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On the Convergence of Generalized Overlapping Domain Decomposition Methods in the Continuous and Discrete cases

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Abstract

With the development of parallel computers, domain decomposition methods (DDM) have been increasingly used as important tools for solving boundary value problems. There exists, in practice, two ideas of decomposition of the domain: with and without overlapping of subdomains. This work is concerned with the analysis of the generalized overlapping (Schwarz) DDM, by using Robin boundary conditions on the interfaces. The nonoverlapping case was studied in [2,3,7,10,11]. We are interested in the study of the convergence of the iteratif process in the continuous and discrete cases. We use an energy method of Lions [7] to prove the convergence of the iteratif process and a generalization of a relaxation procedure first used by Deng [2], to avoid the computation of normal derivatives and to facilitate the application of this method to discrete problems and to get an optimal convergence rate.

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1. Introduction

Domain decomposition methods (DDM) have known since their introduction by Schwarz [13] in 1870 an intense and accelerated development, see the monographs of Smith et al. [14], Quarteroni and Valli [12] and Toselli and Widlund [15]. This is due mainly to the considerable development that has experienced the world of computers. In particular, the

appearance of vector or parallel computing machines (multiprocessors). They become a very active area of research in the domain of numerical analysis of partial differential equations. Up to this moment twenty three conferences have been dedicated to this subject (see the website: www.DDM.org for the conference proceedings). These methods allow primarily to reduce large algebraic systems to systems of small size easy and less expansive from computational point of view. They can also transform boundary value problems laid on irregular geometries, like for example L, T or C shapes, to a set of problems posed on regular and simple subdomains. The order of convergence of the approximate solution using uniform finite elements decreases in the neighborhood of a singularity $\left(\text{angle strictly greater than } \frac{\pi}{2}\right)$. To remedy this situation one can refine the mesh of the triangulation near this singularity. But this leads to a significant increase in the size of the algebraic system obtained by approximation of the problem. Hence the need to use a domain decomposition method to get rid of this singularity. There are two ideas of decomposition of domains, one with and the other without overlapping of subdomains. The first idea called Schwarz method, introduced by Schwarz [13] in 1870 and analyzed in variational form by Lions [6] in 1988, the other called with interfaces, introduced and analyzed by Lions [7] in the continuous case, and by Deng [2,3], Guo and Hou [5], Qin et al. [10,11] in the continuous and discrete cases. The optimal convergence rate achieved is $1 - O(h^{\frac{1}{2}})$, where h is the discretization parameter. See also the works of Lui [8], Chen et al. [1] and the references their in.
In various situations it is better to use overlapping decompositions for faster convergence rate, for example, the case of decomposition into simple subdomains where uniform discretizations are possible. In this paper, we consider the generalized overlapping (Schwarz) DDM, that is with Robin transmission conditions on the interfaces, for a second order boundary value problems on a bounded domain Ω with Dirichlet boundary conditions. To improve the convergence rate of the discrete version of this method, one have to sort the problem of avoiding the computation of normal derivatives in every iteration, this problem is overcomed by a relaxation procedure which was first used by Deng [2], although the first relaxation procedure was introduced by Marini and Quarteroni [9] for simple nonoverlapping DDM.

The outline of the paper is as follows. In section 2 we introduce some necessary notations. We then introduce a model problem and the overlapping generalized DDM and their variational formulations. Section 3 is devoted to proving the convergence in the H^1-norm of the DDM iteratif prossess in the continuous case. In section 4 we get a convergence rate for this DDM in the discrete case depending on h and H (the diameter of subdomain) and the parameter λ of the Robin interface condition. The best choice of the parameter λ in the Robin conditions can lead us to an optimal rate of convergence. An illustratif example is given at the end of this section to support the theory.

2. Notations and Preliminaries

We are interested in the study of the convergence of the generalized overlapping DDM. We give some necessary notations and definitions necessary for the rest of this paper, then we give the variational formulation of a modal problem. Let Ω be an open domain in $\mathbb{R}^N$ with piecewise continuous boundary $\partial\Omega = \Gamma$, on which we define the following functional spaces: The Hilbert space $L^2(\Omega)$ of measurable square integrable functions in Ω, equipped with the inner product:

$$(u, v)_\Omega = \int_\Omega uv dx. \tag{1}$$

The Sobolev space $H^m(\Omega)$ defined by:

$$H^m(\Omega) = \{v \in L^2(\Omega), D^\alpha v \in L^2(\Omega)\ \forall\alpha \text{ such that } |\alpha| \leq m\}$$

where D^α is the weak or distributional derivative. $H^m(\Omega)$, is equipped with the norm:

$$\|v\|_{m,\Omega} = \left(\sum_{|\alpha|\leq m} \int_\Omega \left|D^\alpha v\right|^2 dx\right)^{\frac{1}{2}},$$

is a Hilbert space for the inner product

$$(u, v)_{m,\Omega} = \sum_{|\alpha|\leq m} \int_\Omega D^\alpha u D^\alpha v dx.$$

$H_0^m(\Omega)$ is a subspace of $H^m(\Omega)$ defined by:

$$H_0^m(\Omega) = \{v \in H^m(\Omega) : \Lambda\left(\frac{\partial^i v}{\partial n_i}\right) = 0\ \forall i = 0, 1, \ldots, m-1\},$$

where Λ is the trace operator and $\dfrac{\partial^i}{\partial n_i}$ is the i normal derivative to Ω.

2.1. Problem and DDM Formulations

We consider the following model problem

$$\begin{cases} Lu = f \in L^2(\Omega), \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

where L is a second order elliptic operator, Ω is a bounded domain in $\mathbb{R}^N$ with a sufficiently smooth boundary $\partial\Omega$. The variational formulation of this problem is:

$$\begin{cases} \text{Find } u \in H_0^1(\Omega) \text{ solution of} \\ a_\Omega(u, v) = (f, v)_\Omega \quad \forall v \in H_0^1(\Omega), \end{cases} \tag{3}$$

where

$$\begin{aligned} a_\Omega(u,v) &= \int_\Omega \left(\sum_{k,l} \alpha_{kl}(x) \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_l} + \alpha_0(x) uv \right) dx, \\ (f,v)_\Omega &= \int_\Omega f v dx. \end{aligned}$$

$(\alpha_{kl}, \alpha_0) \in L^\infty(\Omega)$, α_0 is non negative and α_{kl} are symmetric uniformly positive definite.

