Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4495–4509

http://www.ripublication.com/gjpam.htm

Approximate Controllability for the Parabolic Equations of the Fourth Order

Rezzoug Imad

*Larbi Ben M'hidi University, Faculty of Science,
Laboratory of Dynamical Systems and Control,
P.O.Box 358, OEB, Algeria.*

Ayadi Abdelhamid

*Larbi Ben M'hidi University, Faculty of Science,
Laboratory of Dynamical Systems and Control,
P.O.Box 358, OEB, Algeria.*

Abstract

In this paper, we study an approximate controllability problem. This problem appears naturally of approximate sentinel "weakly sentinel". The main tool is a theorem of uniqueness of the solution of ill-posed Cauchy problem for the parabolic equations of the 4th order.

AMS subject classification: Primary 93B05; Secondary 93C20, 92D40.
Keywords: Controllability, Sentinels, Parabolic equation.

1. Setting the problem

1.1. Problem formulation

The notion of sentinel was introduced by J. L. Lions to study systems of incomplete data [22]. The notion permits to distinguish and to analyse two types of incomplete data: the so called pollution terms on which we look for informations, independently of the other type of incomplete data which is the missing terms, and that we do not want to identify.

Typically, the Lions' sentinel is a functional defined from an open set $\mathcal{O}$ on which we consider three functions: the "observation" y_{obs} corresponding to measurements, a given "mean" function h_0, and a control function u to be determined.

Let us remind that Lions' sentinel theory [22] relies on the following three features: the state equation y which is gouverned by a system of PDE, the observation system and some particular evaluation function: the sentinel itself.

For $n = \{2; 3\}$, let Ω be a bounded open subset of $\mathbb{R}^n$ with boundary $\partial\Omega = \Gamma$ of class $\mathcal{C}^2$, $T > 0$, and let $\mathcal{O}$ be an open non empty subset of Ω. Set $\mathcal{Q} = \Omega \times (0, T)$, $\Sigma = \Gamma \times (0, T)$, $\mathcal{U} = \mathcal{O} \times (0, T)$. If Γ_0 is a subset of the border Γ of Ω. We consider the parabolic equation:

$$\left\{\begin{array}{rclll} y' + \Delta^2 y + f(y) & = & 0 & \text{in} & \mathcal{Q} \\ y & = & \xi_0 + \lambda_0 \widehat{\xi_0} & \text{on} & \Sigma_0 = \Gamma_0 \times (0, T) \\ y & = & 0 & \text{on} & \Sigma \backslash \Sigma_0 \\ \dfrac{\partial y}{\partial \nu} & = & \xi_1 + \lambda_1 \widehat{\xi_1} & \text{on} & \Sigma_0 \\ \dfrac{\partial y}{\partial \nu} & = & 0 & \text{on} & \Sigma \backslash \Sigma_0 \\ y(0) & = & y_0 + \tau \widehat{y_0} & \text{on} & \Omega \end{array}\right. \tag{1.1}$$

Where $(.)'$ is the partial derivative with respect to time t.

Remark 1.1. The problem (1.1) admits a unique solution. For the sake of simplicity, we denote $y(x, t; \lambda, \tau) = y(\lambda, \tau)$; $\lambda = \{\lambda_0, \lambda_1\}$.

That supposes that the data ξ_0, ξ_1 are rather regular, and that the terms of pollution "that one wants to estimate" are rather regular. It will be always supposed that the solution y check at least $y \in L^2(\mathcal{Q})$.

Remark 1.2. One will always indicate by y_0 the solution $y(x, t; 0, 0)$; thus

$$\left\{\begin{array}{rclll} y_0' + \Delta^2 y_0 + f(y_0) & = & 0 & \text{in} & \mathcal{Q} \\ y_0 & = & \xi_0 & \text{on} & \Sigma_0 = \Gamma_0 \times (0, T) \\ y_0 & = & 0 & \text{on} & \Sigma \backslash \Sigma_0 \\ \dfrac{\partial y_0}{\partial \nu} & = & \xi_1 & \text{on} & \Sigma_0 \\ \dfrac{\partial y_0}{\partial \nu} & = & 0 & \text{on} & \Sigma \backslash \Sigma_0 \\ y_0(0) & = & y_0 & \text{on} & \Omega \end{array}\right. \tag{1.2}$$

The problem considered here consists in trying to estimate $\lambda_0 \widehat{\xi_0}$ and $\lambda_1 \widehat{\xi_1}$ starting from observations, distributed or borders, without seeking to estimate the tereme lack $\tau \widehat{y_0}$.

