{n+1}{n+l+1}.$$

By (1.2), (1.10), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x)\frac{t^n}{n!} &= \left(\frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1}\right) e^{xt} \\
&= \frac{(e^t-1)^r}{r!}\frac{r!}{t^r}\left(\frac{t}{e^t-1}\right)^r e^{xt}\sum_{n=0}^{\infty} T_n^{(k)}\frac{t^n}{n!} \\
&= \frac{(e^t-1)^r}{r!}\left(\sum_{n=0}^{\infty}\mathbf{B}_n^{(r)}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} T_n^{(k)}\frac{t^n}{n!}\right)\frac{r!}{t^r} \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i}\mathbf{B}_i^{(r)}(x)T_{n-l-i}^{(k)}\right)\frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

**Theorem 3.6.** For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$T_n^{(k)}(x) = \sum_{l=0}^{n}\frac{\binom{n}{l}}{\binom{l+r}{r}} S_2(l+r,r)\sum_{i=0}^{n-l}\binom{n-l}{i} T_{n-l-i}^{(k)}\mathbf{B}_i^{(r)}(x).$$

From (2.1) and Theorem 2.2, we note that

$$\begin{aligned}
&\int_0^1 y^n T_n^{(k)}(x+y)dy = \left.\frac{y^n T_{n+1}^{(k)}(x+y)}{n+1}\right|_0^1 - \int_0^1 \frac{n y^{n-1} T_{n+1}^{(k)}(x+y)}{n+1}dy \\
&= \frac{T_{n+1}^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 \sum_{l=0}^{\infty}\frac{1}{(l+1)^k}\sum_{i=0}^{l+1}\binom{l+1}{i}(-1)^i \mathbf{T}_{n+1}(x+y-i)y^{n-1}dy \\
&= \frac{T_{n+1}^{(k)}(x+1)}{n+1} \\
&- \frac{n}{n+1}\sum_{l=0}^{\infty}\sum_{i=0}^{l+1}\sum_{j=0}^{n+1}\frac{\binom{l+1}{i}\binom{n+1}{l}}{(l+1)^k}(-1)^{n+1+i}\mathbf{T}_{n+1-j}(1-x+i)\int_0^1 y^{n-1}(1-y)^j dy
\end{aligned}$$

$$= \frac{T_{n+1}^{(k)}(x+1)}{n+1} - \frac{n}{n+1} \sum_{l=0}^{\infty} \sum_{i=0}^{l+1} \sum_{j=0}^{n+1} \frac{\binom{l+1}{i}\binom{n+1}{l}}{(l+1)^k} (-1)^{n+1+i} \mathbf{T}_{n+1-j}(1-x+i) B(n, j+1), \tag{3.6}$$

where $B(n, j)$ is beta integral (see [1]).

Therefore, by (3.5) and (3.6), we obtain the following theorem.

**Theorem 3.7.** For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n+1} \binom{n+1}{l} \frac{T_l^{(k)}(x)}{2n-l+1} = \sum_{l=0}^{\infty} \sum_{i=0}^{l+1} \sum_{j=0}^{n+1} \frac{\binom{l+1}{i}\binom{n+1}{l}}{(l+1)^k} (-1)^{n+1+i} \mathbf{T}_{n+1-j}(1-x+i) B(n, j+1).$$

By (1.2), (1.11), (2.1), and by using Cauchy product, we get

$$\begin{aligned}
\sum_{n=0}^{\infty} T_n^{(k)}(x) \frac{t^n}{n!} &= \left( \frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1} \right) e^{xt} \\
&= \frac{(e^t-u)^r}{(1-u)^r} \left( \frac{1-u}{e^t-u} \right)^r e^{xt} \frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1} \\
&= \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u, x) \frac{t^n}{n!} \sum_{i=0}^{r} \binom{r}{i} e^{it} (-u)^{r-i} \frac{1}{(1-u)^r} \frac{2\mathrm{Li}_k(1-e^{-t})}{e^{2t}+1} \\
&= \frac{1}{(1-u)^r} \sum_{i=0}^{r} \binom{r}{i} (-u)^{r-i} \sum_{n=0}^{\infty} \mathbf{H}_n^{(r)}(u, x) \frac{t^n}{n!} \sum_{n=0}^{\infty} T_n^{(k)}(i) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \frac{1}{(1-u)^r} \sum_{i=0}^{r} \binom{r}{i} (-u)^{r-i} \sum_{l=0}^{n} \binom{n}{l} \mathbf{H}_l^{(r)}(u, x) T_{n-l}^{(k)}(i) \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients on both sides, we have the following theorem.