Engquist and Zhao [4] looked at the convergence of the generalized Schwarz alternating method (GSAM) on two subdomains using Fourier analysis (the domain Ω has to be simple: rectangle or annulus). Here we split Ω in a finite number of subdomains $m \geq 2$: $\Omega_1, \ldots, \Omega_m$ such that

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_m$$

where Ω_i, $1 \leq i \leq m$ are bounded open subdomains in $\mathbb{R}^N$, $\gamma_{ij}(\gamma_{ji})$ are the interfaces between Ω_i and Ω_j (Ω_j and Ω_i) respectively i.e. ($\gamma_{ij} = \partial\Omega_i \cap \Omega_j$ and $\gamma_{ji} = \partial\Omega_j \cap \Omega_i$) and $\Omega_i \cap \Omega_j \neq \varnothing$. See as an example figure 1. The generalized overlapping DDM is written as follows: Let (g_{ij}^0) be starting values in $L^2\left(\gamma_{ij}\right)$, $1 \leq i \neq j \leq m$

$$\begin{cases} Lu_i^n = f \text{ in } \Omega_i, 1 \leq i \leq m \\ \dfrac{\partial u_i^n}{\partial n_{ij}} + \lambda_{ij} u_i^n = g_{ij}^n \text{ on } \gamma_{ij}, \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, j \neq i, \\ u_i^n = 0 \text{ on } \partial\Omega_i \cap \partial\Omega = \Gamma_i, \end{cases} \tag{4}$$

where

$$\frac{\partial u}{\partial n_{ij}} = \sum_{k,l} \left(\alpha_{kl}(x) \frac{\partial u}{\partial x_l} \right) n_{ij}^{(k)}, 1 \leq i \leq m. \tag{5}$$

n_{ij} is the unit outward normal vector to $\partial\Omega_i$ on γ_{ij}. We need to avoid the computation of normal derivatives in each iteration. A relaxation procedure is used between Robin values on the interfaces, when $n = 0$, as follows, (see Deng [2]):

$$g_{ij}^{n+1} = 2\lambda_{ij} u_j^n - g_{ji}^n \text{ on } \gamma_{ij}\ ,\ \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, j \neq i \tag{6}$$

$g_{ij}^0 \in L^2\left(\gamma_{ij}\right)$ $(1 \leq i \neq j \leq m)$ is a starting value, $\lambda_{ij} = \lambda_{ji} > 0$ and the measure of γ_{ij} is strictly positive.

Whereas for $n \geq 1$, we use the update:

$$g_{ij}^{n+1} = 2\lambda_{ij} \left(u_j^n - u_i^{n-1} \right) + g_{ij}^{n-1} \text{ on } \gamma_{ij},\ \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, j \neq i. \tag{7}$$

The weak formulation associated with the generalized overlapping DDM (4) is:

$$a_i\left(u_i^n, v\right) + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \lambda_{ij} \int_{\gamma_{ij}} u_i^n v ds = (f,v)_i + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \int_{\gamma_{ij}} g_{ij}^n v ds, \forall v \in V, \tag{8}$$

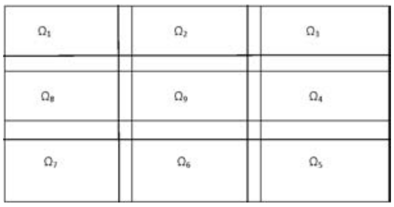

Figure 1: Example of multidomains Schwarz decomposition

where

$$V = H^1_{\Gamma_i}(\Omega_i) = \left\{u \in H^1(\Omega_i)/u = 0 \text{ on } \Gamma_i\right\}.$$

Here $a_i\,(.,.) = a_{\Omega_i}\,(.,.)$ and $(f,v)_i = (f,v)_{\Omega_i}$. The GSAM (with Robin conditions) and this method (with the relaxation procedures) are equivalent. Since when

$$g_{ij}^{n+1} = 2\lambda_{ij}u_j^n - g_{ji}^n \text{ in } L^2\left(\gamma_{ij}\right), \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, j \neq i,$$

Then

$$\begin{aligned} g_{ij}^{n+1} &= 2\lambda_{ij}u_j^n - g_{ji}^n \text{ on } \gamma_{ij} \\ &= 2\lambda_{ij}u_j^n - \left(\frac{\partial u_j^n}{\partial n_{ji}} + \lambda_{ji}u_j^n\right) \text{ on } \gamma_{ij} \\ &= 2\lambda_{ij}u_j^n - \frac{\partial u_j^n}{\partial n_{ji}} - \lambda_{ji}u_j^n \text{ on } \gamma_{ij}. \end{aligned}$$

If $\lambda_{ij} = \lambda_{ji} > 0$, $\gamma_{ij}//\gamma_{ji}$ then $\dfrac{\partial}{\partial n_{ji}} = -\dfrac{\partial}{\partial n_{ij}}$. Therefore

$$g_{ij}^{n+1} = \frac{\partial u_j^n}{\partial n_{ij}} + \lambda_{ij}u_j^n \text{ on } \gamma_{ij}.$$

On the other hand we can take

$$\begin{aligned} g_{ij}^{n+1} &= 2\lambda_{ij}u_j^n - g_{ji}^n \text{ on } \gamma_{ij} \\ &= 2\lambda_{ij}u_j^n - \left(2\lambda_{ij}u_i^{n-1} - g_{ij}^{n-1}\right) \text{ on } \gamma_{ij} \\ &= 2\lambda_{ij}\left(u_j^n - u_i^{n-1}\right) + g_{ij}^{n-1} \text{ on } \gamma_{ij}. \end{aligned}$$

to get the update when $n \geq 1$.

3. Convergence results in the continuous case

In this section, we prove the convergence of the iteratif process (4), (6) and (7) in the H^1-norm for arbitrary initial values as its done in Deng [2] for nonoverlapping DDM and predicted by Lions [7] for overlapping DDM. Let u be the weak solution of the variational model problem (3), We denote by

$$u_i = u_{|\Omega_i}, \; u = (u_i)_{1\le i\le m} \in \prod_{i=1}^{m} H^1_{\Gamma_i}(\Omega_i), \tag{9}$$

$$u^n_{|\Omega_i} = u^n_i, \; u^n = \left(u^n_i\right)_{1\le i\le m} \in \prod_{i=1}^{m} H^1_{\Gamma_i}(\Omega_i), \tag{10}$$

$$e^n = u^n - u, \; e^n = (e^n_i) = \left(u^n_i - u_i\right)_{1\le i\le m} \in \prod_{i=1}^{m} H^1_{\Gamma_i}(\Omega_i), \tag{11}$$