One starts with a distributed observation, therefore a distributed sentinel

1.2. The "Sentinels method"

Proposition 1.3. (definition, existence and uniqueness of the sentinel) Let $h_0 \in L^2(\mathcal{U})$ be a given function, such that:

$$h_0 \geq 0, \int_0^T \int_{\mathcal{O}} h_0 dx dt = 1$$

For any control function $u \in L^2(\mathcal{U})$, set

$$\mathcal{S}(\lambda, \tau) = \int_{Q} (h_0 + u)\, \chi_{\mathcal{O}}\, y(x, t; \lambda, \tau)\, dxdt \tag{1.3}$$

The role of the function u appears in the following definition. We shall say that $\mathcal{S}$ defines a weakly sentinel (for the system (1.1), and definition of h_0) if there exists u such that the functional $\mathcal{S}$ satisfies the following conditions: for all $\epsilon > 0$ there exists $u \in L^2(\mathcal{U})$ such as

$$u \in L^2(\mathcal{U}), \text{ of minimal norm} \tag{1.4}$$

$$\left| \frac{\partial}{\partial \tau} \mathcal{S}(0, 0) \right| \leq \epsilon \tag{1.5}$$

Then $\mathcal{S}(\lambda, \tau)$ defined by (1.3, 1.4, 1.5) exists and is unique (that means the existence and uniqueness of the function u).

Remark 1.4. The function $u = -h_0$ give place to (1.5) so that the problem (1.4, 1.5) admits a single solution, which is defined by h_0.

The problem is thus:

(1) to calculate this solution;

(2) to see whether the corresponding sentinel justifies its name, i.e. gives information on pollution $\lambda_0 \widehat{\xi_0}$ and $\lambda_1 \widehat{\xi_1}$.

1.2.1 Adjoint state

The adjoint state is introduced q by

$$\begin{cases} -q' + \Delta^2 q + f'(y_0)\, q = (h_0 + u)\, \chi_{\mathcal{O}} & \text{in } Q \\ q = 0 & \text{on } \Sigma \\ \dfrac{\partial q}{\partial \nu} = 0 & \text{on } \Sigma \\ q(T) = 0 & \end{cases} \tag{1.6}$$

Where $(.)'$ is the partial derivative with respect to time t, $h_0, u \in L^2(\mathcal{U})$ and $\chi_{\mathcal{O}}$ denotes the characteristic function of $\mathcal{O}$.

Remark 1.5. System (1.6) is a backward parabolic problem. It appears under this form in J.L. Lions sentinels theory as the associated adjoint state.

It is checked at once that

$$\frac{\partial}{\partial \tau} \mathcal{S}(0, 0) = (q(0), \widehat{y_0}) \tag{1.7}$$

So that (1.5) is equivalent to

$$\|q(x, 0)\|_{L^2(\Omega)} \leq \epsilon \tag{1.8}$$

There is thus business with a problem of the type "approximate controllability with zero" (with, an operator of the 4th order in x).

1.2.2 The main result

The main result is the following

Lemma 1.6. Let $v \in L^2(\mathcal{U})$. Then there is no $\rho \in L^2(\mathcal{Q})$, $\rho \neq 0$ such that ρ satisfies

$$\left\{ \begin{array}{rcll} \rho' + \Delta^2 \rho + f'(y_0)\,\rho & = & 0 & \text{in} \quad \mathcal{Q} \\ \rho & = & 0 & \text{on} \quad \Sigma \\ \dfrac{\partial \rho}{\partial \nu} & = & 0 & \text{on} \quad \Sigma \\ \rho \chi_{\mathcal{O}} & = & v & \end{array} \right. \tag{1.9}$$

Proof. If the problem (1.9) admits a solution, then it is given by

$$\rho(x,t) = \sum_{j=1}^{\infty} \alpha_j(t)\, u_j(x) \tag{1.10}$$

Where u_j are eigenfunctions of

$$\left\{ \begin{array}{rcll} -\Delta u & = & \lambda u & \text{in } \Omega, \\ u & = & 0 & \text{on } \Gamma. \end{array} \right. \tag{1.11}$$

Differentiate the solution (1.11) once with respect to t and twice with respect to x and substitute these derivatives into the first equation of (1.9). We then obtain

$$\sum_{j=1}^{\infty} \left(\alpha_j'(t) + \lambda_j \alpha_j(t) \right) u_j(x) = 0 \tag{1.12}$$

Thus,

$$\alpha_j'(t) + \lambda_j \alpha_j(t) = 0 \tag{1.13}$$

Because (u_j) form an orthonormal base of $L^2(\mathcal{Q})$. Furthermore, the function ρ satisfies the boundary conditions if and only if

$$\sum_{j=1}^{\infty} \alpha_j(t)\, u_j(x) = v \chi_{\mathcal{O}} \tag{1.14}$$

As $v \chi_{\mathcal{O}} \in L^2(\mathcal{Q})$ then

$$v \chi_{\mathcal{O}} = \sum_{j=1}^{\infty} \langle v \chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})}\, u_j(x) \tag{1.15}$$

Consequently

$$\alpha_j(t) = \langle v \chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} \tag{1.16}$$