**Theorem 3.8.** For $n \in \mathbb{Z}_+$ and $r \in \mathbb{N}$, we have

$$T_n^{(k)}(x) = \frac{1}{(1-u)^r} \sum_{i=0}^{r} \sum_{l=0}^{n} \binom{r}{i} \binom{n}{l} (-u)^{r-i} \mathbf{H}_l^{(r)}(u, x) T_{n-l}^{(k)}(i).$$

For $n \in \mathbb{N}$ with $n \geq 4$, we obtain

$$\begin{aligned}
\int_0^1 y^n T_n^{(k)}(x+y)dy &= y^{n+1}\frac{T_n^{(k)}(x+y)}{n+1}\Bigg|_0^1 - \int_0^1 ny^{n+1}\frac{T_{n-1}^{(k)}(x+y)}{n+1}dy \\
&= \frac{T_n^{(k)}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n+1}T_{n-1}^{(k)}(x+y)dy \\
&= \frac{T_n^{(k)}(x+1)}{n+1} - \frac{nT_{n-1}^{(k)}(x+1)}{(n+1)(n+2)} + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\int_0^1 y^{n+2}T_{n-2}^{(k)}(x+y)dy \\
&= \frac{T_n^{(k)}(x+1)}{n+1} + (-1)\frac{nT_{n-1}^{(k)}(x+1)}{(n+1)(n+2)} + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\frac{T_{n-2}^{(k)}(x+1)}{n+3} \\
&\quad + (-1)^3\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\int_0^1 y^{n+3}T_{n-3}^{(k)}(x+y)dy \\
&= \frac{T_n^{(k)}(x+1)}{n+1} + (-1)\frac{nT_{n-1}^{(k)}(x+1)}{(n+1)(n+2)} + (-1)^2\frac{n}{n+1}\frac{n-1}{n+2}\frac{T_{n-2}^{(k)}(x+1)}{n+3} \\
&\quad + (-1)^3\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{T_{n-3}^{(k)}(x+1)}{n+4} \\
&\quad + (-1)^4\frac{n}{n+1}\frac{n-1}{n+2}\frac{n-2}{n+3}\frac{n-3}{n+4}\int_0^1 y^{n+4}T_{n-4}^{(k)}(x+y)dy
\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}
\int_0^1 y^n T_n^{(k)}(x+y)dy &= \frac{T_n(x+1)}{n+1} \\
&\quad + \sum_{l=2}^{n}\frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)}T_{n-l+1}^{(k)}(x+1) \qquad (3.7) \\
&\quad + (-1)^n\frac{n!}{(n+1)(n+2)\cdots(2n)}\int_0^1 y^{2n}T_0^{(k)}(x+y)dy
\end{aligned}$$

Hence, by (3.4) and (3.7), we have the following theorem.

**Theorem 3.9.** For $n \in \mathbb{N}$ with $n \geq 2$, we have

$$\begin{aligned}
&\sum_{l=0}^{n}\binom{n}{l}T_{n-l}^{(k)}(x)\frac{1}{n+l+1} \\
&= \frac{T_n^{(k)}(x+1)}{n+1} + \sum_{l=2}^{n}\frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)}T_{n-l+1}^{(k)}(x+1) \\
&\qquad + (-1)^n\frac{n!}{(n+1)(n+2)\cdots(2n)(2n+1)}.
\end{aligned}$$