Due to the linearity of equations (2)-(4)-(6)and (7) we can write the equations for the error e^n as follows:

$$\begin{cases} Le^n_i = 0, \text{ in } \Omega_i, 1 \le i \le m \\ \dfrac{\partial e^n_i}{\partial n_{ij}} + \lambda_{ij} e^n_i = g^n_{ij} \text{ on } \gamma_{ij}, \gamma_{ij} \ne \gamma_{ji}, 1 \le j \le m, \, j \ne i, \\ e^n_i = 0 \text{ on } \Gamma_i, \end{cases} \tag{12}$$

of course with different starting values than for equations (4). When $n = 0$, we use, as before, the update

$$g^{n+1}_{ij} = 2\lambda_{ij} e^n_j - g^n_{ji} \text{ on } \gamma_{ij}, \; \gamma_{ij} \ne \gamma_{ji}, 1 \le j \le m, \, j \ne i, \tag{13}$$

whereas, when $n \ge 1$, we use

$$g^{n+1}_{ij} = 2\lambda_{ij}\left(e^n_j - e^{n-1}_i\right) + g^{n-1}_{ij} \text{ on } \gamma_{ij} \;, \; \gamma_{ij} \ne \gamma_{ji}, 1 \le j \le m, \, j \ne i. \tag{14}$$

The associated weak formulation with equations (12) is:

$$a_i\left(e^n_i, v\right) + \sum_{\substack{1\le j\le m \\ j\ne i}} \lambda_{ij} \int_{\gamma_{ij}} e^n_i v\, ds = \sum_{\substack{1\le j\le m \\ j\ne i}} \int_{\gamma_{ij}} g^n_{ij} v\, ds, \forall v \in H^1_{\Gamma_i}(\Omega_i). \tag{15}$$

By letting $v = e^n_i \in H^1_{\Gamma_i}(\Omega_i)$ in (15), we get:

$$a_i\left(e^n_i, e^n_i\right) = \sum_{\substack{1\le j\le m \\ j\ne i}} \int_{\gamma_{ij}} \left(g^n_{ij} - \lambda_{ij} e^n_i\right) e^n_i\, ds. \tag{16}$$

To prove the convergence of this method, we need the following notations

$$|||g|||^2 = \sum_{1\le i\neq j\le m} \frac{1}{\lambda_{ij}} \int_{\gamma_{ij}} \left|g_{ij}^n\right|^2 ds,\ g = (g_{ij})_{1\le i\neq j\le m}, \tag{17}$$

$$\|v\|_{1,\Omega}^2 = \sum_{i=1}^m \|v_i\|_{H^1(\Omega_i)}^2,\ v = (v_i) \in \prod_{i=1}^m H_{\Gamma_i}^1(\Omega_i), \tag{18}$$

$$a(u,v) = \sum_{i=1}^m a_i(u,v),\ u = (u_i) \in \prod_{i=1}^m H_{\Gamma_i}^1(\Omega_i) \tag{19}$$

We need to define two groups of subdomains, those which meet the boundary of Ω and those which do not i.e.

$$G_1 = \{\cup\Omega_k,\ \text{such that } mes\,(\partial\Omega_k \cap \partial\Omega) > 0\} \tag{20}$$

and

$$G_{r+1} = \left\{\cup\Omega_k, \text{such that } mes\left(\partial\Omega_k \cap \overline{G_r}\right) > 0,\ \partial\Omega_k \cap G_l = \varnothing\ \forall l \le r\right\} \tag{21}$$

We also need the following lemma to prove the main convergence theorem of this section.

Lemma 3.1. Let e^n, g_{ij}^n and g_{ij}^{n+1} be as in (11),(13)-(14) and for $k = n-1, n+1$ let $g^k = \left(g_{ij}^k\right)_{1\le i\neq j\le m}$. Then we have

$$\left|\left\|g^{n+1}\right\|\right|^2 = \left|\left\|g^{n-1}\right\|\right|^2 - 4S_n. \tag{22}$$

where

$$S_n = \sum_{1\le i\neq j\le m} \int_{\gamma_{ij}} \left[\lambda_{ij}\left(e_j^n - e_i^{n-1}\right) + g_{ij}^{n-1}\right]\left(e_i^{n-1} - e_j^n\right) ds. \tag{23}$$

$$S_n \underset{n\to\infty}{\to} 0. \tag{24}$$

Proof. We have from (14), the update for $n \ge 1$ is given by

$$g_{ij}^{n+1} = 2\lambda_{ij}\left(e_j^n - e_i^{n-1}\right) + g_{ij}^{n-1} \text{ on } \gamma_{ij},\ \gamma_{ij} \neq \gamma_{ji}, 1 \le j \le m, j \neq i.$$

We apply the norm (17) to both sides of the above equation to get

$$\begin{aligned}
\left|\left\|g^{n+1}\right\|\right|^2 &= \sum_{1\le i\neq j\le m} \frac{1}{\lambda_{ij}} \int_{\gamma_{ij}} \left|g_{ij}^{n+1}\right|^2 ds = \sum_{1\le i\neq j\le m} \frac{1}{\lambda_{ij}} \int_{\gamma_{ij}} \left|2\lambda_{ij}(e_j^n - e_i^{n-1}) + g_{ij}^{n-1}\right|^2 ds \\
&= \left|\left\|g^{n-1}\right\|\right|^2 + 4\sum_{1\le i\neq j\le m} \lambda_{ij}\left\|(e_j^n - e_i^{n-1}\right\|_{0,\gamma_{ij}}^2 + 4\sum_{1\le i\neq j\le m} \left\langle e_j^n - e_i^{n-1}, g_{ij}^{n-1}\right\rangle_{|\gamma_{ij}} \\
&= \left|\left\|g^{n-1}\right\|\right|^2 - 4\sum_{1\le i\neq j\le m} \int_{\gamma_{ij}} \left(\lambda_{ij}(e_j^n - e_i^{n-1}) + g_{ij}^{n-1}\right)(e_i^{n-1} - e_j^n) ds \\
&= \left|\left\|g^{n-1}\right\|\right|^2 - 4S_n,
\end{aligned}$$

Summing the above equality over n, we get

$$\begin{aligned}\sum_{n=1}^{M} S_n &= \frac{1}{4}\sum_{n=1}^{M}\left(\left|\|g^{n-1}\|\right|^2 - \left|\|g^{n+1}\|\right|^2\right)\\ &= \frac{1}{4}(\left|\|g^{0}\|\right|^2 + \left|\|g^{1}\|\right|^2 - \left|\|g^{M}\|\right|^2 - \left|\|g^{M+1}\|\right|^2),\end{aligned}$$

since $\sum_{n=1}^{M} S_n$ is bounded, then

$$S_n \underset{n\to\infty}{\to} 0. \tag{25}$$

■

The main result of this section is the following convergence Theorem:

Theorem 3.2. Let $u \in H_0^1(\Omega)$ be the solution of (2) and let $u_i^n \in H_{\Gamma_i}^1(\Omega_i)$ $(i = 1, 2, \ldots, m)$ be solutions of (4)-(6) and (7). Then for all $(g_{ij}^0) \in L^2(\gamma_{ij})$, $1 \leq i \neq j \leq m$, we have:

$$\|u^n - u\|_{1,\Omega} = \left(\sum_{i=1}^{m}\|u_i^n - u_i\|_{1,\Omega_i}^2\right)^{\frac{1}{2}} \underset{n\to\infty}{\to 0}\ , \tag{26}$$

Proof. Our aim is to prove that:

$$\|e^n\|_{1,\Omega} = \left(\sum_{i=1}^{m}\|e_i^n\|_{1,\Omega_i}^2\right)^{\frac{1}{2}} \underset{n\to\infty}{\to 0}\ . \tag{27}$$

We have from (14), that

$$-g_{ij}^{n+1} + 2\lambda_{ij}e_j^n = 2\lambda_{ij}e_i^{n-1} - g_{ij}^{n-1} \text{ on } \gamma_{ij}\ .$$

We apply the $L^2(\gamma_{ij})$ $-$norm to both sides of the above equality and divide both sides by λ_{ij} $(\lambda_{ij} = \lambda_{ji} > 0)$, and then sum over $1 \leq i, j \leq m$ to end up with:

$$\sum_{i=1}^{m}\sum_{\substack{1\leq j\leq m\\ j\neq i}}\frac{1}{\lambda_{ij}}\left\|-g_{ij}^{n+1} + 2\lambda_{ij}e_j^n\right\|_{0,\gamma_{ij}}^2 = \sum_{i=1}^{m}\sum_{\substack{1\leq j\leq m\\ j\neq i}}\frac{1}{\lambda_{ij}}\left\|g_{ij}^{n-1}\right\|_{0,\gamma_{ij}}^2 - 4\sum_{i=1}^{m}a_i\left(e_i^{n-1}, e_i^{n-1}\right),$$

i.e.

$$a\left(e^{n-1}, e^{n-1}\right) \leq \frac{1}{4}\left|\|g^{n-1}\|\right|^2.$$

Then, we have from (14) that

$$\begin{aligned} g_{ij}^{n+1} &= 2\lambda_{ij}\left(e_j^n - e_i^{n-1}\right) + g_{ij}^{n-1} \text{ on } \gamma_{ij}\\ &= 2\lambda_{ij}e_j^n - \left(2\lambda_{ij}e_i^{n-1} - g_{ij}^{n-1}\right) \text{ on } \gamma_{ij}\\ &= 2\lambda_{ij}e_j^n - g_{ji}^n \ \text{ on } \gamma_{ij}.\end{aligned}$$

Where

$$g_{ji}^{n} = 2\lambda_{ij}e_i^{n-1} - g_{ij}^{n-1} \text{ on } \gamma_{ij}$$
$$g_{ji}^{n} - \lambda_{ij}e_i^{n-1} = \lambda_{ij}e_i^{n-1} - g_{ij}^{n-1} \text{ on } \gamma_{ij}.$$

We multiply both sides of the above equation by e_i^{n-1}, integrate over γ_{ij} and then sum to get

$$\sum_{1\leq j\neq i\leq m} \int_{\gamma_{ij}} \left(g_{ji}^{n} - \lambda_{ij}e_i^{n-1}\right) e_i^{n-1} ds = -a\left(e^{n-1}, e^{n-1}\right) \leq 0, \tag{28}$$

Since (28) is negative and $\gamma_{ij}//\gamma_{ji}$, $\left|\gamma_{ij}\right| = \left|\gamma_{ji}\right|$, then the sign of the integral remainder always negative if we integrate (28) on γ_{ji}, i.e.

$$\sum_{1\leq i\neq j\leq m} \int_{\gamma_{ji}} (g_{ji}^{n} - \lambda_{ij}e_i^{n-1})e_i^{n-1} ds \leq 0. \tag{29}$$

Then, we use again (14)

$$g_{ij}^{n+1} - 2\lambda_{ij}e_j^{n} = g_{ij}^{n-1} - 2\lambda_{ij}e_i^{n-1} \text{ on } \gamma_{ij}.$$

We apply the $L^2\left(\gamma_{ij}\right)$ −norm, we divide the above equality by λ_{ij} $\left(\lambda_{ij} = \lambda_{ji} > 0\right)$ and sum, to get

$$\left|\left\|g^{n+1}\right\|\right|^2 - \left|\left\|g^{n-1}\right\|\right|^2 + 4a\left(e^{n-1}, e^{n-1}\right) = 4\sum_{1\leq i\neq j\leq m} \int_{\gamma_{ij}} (g_{ij}^{n+1} - \lambda_{ij}e_j^{n})e_j^{n} ds,$$

by (29) and (22), we get

$$a\left(e^{n-1}, e^{n-1}\right) \leq S_n. \tag{30}$$

Our aim now is to prove that

$$\left|\left\|g^{n}\right\|\right|^2 \underset{n\to\infty}{\rightarrow} 0\ . \tag{31}$$

From (22)-(24) and (30), we use the fact that

$$\frac{\left|\left\|g^{n+1}\right\|\right|^2}{\left|\left\|g^{n-1}\right\|\right|^2} \leq 1 - \frac{4S_n}{\left|\left\|g^{n-1}\right\|\right|^2 + 1} = M_n \leq 1.$$

To get

$$\lim_{n\to\infty} \left|\left\|g^{n}\right\|\right|^2 = \begin{cases} \lim\limits_{n\to\infty} M_n \times M_{n-1} \times \cdots \times M_1 \left|\left\|g^{0}\right\|\right|^2 = 0, (n \text{ even})\ . \\ \lim\limits_{n\to\infty} M_n \times M_{n-1} \times ... \times M_2 \left|\left\|g^{1}\right\|\right|^2 = 0, (n \text{ odd})\ . \end{cases}$$

From which

$$\lim_{n\to\infty} \left|\left\| g^n \right\|\right|^2 = \lim_{n\to\infty} \sum_{1\le i\ne j\le m} \frac{1}{\lambda_{ij}} \left\| g_{ij}^n \right\|_{0,\gamma_{ij}}^2 = 0, \forall n \in \mathbb{N}. \tag{32}$$