Finally, we have

$$\left\{ \begin{array}{rcl} \alpha'_j(t) + \lambda_j \alpha_j(t) & = & 0 \\ \alpha_j(t) & = & \langle v\chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} \end{array} \right. \text{ in } (0,T), \tag{1.17}$$

Then the solution of the first order linear is given by

$$\alpha_j(t) = \langle v\chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} e^{\lambda_j t} \tag{1.18}$$

Consequently, if the problem (1.9) admits a solution, it is necessarily in the form:

$$\rho(x,t) = \sum_{j=1}^{\infty} \langle v\chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} e^{\lambda_j t} u_j(x) \tag{1.19}$$

We prove now that $\rho \notin L^2(\mathcal{Q})$. Indeed,

$$\begin{aligned} \int_0^T |\alpha_j(t)|^2 dt &= \left| \langle v\chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} \right|^2 \int_0^T e^{2\lambda_j t} dt \\ &= \left| \langle v\chi_{\mathcal{O}}, u_j \rangle_{L^2(\mathcal{Q})} \right|^2 \left[\frac{-1}{2\lambda_j} + \frac{1}{2\lambda_j} e^{2\lambda_j T} \right] \end{aligned} \tag{1.20}$$

But, λ_j is the eigenvalue of problem (1.11), then $\lambda_j \underset{j \longmapsto \infty}{\longrightarrow} \infty$. Consequently,

$$\int_0^T |\alpha_j(t)|^2 dt \underset{\longmapsto \infty}{\longrightarrow} \infty \tag{1.21}$$

Which means that the series whose general term $\alpha_j(t)$ is not normally convergent. So, problem (1.9) admits no solution. ■

Theorem 1.7. For $\epsilon > 0$, $h_0 \in L^2(\mathcal{U})$, there existe some control u and some state q such that (1.6) and (1.8) hold. Moreover, there exists a unique pair $(\widehat{u}\chi_{\mathcal{O}}, \widehat{q})$ with $\widehat{u}$ of minimal norm in $L^2(\mathcal{U})$, i.e. such that (1.6, 1.8) and (1.4) hold.

Proof. Let q be a solution of the system (1.6) and q_0 a solution of the following system:

$$\left\{ \begin{array}{rcll} -q'_0 + \Delta^2 q_0 + f'(y_0) q_0 & = & h_0 \chi_{\mathcal{O}} & \text{in } \mathcal{Q} \\ q_0 & = & 0 & \text{on } \Sigma \\ \dfrac{\partial q_0}{\partial \nu} & = & 0 & \text{on } \Sigma \\ q_0(T) & = & 0 & \end{array} \right. \tag{1.22}$$

We put

$$q = q_0 + z \tag{1.23}$$

Then, z is the solution of the following problem:

$$\begin{cases} -z' + \Delta^2 z + f'(y_0) z = u\chi_{\mathcal{O}} & \text{in } \mathcal{Q} \\ z = 0 & \text{on } \Sigma \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \Sigma \\ z(T) = 0 \end{cases} \tag{1.24}$$

We now introduce the set of states reachable at time 0 defined by

$$\mathcal{F}(0) = \left\{ z(u, 0) \text{ such as } u \in L^2(\mathcal{U}) \right\}. \tag{1.25}$$

It is clear that $\mathcal{F}(0)$ is a vector subspace of $L^2(\Omega)$. According to the HAHN-BANACH theorem, it will be dense in $L^2(\Omega)$ if and only if its orthogonal in $L^2(\Omega)$ is reduced to zero. As $\{0\} \subset \mathcal{F}^{\perp}(0)$, it remains to show that $\mathcal{F}^{\perp}(0) \subset \{0\}$. Let $\rho^0 \in \mathcal{F}^{\perp}(0)$, then

$$\left\langle \rho^0, z(0) \right\rangle_{L^2(\Omega)} = \int_{\Omega} \rho^0 z(0)\, dx = 0 \tag{1.26}$$

Where z is solution of (1.24). It is therefore natural to define the adjoint ρ of z, this is the solution of the following problem:

$$\begin{cases} \rho' + \Delta^2 \rho + f'(y_0) \rho = 0 & \text{in } \mathcal{Q} \\ \rho = 0 & \text{on } \Sigma \\ \dfrac{\partial \rho}{\partial \nu} = 0 & \text{on } \Sigma \\ \rho(0) = \rho^0 \end{cases} \tag{1.27}$$

Where ρ is solution of (1.27).