## 4. Distribution of zeros of the poly-tangent polynomials

This section aims to demonstrate the benefit of using numerical investigation to support theoretical prediction and to discover new interesting pattern of the zeros of the poly-tangent polynomials $T_n^{(k)}(x)$. The poly-tangent polynomials $T_n^{(k)}(x)$ can be determined explicitly. A few of them are

$$\begin{aligned}
T_0^{(k)}(x) &= 0,\\
T_1^{(k)}(x) &= 1,\\
T_2^{(k)}(x) &= -3 + 2^{1-k} + 2x\\
T_3^{(k)}(x) &= 4 - 3 \cdot 2^{2-k} + 2 \cdot 3^{1-k} - 9x + 3 \cdot 2^{1-k}x + 3x^2,\\
T_4^{(k)}(x) &= 3 + 3 \cdot 2^{3-2k} + 7 \cdot 2^{1-k} + 3 \cdot 2^{3-k} - 8 \cdot 3^{1-k} - 4 \cdot 3^{2-k}\\
&\quad + 16x - 3 \cdot 2^{4-k}x + 8 \cdot 3^{1-k}x - 18x^2 + 3 \cdot 2^{2-k}x^2 + 4x^3,\\
T_5^{(k)}(x) &= -14 - 15 \cdot 2^{3-2k} - 15 \cdot 2^{4-2k} - 25 \cdot 2^{2-k} + 5 \cdot 2^{3-k} + 50 \cdot 3^{1-k}\\
&\quad + 20 \cdot 3^{2-k} + 24 \cdot 5^{1-k} + 15x + 15 \cdot 2^{3-2k}x + 35 \cdot 2^{1-k}x + 15 \cdot 2^{3-k}x\\
&\quad - 40 \cdot 3^{1-k}x - 20 \cdot 3^{2-k}x + 40x^2 - 15 \cdot 2^{3-k}x^2 + 20 \cdot 3^{1-k}x^2 - 30x^3\\
&\quad + 5 \cdot 2^{2-k}x^3 + 5x^4.
\end{aligned}$$

We investigate the beautiful zeros of the poly-tangent polynomials $T_n^{(k)}(x)$ by using a computer. We plot the zeros of the poly-tangent polynomials $T_n^{(k)}(x)$ for $n = 30$, $k = -3, -1, 1, 3$ and $x \in \mathbb{C}$ (Figure 1).

In Figure 1(top-left), we choose $n = 30$ and $k = 1$. In Figure 1(top-right), we choose $n = 30$ and $k = 3$. In Figure 1(bottom-left), we choose $n = 30$ and $k = -1$. In Figure 1(bottom-right), we choose $n = 30$ and $k = -3$. Stacks of zeros of $T_n^{(k)}(x)$ for $1 \leq n \leq 50$ from a 3-D structure are presented (Figure 2).

In Figure 2(left), we choose $k = -3$. In Figure 2(middle), we choose $k = 1$. In Figure 2(right), we choose $k = 3$. Our numerical results for approximate solutions of real zeros of $T_n^{(k)}(x)$ are displayed (Tables 1, 2).

The plot of real zeros of $T_n^{(k)}(x)$ for $1 \leq n \leq 40$ structure are presented (Figure 3).

In Figure 3(left), we choose $k = -3$. In Figure 3(middle),we choose $k = 1$. In Figure 3(right), we choose $k = 3$.

We observe a remarkable regular structure of the complex roots of the poly-tangent polynomials $T_n^{(k)}(x)$. We also hope to verify a remarkable regular structure of the complex roots of the poly-tangent polynomials $T_n^{(k)}(x)$ (Table 1).

Next, we calculated an approximate solution satisfying poly-tangent polynomials $T_n^{(k)}(x) = 0$ for $x \in \mathbb{R}$. The results are given in Table 2 and Table 3.

By numerical computations, we will make a series of the following conjectures:

**Conjecture 4.1.** Prove that $T_n^{(1)}(x), x \in \mathbb{C}$, has $Re(x) = 1$ and $Im(x) = 0$ reflection symmetry analytic complex functions. However, $T_n^{(k)}(x), k \neq 1$, has not $Re(x) = a$ reflection symmetry for $a \in \mathbb{R}$.