Therefore

$$\lim_{n\to\infty} a\left(e^n, e^n\right) = \lim_{n\to\infty} \sum_{i=1}^{m} a\left(e_i^n, e_i^n\right) = 0. \tag{33}$$

Because $a\,(.,.)$ is coersive, there exists $\mu > 0$ such that:

$$\mu \left\| e_i^n \right\|_{1,\Omega_i}^2 \le a\left(e_i^n, e_i^n\right)$$

Then for $\alpha_0\,(x) \ge C_0 > 0$, and (33) we get (27) and

$$\left\| e_i^n \right\|_{1,\Omega_i} \le C. \tag{34}$$

In general, $\alpha_0\,(x) \ge 0$, from (33) we get:

$$\left\| \nabla e_i^n \right\|_{0,\Omega_i} \underset{n\to\infty}{\to 0}. \tag{35}$$

Hence, for all $i, \Omega_i \subset G_1$:

$$e_i^n = 0, \text{ in } \Gamma_i \ \ (mes\,(\Gamma_i) > 0)\,.$$

Using (35) and the Poincaré inequality we get

$$\left\| e_i^n \right\|_{1,\Omega_i} \le C \left\| \nabla e_i^n \right\|_{0,\Omega_i} \underset{n\to\infty}{\to 0}, \forall i, \Omega_i \subset G_1. \tag{36}$$

Now, For all $i, \Omega_i \subset G_2$, we have

$$\left\| g_{ij}^n \right\|_{0,\gamma_{ij}} \underset{n\to\infty}{\to 0}, 1 \le j \le m, j \ne i. \tag{37}$$

and

$$\left\| \nabla e_i^n \right\|_{0,\Omega_i} \to 0, 1 \le j \le m, j \ne i.$$

We also have from the trace theorem that:

$$\left\| e_i^n \right\|_{0,\gamma_{ij}} \le C \left\| e_i^n \right\|_{1,\Omega_i} \tag{38}$$

From (16), (32) and (34) we can get for $i, \Omega_i \subset G_2$, $\left\| e_i^n \right\|_{0,\gamma_{ij}} \to 0$. We can take for $v \in H_{\Gamma_i}^1\,(\Omega_i)$:

$$v = \begin{cases} e_i & \text{on } \gamma_{ij}, \forall j \in G_1 \\ 0, & \text{on } \partial\Omega_i \backslash \gamma_{ij}, \end{cases} \tag{39}$$

and then use the Poincaré inequality to get

$$\left\| e_i^n \right\|_{1,\Omega_i} \le C \left(\left\| \nabla e_i^n \right\|_{0,\Omega_i} + \sum_{\Omega_j \subset G_1} \left\| e_i^n \right\|_{0,\gamma_{ij}} \right) \to 0. \tag{40}$$

■

4. Convergence results in the discrete case

To obtain a numerical approximation of u, we replace the problem (2) by its discrete analog, for this we need to introduce triangulation τ_h of $\overline{\Omega}$, using closed triangles $t_k \in \tau_h$ of diameter: $diam(t_k) \leq h_k$. where h_k : is the diameter of the element t_k and

$$h = \max(h_k).$$

$$H : \text{the diameter of subdomain.}$$

We then construct the finite element space $V^h \subset H_0^1(\Omega)$ of dimension N_h, with $(\varphi_1, \varphi_2, .., \varphi_{N_h})$ is its base.

$$V^h = \{v^h \in C^0(\overline{\Omega}), v^h_{|t_k} = P_l, l \geq 1, \forall t_k \in \tau_k, v^h_{|\Gamma} = 0\}. \tag{41}$$

the finite element approximation u^h of u in this space is written as

$$u^h = \sum_{k=1}^{N_h} u_k^h \varphi_k$$

The discrete variational formulation associated to (2) is

$$a_\Omega\left(u^h, v^h\right) = \left(f, v^h\right)_\Omega, \forall v^h \in V^h \tag{42}$$

We need to define T_i, T_{ij} two trace spaces such that

$$T_i = V^h_{|\partial\Omega_i}, T_{ij} = V^h_{|\gamma_{ij}}.$$

We introduce r_i, r_{ij} resctriction operators such that

$$\begin{cases} r_i : V_i^h \to T_i, \quad r_{ij} : V_i^h \to T_{ij}, \\ r_i v^h = v^h_{|\partial\Omega_i}, r_{ij} v^h = v^h_{|\gamma_{ij}}. \end{cases}$$

We also define s_i, s_{ij} to be linear operators as in [11], for $\forall w_i^h \in T_i$

$$\begin{aligned} s_i &: T_i \to V_i^h \\ s_i w_i^h &= \begin{cases} w_i^h \text{ freedom on } \partial\Omega_i \\ 0 \text{ other freedom} \end{cases}. \end{aligned}$$

and $\forall w_{ij}^h \in T_{ij}$

$$\begin{aligned} s_{ij} &: T_{ij} \to V_i^h \\ s_{ij} w_{ij}^h &= \begin{cases} w_{ij}^h \text{ freedom on } \gamma_{ij} \\ 0 \text{ other freedom} \end{cases}. \end{aligned}$$

The discrete generalized overlapping DDM iterative procedure is written as

$$\begin{cases} Lu_i^{n,h} = f \ \text{ in } \Omega_i, 1 \leq i \leq m \\ \dfrac{\partial u_i^{n,h}}{\partial n_{ij}} + \lambda_{ij} u_i^{n,h} = g_{ij}^{n,h} \ \text{ on } \gamma_{ij}, \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, \, j \neq i, \\ u_i^{n,h} = 0 \ \text{ on } \Gamma_i, \end{cases} \tag{43}$$

$$g_{ij}^{n+1,h} = 2\lambda_{ij} u_j^{n,h} - g_{ji}^{n,h} \text{ on } \gamma\text{ij}, \ \gamma_{ij} \neq \gamma_{ji}, 1 \leq \underset{j \neq i}{j} \leq m, \tag{44}$$

where $g_{ij}^{0,h} \in L^2\left(\gamma_{ij}\right)$ $(1 \leq j \leq m)$ is an initial value, the relation (44) is the update when $n = 0$. From the step $n \geq 1$, the following update is used:

$$g_{ij}^{n+1,h} = 2\lambda_{ij} \left(u_j^{n,h} - u_i^{n-1,h} \right) + g_{ij}^{n-1,h} \text{ on } \gamma_{ij} \ , \ \gamma_{ij} \neq \gamma_{ji}, 1 \leq \underset{j \neq i}{j} \leq m. \tag{45}$$

In variational form (43) is written as follows:
Let $g_{ij}^{0,h} \in L^2\left(\gamma_{ij}\right) \cap V^h|_{\gamma_{ij}}, \ 1 \leq i \neq j \ \leq m$, starting values we get $u_i^{n,h} \in V^h|_{\Omega_i}, i = 1, 2, \ldots, m$, solution of,

$$a_i\left(u_i^{n,h}, v^h\right) + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \lambda_{ij} \int_{\gamma_{ij}} u_i^{n,h} v^h ds = \left(f, v^h\right)_i + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \int_{\gamma_{ij}} g_{ij}^{n,h} v^h ds, \ \forall v^h \in V_i^h.$$

We need the following preliminary results on subdomains.