Now multiply the first equation of system (1.27) by z. After integration by parts in $\mathcal{Q}$, it comes

$$\begin{aligned} 0 = & \int\int_{\Omega\times(0,T)} \rho\left(-z' + \Delta^2 z + f'(y_0) z\right) dxdt + \int_{\Omega} \rho(T)\, z(T)\, dx \\ & - \int_{\Omega} \rho(0)\, z(0)\, dx - \int\int_{\Gamma\times(0,T)} \frac{\partial \rho}{\partial \nu} z\, d\Gamma dt + \int\int_{\Gamma\times(0,T)} \rho \frac{\partial z}{\partial \nu} d\Gamma dt \end{aligned} \tag{1.28}$$

Since z and ρ are solutions of (1.24) and (1.27) respectively, (1.28) becomes

$$\int\int_{\Omega\times(0,T)} \rho u \chi_{\mathcal{O}} dxdt - \int_{\Omega} \rho^0 z(0)\, dx = 0 \tag{1.29}$$

This is equivalent to

$$\int\int_{\Omega\times(0,T)} \rho u \chi_{\mathcal{O}} dxdt = 0 \tag{1.30}$$

Because, $\rho^0 \in \mathcal{F}^{\perp}(0)$ et $z(0) \in \mathcal{F}(0)$. Finally, we have

$$\rho \chi_{\mathcal{O}} = 0 \tag{1.31}$$

Therefore, ρ satisfies (1.27) and (1.31) and by applying **Lemma 6**, we deduce that

$$\rho = 0 \quad \text{in} \quad \Omega \times (0, T)$$

As a consequence, $\rho^0 = 0$ which shows that $\mathcal{F}^{\perp}(0) = \{0\}$. ∎

2. Characterization of optimal control

In this section, we will characterize the optimal control using a result of **Fenchel-Rockafellar** duality.

The optimality system satisfied by $(\widehat{u}, \widehat{q})$ is established. Let $\rho^0 \in L^2(\Omega)$ and ρ the associated solution of

$$\begin{cases} \rho' + \Delta^2 \rho + f'(y_0)\rho & = & 0 & \text{in} & Q \\ \rho & = & 0 & \text{on} & \Sigma \\ \dfrac{\partial \rho}{\partial \nu} & = & 0 & \text{on} & \Sigma \\ \rho(0) & = & \rho^0 & \text{in} & \Omega \end{cases} \tag{2.32}$$

We now introduce the functional J_ϵ defined by

$$J_\epsilon\left(\rho^0\right) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\rho|^2 \, dxdt + \epsilon \left\|\rho^0\right\|_{L^2(\Omega)} + \int_0^T \int_{\mathcal{O}} h_0 \rho dxdt \tag{2.33}$$

Consider the following unconstrained problem:

$$(P_\epsilon) : \begin{cases} \min J_\epsilon\left(\rho^0\right) \\ \rho^0 \in L^2(\Omega) \end{cases} \tag{2.34}$$

Then, we have

Proposition 2.1. The functional J_ϵ defined in (2.2) is coercive.

Proof. To prove that J_ϵ is coercive, it suffices to show the following relation:

$$\lim_{\|\rho^0\|_{L^2(\Omega)} \to \infty} \frac{J_\epsilon\left(\rho^0\right)}{\left\|\rho^0\right\|_{L^2(\Omega)}} \geq \epsilon \tag{2.35}$$

Let $\left(\rho_j^0\right) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system (2.1) with $\left\|\rho_j^0\right\|_{L^2(\Omega)} \longrightarrow \infty$. We normalize them as follows

$$\widetilde{\rho}_j^0 = \frac{\rho_j^0}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \tag{2.36}$$

So $\left\|\widetilde{\rho}_j^0\right\|_{L^2(\Omega)} \leq 1$. On the other hand, let $\widetilde{\rho}_j$ be the solution of (2.1) with initial data $\widetilde{\rho}_j^0$. Then, we have

$$\begin{aligned}\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} &= \frac{1}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}\left[\frac{1}{2}\int_0^T\int_{\mathcal{O}}\left|\rho_j\right|^2 dxdt + \int_0^T\int_{\mathcal{O}} h_0\rho_j dxdt\right] + \epsilon \\ &= \frac{1}{2}\int_0^T\int_{\mathcal{O}} \rho_j.\widetilde{\rho}_j dxdt + \epsilon + \int_0^T\int_{\mathcal{O}} h_0\widetilde{\rho}_j dxdt \end{aligned} \tag{2.37}$$

We now show that the last integral in equation (2.6) is bounded. Indeed, we know that ρ_j is the solution of the problem

$$\left\{\begin{array}{rcll} \rho_j' + \Delta^2\rho_j + f'(y_0)\rho_j &=& 0 & \text{in} \quad \mathcal{Q} \\ \rho_j &=& 0 & \text{on} \quad \Sigma \\ \dfrac{\partial\rho_j}{\partial\nu} &=& 0 & \text{on} \quad \Sigma \\ \rho_j(0) &=& \rho_j^0 & \text{in} \quad \Omega \end{array}\right. \tag{2.38}$$

Multiplying the first equation of system (2.7) by ρ_j then **integrating by parts** on $\mathcal{Q}$, yields

$$\begin{aligned} 0 &= \int_0^T\int_\Omega\left(\rho_j' + \Delta^2\rho_j + f'(y_0)\rho_j\right)\rho_j dxdt = \frac{1}{2}\left\|\rho_j(T)\right\|_{L^2(\Omega)}^2 \\ &-\frac{1}{2}\left\|\rho_j^0\right\|_{L^2(\Omega)}^2 + \left\|\nabla\rho_j\right\|_{L^2(\mathcal{Q})}^2 \end{aligned} \tag{2.39}$$