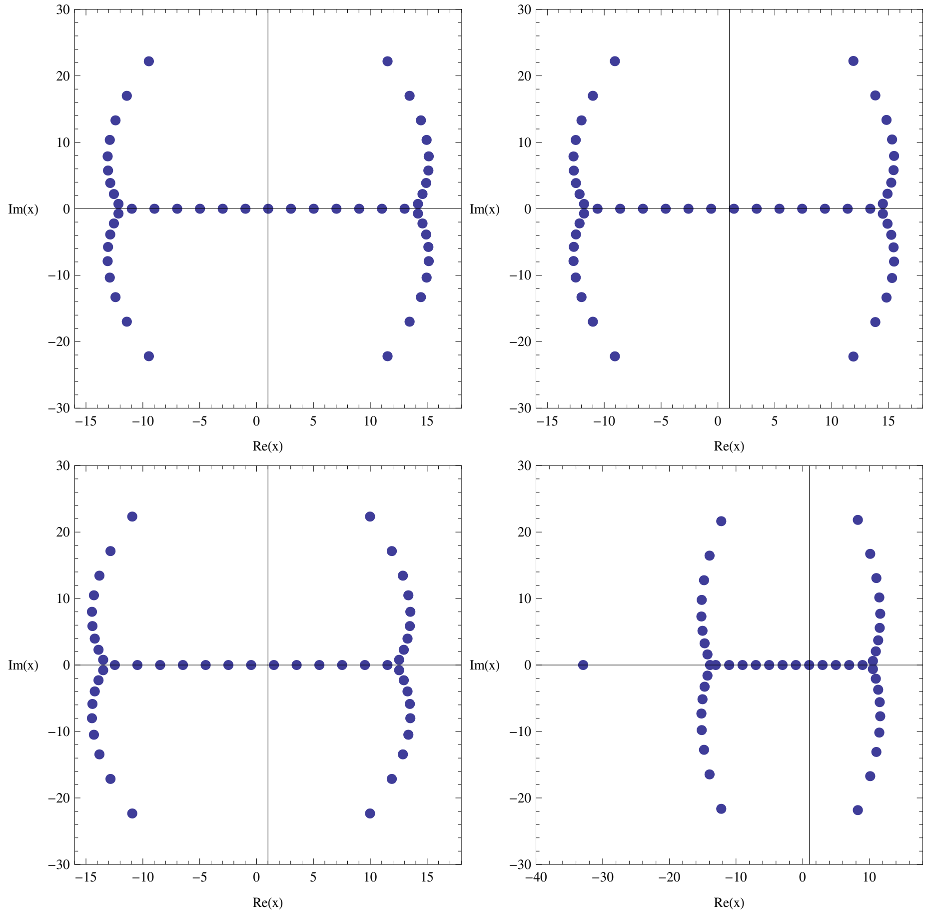


Figure 1: Zeros of $T_n^{(k)}(x)$

Using computers, many more values of $n$ have been checked. It still remains unknown if the conjecture fails or holds for any value $n$ (see Figures 1, 2, 3). We are able to decide if $T_n^{(k)}(x)) = 0$ has $n-1$ distinct solutions (see Tables 1, 2, 3).