Lemma 4.1. If u^h is the solution of the discrete problem on Ω and $u_i^h = u^h_{|\Omega_i}$, then $\exists g_{ij}^h \in T_{ij}$ such that $\forall v_i^h \in V_i^h$:

$$a_i\left(u_i^h, v^h\right) + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \lambda_{ij} \int_{\gamma_{ij}} u_i^h v_i^h ds = \int_{\Omega_i} f v^h dx + \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \int_{\gamma_{ij}} g_{ij}^h v_i^h ds,$$

Our aim is to obtain a convergence rate depending on h and H., We need first to define the convergence rate.

Definition 4.2. Let u^h be the finite element solution to problem (42) and $u^{n,h}$ is the finite element solution to the generalized overlapping DDM at the $n-th$ iteration. If

$$\left\| u^{n,h} - u^h \right\| \leq C A^n \left\| u^{0,h} - u^h \right\| \tag{46}$$

where $\|.\|$ is a certain Sobolev norm, $A \in [0, 1)$ and C is a constant independent n, then (46) is a geometric convergence and A is the convergence rate.

Lemma 4.3. [11] Let r_{ij} be the trace operator s_{ij} the operator defined previously, then we have the following inequalities $\forall v_i^h \in V_i^h$

$$\begin{aligned} \left\| r_{ij} v_i^h \right\|_{0,\gamma_{ij}} &\leq C \left\| v_{i|\gamma ij}^h \right\|_{0,\gamma_{ij}}, \\ \left\| s_{ij} w_{ij}^h \right\|_{0,\Omega_i} &\leq C h^{\frac{1}{2}} \left\| w_{ij}^h \right\|_{0,\gamma_{ij}}, \\ \left| s_{ij} w_{ij}^h \right|_{1,\Omega_i} &\leq C h^{-\frac{1}{2}} \left\| v_{ij}^h \right\|_{0,\gamma_{ij}} \end{aligned}$$

where C is a generic constant independent of h.

Theorem 4.4. [11] If the diameter of each Ω_i, $1 \leq i \leq m$ is H, then $\forall v_i \in V_i^h$ we have

$$\left\| v_i^h \right\|_{0,\partial\Omega_i}^2 \leq CH \left\| v_i^h \right\|_{1,\Omega_i}^2,$$

where C is a generic constant independent of Ω_i.

We define the following errors :

$$E_{ij}^{n,h} = g_{ij}^{n,h} - g_{ij}^h \in T_{ij} \text{ and } e_i^{n,h} = u_i^{n,h} - u_i^h \tag{47}$$

where $e^{n,h} = (e_1^{n,h}, \ldots, e_m^{n,h}) \in \prod_{i=1}^{m} V_i^h$ and

$$\left|\left\| E_{ij}^{n,h} \right\|\right|^2 = \frac{1}{\lambda_{ij}} \left\| E_{ij}^{n,h} \right\|_{0,\gamma_{ij}}^2, \left|\left\| E^{n,h} \right\|\right|^2 = \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \left|\left\| E_{ij}^{n,h} \right\|\right|^2. \tag{48}$$

Then, the discrete error at the n-th iteration $e^{n,h}$ satisfies:

$$\begin{cases} Le_i^{n,h} = 0 \text{ in } \Omega_i, 1 \leq i \leq m \\ \dfrac{\partial e_i^{n,h}}{\partial n_{ij}} + \lambda_{ij} e_i^{n,h} = E_{ij}^{n,h} \text{ on } \gamma_{ij}, \gamma_{ij} \neq \gamma_{ji}, 1 \leq j \leq m, j \neq i, \\ e_i^{n,h} = 0 \text{ on } \Gamma_i, \end{cases} \tag{49}$$

We use, as before, the updates

$$E_{ij}^{n+1,h} = 2\lambda_{ij} e_j^{n,h} - E_{ji}^{n,h} \text{ on } \gamma_{ij}, 1 \leq \underset{j \neq i}{j} \leq m, \text{ when } n = 0, \tag{50}$$

and

$$E_{ij}^{n+1,h} = 2\lambda_{ij} \left(e_j^{n,h} - e_i^{n-1,h} \right) + E_{ij}^{n-1,h} \text{ on } \gamma_{ij}, 1 \leq \underset{j \neq i}{j} \leq m, \text{ when } n \geq 1. \tag{51}$$

The variational formulation of (49) takes the form :

$$a_i\left(e_i^{n,h}, v_i^h\right) + \sum_{\substack{1\leq j\leq m\\ j\neq i}} \lambda_{ij} \int_{\gamma_{ij}} e_i^{n,h} v_i^h ds = \sum_{\substack{1\leq j\leq m\\ j\neq i}} \int_{\gamma_{ij}} E_{ij}^{n,h} v^h ds, \forall v^h \in V_i^h. \tag{52}$$

Before announcing the main theorem of this section we need the following preliminary result.

Lemma 4.5. Let $e_i^{n,h}$, $E_{ij}^{n-1,h}$, $E_{ij}^{n+1,h}$ be the errors defined as in (47),(50) and (51)

$$E^{p,h} = \left(E_{ij}^{p,h}\right)_{1\leq i\neq j\leq m}, \; p = n-1, \; n+1,$$

then we have

$$|\|E^{n+1,h}\||^2 = |\|E^{n-1,h}\||^2 - 4S_{n,h}, \tag{53}$$

and

$$-4S_{n,h} \leq 0$$

where

$$S_{n,h} = \sum_{1\leq i\neq j\leq m} \int_{\gamma_{ij}} \left(\lambda_{ij}(e_j^{n,h} - e_i^{n-1,h}) + E_{ij}^{n-1,h}\right)(e_i^{n-1,h} - e_j^{n,h}) ds. \tag{54}$$

The proof is similar to that for lemma 3.1.