By the Poincaré inequality, (2.8) becomes,

$$C_0\left\|\rho_j\right\|_{L^2(\mathcal{Q})}^2 \leq \left\|\nabla\rho_j\right\|_{L^2(\mathcal{Q})}^2 \leq \frac{1}{2}\left\|\rho_j^0\right\|_{L^2(\Omega)}^2 \tag{2.40}$$

Now, by Cauchy Schwartz inequality, one finds

$$\int_0^T\int_{\mathcal{O}}\frac{h_0\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}dxdt \leq C_1\frac{\left\|\rho_j\right\|_{L^2(\mathcal{Q})}}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \tag{2.41}$$

From (2.9), (2.10), we conclude that

$$\int_0^T\int_{\mathcal{O}}\frac{h_0\rho}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}dxdt \leq C \tag{2.42}$$

Returning to relation (2.6), two cases can occur:

1. $\int_0^T \int_{\mathcal{O}} |\widetilde{\rho}_j|^2 \, dxdt > 0$. In this case, we immediately obtain

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \underset{\left\|\rho_j^0\right\|_{L^2(\Omega)} \longmapsto +\infty}{\longrightarrow} +\infty. \tag{2.43}$$

2. $\int_0^T \int_{\mathcal{O}} |\widetilde{\rho}_j|^2 \, dxdt = 0$.In this case, since $\left(\widetilde{\rho}_j^0\right)_j$ is bounded in $L^2(\Omega)$, we can extract a subsequence $\left(\widetilde{\rho}_j^0\right)_j$ such that:

$$\begin{cases} \widetilde{\rho}_j^0 \rightharpoonup \psi^0 \text{ weakly in } L^2(\Omega), \\ \widetilde{\rho}_j \rightharpoonup \psi \text{ weakly in } L^2\left(0, T; H_0^1(\Omega)\right). \end{cases} \tag{2.44}$$

Where ψ is solution of system (2.1) with initial data ψ^0. Moreover, by lower semi continuity of the norm, it comes

$$\int_0^T \int_{\mathcal{O}} |\psi|^2 \, dxdt \leq \liminf \int_0^T \int_{\mathcal{O}} |\widetilde{\rho}_j|^2 \, dxdt = 0 \tag{2.45}$$

Therefore,

$$\psi = 0 \quad \text{in} \quad \mathcal{O} \times (0, T) \tag{2.46}$$

And as ψ is solution of (2.1), and in view of (2.15), we have

$$\psi = 0 \quad \text{in} \quad \Omega \times (0, T) \tag{2.47}$$

Thus,

$$\widetilde{\rho}_j \rightharpoonup 0 \text{ weakly in } L^2\left(0, T; H_0^1(\Omega)\right). \tag{2.48}$$

Moreover, from inequality (2.9), we deduce that $\left(\dfrac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}\right)_j$ is bounded in $L^2\left(0, T; H_0^1(\Omega)\right)$. Hence

$$\frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \rightharpoonup \xi \text{ in } L^2\left(0, T; H_0^1(\Omega)\right) \tag{2.49}$$

But,

$$\widetilde{\rho}_j = \frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \rightharpoonup 0 \tag{2.50}$$

From (2.18) and (2.19), we conclude that

$$\xi' + \Delta^2\xi + f'(y_0)\,\xi = 0 \quad \text{in} \quad L^2(\mathcal{Q}) \tag{2.51}$$

So by **Lemma 6**, it comes

$$\xi = 0 \quad \text{in} \quad \mathcal{Q} \tag{2.52}$$

As a consequence,

$$\widetilde{\rho}_j = \frac{\rho_j}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \longmapsto 0 \tag{2.53}$$

But,

$$\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} = \frac{1}{\left\|\rho_j^0\right\|_{L^2(\Omega)}}\left[\frac{1}{2}\int_0^T\int_{\mathcal{O}}\left|\rho_j\right|^2 dxdt + \int_0^T\int_{\mathcal{O}} h_0\rho_j dxdt\right] + \epsilon \tag{2.54}$$

Thus,

$$\liminf_{j\longmapsto+\infty}\frac{J_\epsilon\left(\rho_j^0\right)}{\left\|\rho_j^0\right\|_{L^2(\Omega)}} \geq \epsilon \tag{2.55}$$

Hence relation (2.4) is satisfied. ∎

Theorem 2.2. Problem (2.3) has a unique solution $\widehat{\rho}^0 \in L^2(\Omega)$. Furthermore, if $\widehat{\rho}$ is the solution of (2.1) associated to $\widehat{\rho}^0$, then $(\widehat{u} = \widehat{\rho}, q)$ is solution such that (1.6), (1.8) and (1.4) hold.