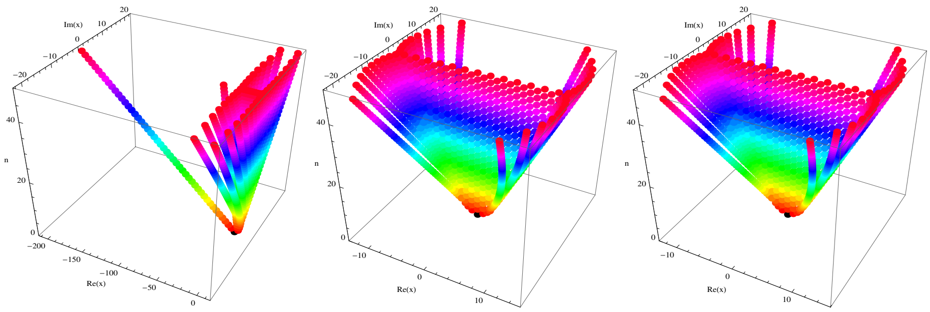

Figure 2: Stacks of zeros of $T_n^{(k)}(x)$ for $1 \leq n \leq 50$

Table 1: Numbers of real and complex zeros of $T_n^{(k)}(x)$

| degree $n$ | $k = 1$ | | $k = 3$ | | $k = -3$ | |
|---|---|---|---|---|---|---|
| | real zeros | complex zeros | real zeros | complex zeros | real zeros | complex zeros |
| 2 | 1 | 0 | 1 | 0 | 1 | 0 |
| 3 | 2 | 0 | 2 | 0 | 2 | 0 |
| 4 | 3 | 0 | 3 | 0 | 3 | 0 |
| 5 | 4 | 0 | 4 | 0 | 4 | 0 |
| 6 | 5 | 0 | 3 | 2 | 5 | 0 |
| 7 | 2 | 4 | 2 | 4 | 4 | 2 |
| 8 | 3 | 4 | 3 | 4 | 5 | 2 |
| 9 | 4 | 4 | 4 | 4 | 4 | 4 |
| 10 | 5 | 4 | 5 | 4 | 5 | 4 |
| 11 | 6 | 4 | 6 | 4 | 6 | 4 |
| 12 | 3 | 8 | 3 | 8 | 5 | 6 |

**Table 2.** Approximate solutions of $T_n^{(k)}(x) = 0$, $k = 3$

| degree $n$ | $x$ |
|---|---|
| 2 | 1.3750 |
| 3 | 0.38343, 2.3666 |
| 4 | −0.34576, 1.3816, 3.0891 |
| 5 | −0.84845, 0.38511, 2.3849, 3.5784 |
| 6 | −0.88180, −0.81408, 1.3832 |
| 7 | 0.38363, 2.3836 |

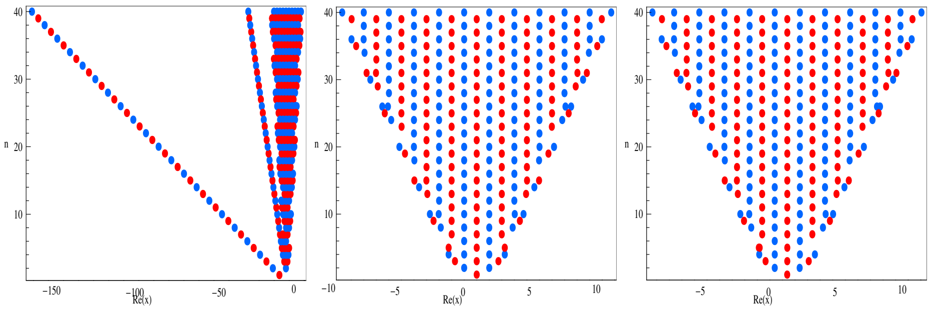

Figure 3: Real zeros of $T_n^{(k)}(x)$ for $1 \leq n \leq 40$

**Table 3.** Approximate solutions of $T_n^{(k)}(x) = 0$, $k = -3$

| degree $n$ | $x$ |
|---|---|
| 2 | $-6.5000$ |
| 3 | $-10.849$, $-2.1507$ |
| 4 | $-15.112$, $-3.2544$, $-1.1336$ |
| 5 | $-19.350$, $-4.1433$, $-2.0488$, $-0.45781$ |
| 6 | $-23.578$, $-4.8783$, $-2.9605$, $-1.0492$, $-0.034053$ |
| 7 | $-27.801$, $-5.5144$, $-3.7113$, $-2.0584$ |

**Conjecture 4.2.** Prove that $T_n^{(k)}(x) = 0$ has $n-1$ distinct solutions.

Since $n-1$ is the degree of the polynomial $T_n^{(k)}(x)$, the number of real zeros $R_{T_n^{(k)}(x)}$ lying on the real plane $Im(x) = 0$ is then $R_{T_n^{(k)}(x)} = n - C_{T_n^{(k)}(x)}$, where $C_{T_n^{(k)}(x)}$ denotes complex zeros. See Table 1 for tabulated values of $R_{T_n^{(k)}(x)}$ and $C_{T_n^{(k)}(x)}$. The author has no doubt that investigations along these lines will lead to a new approach employing numerical method in the research field of the poly-tangent polynomials $T_n^{(k)}(x)$ which appear in mathematics and physics.

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