Theorem 4.6. Let $\lambda_{ij} = \lambda_{ji} > 0, \forall i \neq j$, we have the following convergence results for $\alpha_0(x), \alpha_{kl}(x) > 0$:

$$a\left(e^{n,h}, e^{n,h}\right) \leq \frac{1}{4} |\|E^{n,h}\||^2,$$

$$|\|E^{n,h}\||^2 \leq \begin{cases} \left[1 - C(h^{-\frac{1}{2}}\lambda^{-\frac{1}{2}} + h^{\frac{1}{2}}\lambda^{-\frac{1}{2}} + H^{\frac{1}{2}}\lambda^{\frac{1}{2}})^{-2}\right]^{\frac{n}{2}} |\|E^{0,h}\||^2, (n \text{ even}) \; . \\ \left[1 - C(h^{-\frac{1}{2}}\lambda^{-\frac{1}{2}} + h^{\frac{1}{2}}\lambda^{-\frac{1}{2}} + H^{\frac{1}{2}}\lambda^{\frac{1}{2}})^{-2}\right]^{\frac{n+1}{2}} |\|E^{1,h}\||^2, (n \text{ odd}) \; . \end{cases}$$

where C is a generic constant independent of h and H.

Proof. In a similar way to that in the proof of theorem 3.2 we can prove that

$$a\left(e^{n-1,h}, e^{n-1,h}\right) \leq \frac{1}{4} |\|E^{n-1,h}\||^2. \tag{55}$$

and

$$a\left(e^{n-1,h}, e^{n-1,h}\right) \leq S_{n,h}. \tag{56}$$

Now, if we take $v_i^h = s_{ij} E_{ij}^{n-1,h}$ in the variational formulation (52) we get

$$\left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 = a_i \left(e_i^{n-1,h}, s_{ij} E_{ij}^{n-1,h} \right) + \lambda_{ij} \int\limits_{\gamma_{ij}} e_i^{n-1,h} E_{ij}^{n-1,h} ds.$$

Now we use the Cauchy-Schwarz inequality, lemma 4.3 and the trace theorem 4.4 to obtain

$$\left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 \leq \max \alpha_{kl}(x) \left| e_i^{n-1,h} \right|_{1,\Omega_i} \left| s_{ij} E_{ij}^{n-1,h} \right|_{1,\Omega_i}$$

$$+ \max \alpha_0(x) \left\| e_i^{n-1,h} \right\|_{0,\Omega_i} \left\| s_{ij} E_{ij}^{n-1,h} \right\|_{0,\Omega_i} + \lambda_{ij} \left\| e_i^{n-1,h} \right\|_{0,\gamma_{ij}} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}},$$

$$\left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 \leq C h^{\frac{-1}{2}} \max \alpha(x) \left| e_i^{n-1,h} \right|_{1,\Omega_i} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}$$

$$+ C h^{\frac{1}{2}} \max \alpha_0(x) \left\| e_i^{n-1,h} \right\|_{0,\Omega_i} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}} + C \lambda_{ij} H^{\frac{1}{2}} \left\| e_i^{n-1,h} \right\|_{1,\Omega_i} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}},$$

Because $\left\| e_i^{n-1,h} \right\|_{1,\Omega_i}^2 = \left| e_i^{n-1,h} \right|_{1,\Omega_i}^2 + \left\| e_i^{n-1,h} \right\|_{0,\Omega_i}^2$, then we have

$$\left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 \leq C \left[h^{-\frac{1}{2}} \max \alpha_{kl}(x) + h^{\frac{1}{2}} \max \alpha_0(x) + \lambda_{ij} H^{\frac{1}{2}} \right] \left\| e_i^{n-1,h} \right\|_{1,\Omega_i} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}.$$

Using the fact that $a_i(.,.)$ is coercive, i.e. $\exists \beta > 0$ such that

$$a_i \left(e_i^{n-1,h}, e_i^{n-1,h} \right) \geq \beta \left\| e_i^{n-1,h} \right\|_{1,\Omega_i}^2,$$

we have

$$\left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 \leq C \left[h^{\frac{-1}{2}} \max \alpha(x) + h^{\frac{1}{2}} \max \alpha_0(x) + \lambda_{ij} H^{\frac{1}{2}} \right] a_i^{\frac{1}{2}} \left(e_i^{n-1,h}, e_i^{n-1,h} \right) \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}},$$

we take $\lambda_{ij} = \lambda$, we multiply the inequality by $\lambda^{-\frac{1}{2}}$ to obtain

$$\begin{aligned} \frac{1}{\lambda^{\frac{1}{2}}} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}} &\leq C \left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right] a_i^{\frac{1}{2}} \left(e_i^{n-1,h}, e_i^{n-1,h} \right), \\ \frac{1}{\lambda} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 &\leq C \left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{\frac{-1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^2 a_i \left(e_i^{n-1,h}, e_i^{n-1,h} \right). \end{aligned}$$

Summing over $i,\ j;\ 1 \leq i,\ j \leq m,\ j \neq i$ to get

$$\sum_{i=1}^{m} \sum_{\substack{1 \leq j \leq m \\ j \neq i}} \frac{1}{\lambda} \left\| E_{ij}^{n-1,h} \right\|_{0,\gamma_{ij}}^2 \leq C \left[h^{\frac{-1}{2}} \lambda^{-\frac{1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^2 \sum_{i=1}^{m} a_i \left(e_i^{n-1,h}, e_i^{n-1,h} \right),$$

i.e.

$$\left|\left\| E^{n-1,h} \right\|\right|^2 \leq C \left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{\frac{-1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^2 a\left(e^{n-1,h}, e^{n-1,h}\right).$$

Hence

$$-a\left(e^{n-1,h}, e^{n-1,h}\right) \leq -\frac{C}{\left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{\frac{-1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^2} \left|\left\| E^{n-1,h} \right\|\right|^2, \qquad (57)$$

and we have from (56) that

$$-4S_{n,h} \leq -C \left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^{-2} \left|\left\| E^{n-1,h} \right\|\right|^2$$

$$\left|\left\| E^{n+1,h} \right\|\right|^2 - \left|\left\| E^{n-1,h} \right\|\right|^2 \leq -C \left[h^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^{-2} \left|\left\| E^{n-1,h} \right\|\right|^2.$$

Hence

$$\left|\left\| E^{n+1,h} \right\|\right|^2 \leq \left(1 - C \left[h^{\frac{-1}{2}} \lambda^{\frac{-1}{2}} + h^{\frac{1}{2}} \lambda^{\frac{-1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^{-2} \right) \left|\left\| E^{n-1,h} \right\|\right|^2.$$