Proof. As J_ϵ attains its minimum value at $\widehat{\rho}^0 \in L^2(\Omega)$, then, for any $\psi^0 \in L^2(\Omega)$ and any $r \in \mathbb{R}$ we have

$$J_\epsilon\left(\widehat{\rho}^0\right) \leq J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) \Longrightarrow J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) - J_\epsilon\left(\widehat{\rho}^0\right) \geq 0 \tag{2.56}$$

On the other hand,

$$J_\epsilon\left(\widehat{\rho}^0\right) = \frac{1}{2}\int_0^T\int_{\mathcal{O}}|\widehat{\rho}|^2 dxdt + \epsilon\left\|\widehat{\rho}^0\right\|_{L^2(\Omega)} + \int_0^T\int_{\mathcal{O}} h_0\widehat{\rho} dxdt$$

$$\begin{aligned} J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) &= \frac{1}{2}\int_0^T\int_{\mathcal{O}}|\widehat{\rho}|^2 dxdt + \frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2 dxdt \\ &\quad + r\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi dxdt + \sqrt{\varepsilon}\left\|\widehat{\rho}^0 + r\psi^0\right\|_{L^2(\Omega)} \\ &\quad + \int_0^T\int_{\mathcal{O}} h_0\left(\widehat{\rho} + r\psi\right) dxdt \end{aligned} \tag{2.57}$$

Substituting (2.26) in (2.25) and after simplifications, we find

$$\begin{aligned} 0 &\leq J_\epsilon\left(\widehat{\rho}^0 + r\psi^0\right) - J_\varepsilon\left(\widehat{\rho}^0\right) \qquad (2.58)\\ 0 &\leq \frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2\,dxdt + \epsilon\left[\left\|\widehat{\rho}^0 + r\psi^0\right\|_{L^2(\Omega)} - \left\|\widehat{\rho}^0\right\|_{L^2(\Omega)}\right]\\ &\quad + r\left[\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi\,dxdt + \int_0^T\int_{\mathcal{O}} h_0\psi\,dxdt\right] \end{aligned}$$

On the other hand,

$$\left\|\widehat{\rho}^0 + r\psi^0\right\|_{L^2(\Omega)} - \left\|\widehat{\rho}^0\right\|_{L^2(\Omega)} \leq |r|\,.\,\left\|\psi^0\right\|_{L^2(\Omega)} \qquad (2.59)$$

From (2.27) and (2.28), we obtain for any $\psi^0 \in L^2(\Omega)$ and $r \in \mathbb{R}$,

$$\begin{aligned} 0 &\leq \frac{r^2}{2}\int_0^T\int_{\mathcal{O}}|\psi|^2\,dxdt + \epsilon\,|r|\,.\,\left\|\psi^0\right\|_{L^2(\Omega)}\\ &\quad + r\left[\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi\,dxdt + \int_0^T\int_{\mathcal{O}} h_0\psi\,dxdt\right] \end{aligned}$$

Dividing by $r > 0$ and by passing to the limit $r \to 0$, we obtain

$$\epsilon.\left\|\psi^0\right\|_{L^2(\Omega)} + \int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi\,dxdt + \int_0^T\int_{\mathcal{O}} h_0\psi\,dxdt \geq 0$$

The same calculations with $r < 0$ give

$$\left|\int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi\,dxdt + \int_0^T\int_{\mathcal{O}} h_0\psi\,dxdt\right| \leq \epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\ \forall\psi^0 \in L^2(\Omega)\,.$$

Therefore,

$$\left|\int_0^T\int_{\mathcal{O}} h_0\psi\,dxdt + \int_0^T\int_{\mathcal{O}}\widehat{\rho}\psi\,dxdt\right| \leq \epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\ \forall\psi^0 \in L^2(\Omega)\,.$$

Therefore,

$$\left|\int_0^T\int_{\mathcal{O}}\left(h_0 + \widehat{\rho}\right)\psi\,dxdt\right| \leq \epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\ \forall\psi^0 \in L^2(\Omega)\,.$$

Alors if we take $\widehat{u} = \widehat{\rho}\chi_{\mathcal{O}}$ in (1.6) and we multiply the first equation of the system (1.6) by ψ solution of (2.1) and we get after integration by parts over $\mathcal{Q}$,

$$\int_\Omega q(0)\psi^0 dx = \int_0^T\int_{\mathcal{O}}\left(h_0 + \widehat{\rho}\right)\psi\,dxdt \qquad (2.60)$$

It comes from the last two relations:

$$\left|\int_\Omega q(0)\psi^0 dx\right| \leq \epsilon\left\|\psi^0\right\|_{L^2(\Omega)};\ \forall\psi^0 \in L^2(\Omega)\,.$$

Consequently,

$$\|q(x,0)\|_{L^2(\Omega)} \leq \epsilon. \qquad (2.61)$$

■

3. A use of the concept of sentinel: Detection of pollution and Furtivity

It is noted that

$$\mathcal{S}(\lambda, \tau) \simeq \mathcal{S}(0,0) + \lambda_0 \frac{\partial \mathcal{S}}{\partial \lambda_0}(0,0) + \lambda_1 \frac{\partial \mathcal{S}}{\partial \lambda_1}(0,0) \tag{3.62}$$

And

$$\text{observation of } y = y\chi_{\mathcal{O}} = \text{function } m_0(x,t) \text{ of } L^2(\mathcal{O} \times (0,T)) \tag{3.63}$$

With the notation (3.2) for the observation of y, and while using (1.3), one thus has

$$\begin{aligned} \mathcal{S}(\lambda, \tau) &= \int\int_{\mathcal{O}\times(0,T)} (h_0 + u)(m_0 - y_0)\, dxdt \simeq \lambda_0 \frac{\partial \mathcal{S}}{\partial \lambda_0}(0,0) + \lambda_1 \frac{\partial \mathcal{S}}{\partial \lambda_1}(0,0) \\ &= \int\int_{\mathcal{O}\times(0,T)} (h_0 + u)\left(\lambda_0 y_{\lambda_0} + \lambda_1 y_{\lambda_1}\right) dxdt \end{aligned} \tag{3.64}$$

In (3.3), y_{λ_0} and y_{λ_1} are defined by

$$\left\{ \begin{array}{rcll} y'_{\lambda_0} + \Delta^2 y_{\lambda_0} + f'(y_0)\, y_{\lambda_0} & = & 0 & \text{in } \mathcal{Q} \\ y_{\lambda_0} & = & \widehat{\xi_0} & \text{on } \Sigma_0 \\ y_{\lambda_0} & = & 0 & \text{on } \Sigma\backslash\Sigma_0 \\ \dfrac{\partial y_{\lambda_0}}{\partial \nu} & = & 0 & \text{on } \Sigma \\ y_{\lambda_0} & = & 0 & \end{array} \right. \tag{3.65}$$

And

$$\left\{ \begin{array}{rcll} y'_{\lambda_1} + \Delta^2 y_{\lambda_1} + f'(y_0)\, y_{\lambda_1} & = & 0 & \text{in } \mathcal{Q} \\ \dfrac{\partial y_{\lambda_1}}{\partial \nu} & = & \widehat{\xi_1} & \text{on } \Sigma_0 \\ \dfrac{\partial y_{\lambda_1}}{\partial \nu} & = & 0 & \text{on } \Sigma\backslash\Sigma_0 \\ y_{\lambda_1} & = & 0 & \end{array} \right. \tag{3.66}$$

That is to say $q(h_0)$ the state adjoint correspondent with $u = \rho\chi_{\mathcal{O}}$.

By multiplying the corresponding equation (1.6) by y_{λ_0} then by y_{λ_1}, one finds, after integrations by parts, that

$$\int\int_{\mathcal{O}\times(0,T)} (h_0 + \rho)\, y_{\lambda_0} dxdt = \int_{\Sigma_0} \frac{\partial}{\partial \nu} \Delta q(h_0)\, \widehat{\xi_0} d\Sigma, \tag{3.67}$$

And

$$\int\int_{\mathcal{O}\times(0,T)} (h_0 + \rho)\, y_{\lambda_1} dxdt = -\int_{\Sigma_0} \Delta q(h_0)\, \widehat{\xi_1} d\Sigma. \tag{3.68}$$

Consequently

$$\int\int_{\mathcal{O}\times(0,T)} (h_0 + u)\left(\lambda_0 y_{\lambda_0} + \lambda_1 y_{\lambda_1}\right) dxdt$$
$$= \int_{\Sigma_0} \left[\lambda_0 \frac{\partial}{\partial \nu} \Delta q\,(h_0)\,\widehat{\xi_0} - \lambda_1 \Delta q\,(h_0)\,\widehat{\xi_1}\right] d\Sigma \qquad (3.69)$$

It is the quantity (3.8) which is estimated by the 1st member of (3.3).

Pollution $\left\{\lambda_0\widehat{\xi_0}, \lambda_1\widehat{\xi_1}\right\}$ is furtive for the sentinel defined by h_0 if

$$\int_{\Sigma_0} \left[\frac{\partial}{\partial \nu} \Delta q\,(h_0)\,\lambda_0\widehat{\xi_0} - \Delta q\,(h_0)\,\lambda_1\widehat{\xi_1}\right] d\Sigma = 0 \qquad (3.70)$$

There are thus always furtive pollution for a sentinel.

4. Concluding remarks

In this paper we have presented an efficient method to estimate the pollution terms in the parabolic equations of the 4th order with missing initial data condition and perturbuted term or pollution term. The theory used for the identification needs the sentinels method by Lions [22]. And finally, we give the characterization of the weakly sentinel, which permits to identify the pollution parameters.

Acknowledgements

The authors thank the referees for their careful reading and their precious comments. Their help is much appreciated.