In general, we can write

$$\left|\left\| E^{n,h} \right\|\right|^2 \leq \begin{cases} \left(1 - C \left[h^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^{-2} \right)^{\frac{n}{2}} \left|\left\| E^{0,h} \right\|\right|^2, & (n \text{ even}), \\ \left(1 - C \left[h^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} + h^{\frac{1}{2}} \lambda^{-\frac{1}{2}} + \lambda^{\frac{1}{2}} H^{\frac{1}{2}} \right]^{-2} \right)^{\frac{n+1}{2}} \left|\left\| E^{1,h} \right\|\right|^2, & (n \text{ odd}). \end{cases}$$

■

Corollary 4.7. For a choice of

$$\lambda = O(H^{\frac{1}{2}} h^{-\frac{1}{2}})$$

we have

$$\left|\left\| E^{n,h} \right\|\right|^2 \leq \begin{cases} \left(1 - C \frac{h^{\frac{1}{2}} H^{\frac{1}{2}}}{(1+h+H)^2} \right)^{\frac{n}{2}} \left|\left\| E^{0,h} \right\|\right|^2, & (n \text{ even}). \\ \left(1 - C \frac{h^{\frac{1}{2}} H^{\frac{1}{2}}}{(1+h+H)^2} \right)^{\frac{n+1}{2}} \left|\left\| E^{1,h} \right\|\right|^2, & (n \text{ odd}). \end{cases}$$

Table 1: Computed errors using the optimal parameter λ_1 computed using the diameter of subdomains.

n	10	9	8	8	7	7
h	7.22656 (−2)	4.98827 (−2)	3.56725 (−2)	3.16248 (−2)	2.39612 (−2)	2.02743 (−2)
λ_1	2.61713	3.15004	3.72499	3.95619	4.54503	4.94104
E_1	7.28185 (−4)	6.06232 (−4)	9.62553 (−4)	6.97588 (−4)	1.6411 (−3)	1.51838 (−3)
E_2	6.42793 (−4)	4.21228 (−4)	2.42216 (−4)	4.88872 (−4)	9.88321 (−4)	8.73636 (−4)
E_3	7.68888 (−4)	6.19124 (−4)	1.00608 (−3)	7.2757 (−4)	1.6747 (−3)	1.5305 (−3)
E_4	7.46416 (−4)	5.77527 (−4)	3.36459 (−4)	5.0681 (−4)	1.00272 (−3)	8.5534 (−4)
E_5	9.65344 (−4)	7.80042 (−4)	1.09302 (−3)	8.09513 (−4)	1.75415 (−3)	1.57165 (−3)
E_6	8.27126 (−4)	5.35709 (−4)	3.29798 (−4)	5.05156 (−4)	9.99831 (−4)	8.55825 (−4)
E_7	8.10454 (−4)	6.09655 (−4)	9.97034 (−4)	7.31062 (−4)	1.6765 (−3)	1.52733 (−3)
E_8	6.38624 (−4)	4.10186 (−4)	2.61681 (−4)	4.85569 (−4)	9.9218 (−4)	8.70073 (−4)
E_9	7.6058 (−4)	4.7342 (−4)	3.61371 (−4)	3.88561 (−4)	4.4932 (−4)	4.84113 (−4)

Table 2: Computed errors using the optimal parameter λ_2 computed using the size of the overlap.

n	10	9	8	8	7	7
h	7.22656 (−2)	4.98827 (−2)	3.56725 (−2)	3.16248 (−2)	2.39612 (−2)	2.02743 (−2)
λ_2	2.24434	2.70134	3.19439	3.39266	3.89762	4.23722
E_1	1.31766 (−3)	1.35437 (−3)	1.75157 (−3)	1.49889 (−3)	2.03904 (−3)	1.81746 (−3)
E_2	1.1692 (−3)	1.05351 (−3)	9.19297 (−4)	6.21742 (−4)	1.47993 (−3)	1.19704 (−3)
E_3	1.40724 (−3)	1.39505 (−3)	1.82353 (−3)	1.55541 (−3)	2.10579 (−3)	1.87078 (−3)
E_4	1.30072 (−3)	1.19023 (−3)	1.02439 (−3)	7.15879 (−4)	1.54542 (−3)	1.22824 (−3)
E_5	1.6002 (−3)	1.57622 (−3)	1.95841 (−3)	1.68027 (−3)	2.24863 (−3)	1.97966 (−3)
E_6	1.33054 (−3)	1.1706 (−3)	1.01985 (−3)	7.16062 (−4)	1.54357 (−3)	1.22825 (−3)
E_7	1.41778 (−3)	1.39459 (−3)	1.81453 (−3)	1.55687 (−3)	2.10757 (−3)	1.86765 (−3)
E_8	1.16197 (−3)	1.05183 (−3)	9.17917 (−4)	6.26358 (−4)	1.48284 (−3)	1.19338 (−3)
E_9	8.19311 (−4)	5.90348 (−4)	3.86911 (−4)	3.19053 (−4)	4.37793 (−4)	4.072 (−4)

4.1. Numerical results

As an illustration of the theoretical results obtained in this paper and to be able to use the generalized overlapping (Schwarz) DDM, we consider the following problem

$$\begin{cases} -\Delta u + u = f \in L^2(\Omega). \\ u = 0 \text{ on } \Gamma. \end{cases}$$

We take $\Omega = [0, 0.85]^2$. The exact solution is taken to be

$$u\,(x, y) = xy\,(x - 0.85)\,(y - 0.85)\,e^{xy}$$

We note by n the iteration number, $Er_i = \left\| u - u_i^{h,n} \right\|_{1,\Omega_i}$, $1 \le i \le 9$, the error between the exact solution and discrete DDM approximation in the $H^1 - norm$. We compute the discrete DDM approximations $u_i^{h,n}$, on $\Omega_i, 1 \le i \le 9$, for h is the mesh size, the maximum of h_i over $\Omega_i, 1 \leq i \leq 9$, $H_1 = 0.494975$ is the diameter of each subdomain and $H_2 = 0.364005$ is the diameter of each overlap. $\lambda_i = H_i^{\frac{1}{2}} h^{-\frac{1}{2}}$ is the relaxation parameter. We take $\Omega_1 = [0, 0.35]^2$ and by uniformity we construct the other subdomains. In the first table, we use the diameter H_1 of subdomains, to compute the optimal relaxation parameter λ_1. Whereas in the second table, we use the diameter H_2 of the overlap, to compute the optimal relaxation parameter λ_2.

We can remark that the results of the first table (using the size of subdomains to compute the optimal parameter λ_1) are a bit better than that of the second table (where we used the size of the overlap to compute the optimal parameter λ_2).

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