References

[1] I. Rezzoug, *Étude théorique et numérique des problèmes d'identification des systèmes gouvernés par des équations aux dérivées partielles*, Thèse de doctorat, Université de Oum El Bouaghi, Algérie, (2014).

[2] I. Rezzoug, A. Ayadi, *Sentinels for the identification of pollution in domains with missing data*, ADSA. ISSN 0973-5321, Volume 7, Number 2, pp. 439–449, (2013).

[3] Berhail Amel, *Étude des systèmes hyperboliques à données manquantes*, Thèse de doctorat, Université de mentouri, Constantine, (2013).

[4] T. Chahnaz Zakia, *Détection de la pollution et identification des défauts en élasticité linéaire*, Thèse de doctorat, Université d'Oran, (2013).

[5] I. Rezzoug, A. Ayadi, *Weakly sentinels for the distributed system with pollution terms in the boundary*. FEJ of AM. V. 63, N1, P 25–37, (2012).

[6] I. Rezzoug, A. Ayadi, *Weakly sentinels for the distributed system with missing terms and with pollution in the boundary conditions*. IJ of MA. V. 6, N1, no 45, P 2245–2256, (2012).

[7] I. Rezzoug, *Identification d'une partie de la frontière inconnue d'une membrane, Thèse de magister*, Université de Oum El Bouaghi, Algérie, (2009).

[8] G. Mophou., J. Velin, *A null controllability problem with constraint on the control deriving from boundary discriminating sentinels*, J. Nolinear Analysis, No 71, 910–924, (2009).

[9] G. Massengo., J. P. Puel, *Boundary sentinels with given sensitivity*. Rev. Mat. Complut. Vol 22, N 1, 165–185, (2009).

[10] M. Dalah, *Étude des problèmes paraboliques à données manquantes, Thèse de doctorat*, Université de mentouri de Constantine, Algérie, (2008).

[11] O. Nakoulima, *A revision of J.L.Lions notion of sentinels*, Portugal. Math. (N.S). Vol. 65, Fasc. 1, 1–22, (2008).

[12] G. Massengo., O. Nakoulima, *Sentinels with given sensitivity*. Euro. Jnl of Appl. Math, Vol. 19, 21–40, (2008).

[13] E.H. Zerrik., A. Afifi., A. El Jai, *Systèmes dynamiques, Analyse régionale des systèmes linéaires distribués*. Tome 2, Presses Universitaires de Perpignan, (2008).

[14] A. Traor, B. Mampassi, B. Saley, *A numerical approach of the sentinel method for distributed parameter systems*, C. Euro. J. Mathe, Vol 5, N 4, 751–763, (2007).

[15] Y. Miloudi, O. Nakoulima, A. Omrane, *a method for detecting pollution in dissipative systems with incomplete data*. ESAIM: PROCEEDINGS, April, 67–79, (2007).

[16] A. Ayadi, M. Djebarni, T. Laib T., *Sentinelles faibles, Sci: Tech. A-N*°24, univ Mentouri Constantine, 07–10, (2006).

[17] O. Nakoulima, *Contrôlabilité à zéro avec contraintes sur le contrôle*. Université Antilles-Guyane, DMI, Campus de Fouillole, 97159 Pointe à Pitre cedex, Guadeloupe, 405–410, (2004).

[18] J. H. Ortega and E. Zuazua, *On a constrained approximate controllability problem for the heat equation*, Journal of optimization theory and applications 108 no. 1, pp. 29–63, (2001).

[19] O. Bodart, P. Demeestère, *Sentinels for the identification of an unknown boundary*. Université de Technologie de Compiègne (France). M3AS, 7(6), 871–885, (1997).

[20] J. S. Paulin, M. Vanninathan, *Sentinelles et pollutions frontières dans des domaines minces*, C. R. Acad. Sci. Paris, t. 325, Série I, 1299-1304, (1997).

[21] E.H. Zerrik, *Analyse régionale des systèmes distribués. Thèse*. Univ. Mohammed $\vee$. Maroc, (1993).

[22] J.L. Lions, *Sentinelles pour les systèmes distribués à données incomplètes*. Masson, Paris, (1992).

[23] J. S. Saut, B. Scheurer, *Unique continuation for some evolution equations*, J. Deff. Equa., Vol 66, pp. 118–139, (1987).

[24] A. EL Jai., A.J. Pritchard, *Capteurs et actionneurs dans l'analyse des systèmes distribués*. Masson. RMA 3. Paris, (1986).

[25] J. C. Saut ET B. Scheurer, *Remarques sur un théorème de prolongement unique de MIZOHATA*. C.R.A.S. Paris, p. 307–310, (1983).

[26] J. L. Lions, *contrôle optimal des systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Gauthier-Villars, Paris, (1968).

[27] Z. Mizohata, *Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques*, Mem. Coll. Sc. Univ. Kyoto, Série A, Vol 31, 219–239, (1958).