Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4443-4453

http://www.ripublication.com/gjpam.htm

A Predator-Prey Mathematical Model in a Limited Area

Eugeny Petrovich Kolpak

Saint Petersburg State University, Faculty of Applied Mathematics and Control Process, 199034, Saint Petersburg, Universitetskaya nab., 7/9, Russian Federation.

Mariia Vladimirovna Stolbovaia

Saint Petersburg State University, Faculty of Applied Mathematics and Control Process, 199034, Saint Petersburg, Universitetskaya nab., 7/9, Russian Federation.

Inna Sergeevna Frantsuzova

Saint Petersburg State University, Faculty of Applied Mathematics and Control Process, 199034, Saint Petersburg, Universitetskaya nab., 7/9, Russian Federation.

Abstract

A predator-prey mathematical model is a boundary-value problem for a system of two non-linear differential equations in partial derivatives. A stationary state stability is studied. A variational method is used to build a numerical solution. The dependence of an amplitude and a frequency of damped vibrations on parameters, characterizing the mobility of species, is estimated.

AMS subject classification:

Keywords: population, boundary-value problems, mathematical modeling.

INTRODUCTION

Most published papers represent a predator-prey mathematical model as a Cauchy problem for the system of ordinary differential equations [1-11]. These models do not take into account spatial distribution of species. Actual populations live in limited areas with different environment characteristics in its various parts [12-23]. For various reasons (for example, in search of food or habitat availability), some species are inclined to migration through area. Models, considering spatial distribution of the species, describe the population density, and species are believed to be distributed in area. Habitat is believed to be solid, allowing to use the system of differential equations in partial derivatives, widely used at the development of mathematical models of continuous environment with linear properties [24-33].

A PREDATOR-PREY POINT MODEL

A mathematical model describing dynamics of two populations interacting under predator-prey basis was offered by Lotka and Volterra. [3, 24]

$$\begin{aligned} \frac{du}{d\tau} &= c_1 u - a_{12} uv, \\ \frac{dv}{d\tau} &= -c_2 v + a_{21} uv. \end{aligned} \tag{1}$$

In these equations u and v - prey and predator populations are shown correspondingly, c_1 - the specific rate of prey population growth in predator's absence, a_{12} - constant, characterizing the rate of predators' consumption of prey species, c_2 - the specific rate of predator's mortality, a_{21} - constant characterizing the rate of increased number of predators due to preys' death. A fixed point of the equation system (1) is $u_* = c_2 / a_{21}$, $v_* = c_1 / a_{12}$. This fixed point is stable, host harmonic fluctuations of both populations with a frequency of $\omega = \sqrt{c_1 c_2}$ in its neighborhood [24].

A Volterra model explains one of the reasons for fluctuations in the "predator-prey" system, but it usually does not harmonize with experimental data. For example, the papers [14, 17] contain the data on the population of different types of predators and preys. The authors note that the preys' and predators' population extremum coincide in the dynamics - a Volterra model does not provide this result. The population change dynamics cannot be accurately described in the predator-prey system, because it is hard to simultaneously estimate the values of all constants in (1) for the real population [12-18]. The most accurate estimates are possible for birth rate c_1 and mortality rate c_2 and, consequently, for the fluctuations frequency [15]. According to statistics on species by square or individual areas [15], values $u_* = c_2 / a_{21}$ and $v_* = c_1 / a_{12}$ and, consequently, constants a_{12} and a_{21} can be estimated.

A PREY-PREDATOR MODEL IN THE LINEAR AREA

Examples of linear areas are pipelines, roadsides, forest clearings [13, 15, 17, 24]. To create models of populations diffusive in space, the continuum mechanics and physics methods are used [24, 28-31]. A predator-prey mathematical model (1) on the interval with the change of variables [24]

$$u = \frac{c_1}{a_{21}} u_1 ,\ v = \frac{c_1}{a_{12}} u_2 ,\ \tau = t / c_1$$

are represented by a system of two equations evolution [24, 31, 32]

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= D_1 \frac{\partial^2 u_1}{\partial x^2} + u_1 - u_1 u_2 , \\ \frac{\partial u_2}{\partial t} &= D_2 \frac{\partial^2 u_2}{\partial x^2} - \gamma u_2 + u_1 u_2 . \end{aligned} \tag{2}$$

In these equations x is a coordinate, t is time $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ is linear population density, D_1 and D_2 - parameters characterizing the species' mobility, $\gamma = c_2 / c_1$.

The value of functions $u_1 = u_1(t, x)$ and $u_2 = u_2(t, x)$ is set as initial conditions at the initial time:

at $t = 0$ $u_1(x) = u_{10}(x)$, $u_2(x) = u_{20}(x)$.

Two following options are considered as boundary conditions for a segment of l length:

$$\left.\frac{\partial u_1}{\partial x}\right|_{x=0} = \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0 ,\ \left.\frac{\partial u_2}{\partial x}\right|_{x=0} = \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 \tag{3}$$

and

$$u_1|_{x=0} = 0,\quad \left.\frac{\partial u_1}{\partial x}\right|_{x=l} = 0 ,\ u_2|_{x=0} = 0,\quad \left.\frac{\partial u_2}{\partial x}\right|_{x=l} = 0 . \tag{4}$$

The condition of bringing functions u_1 and u_2 to zero on a segment border corresponds to inability of population to exist at this point, and the condition of bringing derivatives $\partial u_1 / \partial x$ and $\partial u_2 / \partial x$ to zero (condition of environment filling [24]) allows the free growth of population.

Total population of preys ($M_1(t)$) and predators ($M_2(t)$) on the segment at the time point t are calculated by the formulas

$$M_1 = \int_0^l u_1(t,x)\,dx\,,\quad M_2 = \int_0^l u_2(t,x)\,dx\,.$$

STABILITY OF SOLUTIONS

The trivial solution $u_1 = 0$, $u_2 = 0$ satisfies the system of equations (2) at boundary conditions (3) or (4) in the stationary case. Disturbance of this equilibrium state is represented as [30-33] $u_1 = \delta u_1$, $u_2 = \delta u_2$, where δu_1 and δu_2 are small compared to the unit value: $0 \le \delta u_1 \ll 1$, $0 \le \delta u_2 \ll 1$. Then the equations (2) with accuracy of the second order infinitesimal [30, 31] are brought to:

$$\begin{aligned} \frac{\partial \delta u_1}{\partial t} &= D_1 \frac{\partial^2 \delta u_1}{\partial x^2} + \delta u_1, \\ \frac{\partial \delta u_2}{\partial t} &= D_2 \frac{\partial^2 \delta u_2}{\partial x^2} - \gamma \delta u_2. \end{aligned} \qquad (5)$$

The solution of the first equation satisfying the boundary conditions (3) is represented as a trigonometric series

$$\delta u_1 = \sum_{k=0}^{\infty} A_k(t) \cos k\pi \frac{x}{l}\,.$$

Hence, expansion coefficients have to satisfy the equations [30, 31]

$$\frac{dA_0}{dt} = A_0\,, \qquad \frac{dA_k}{dt} = -D_1 \left(\frac{k\pi}{l}\right)^2 A_k + A_k \quad k = 1, 2, \dots.$$

It is seen from the first equation that $A_0(t) = A_0(0)e^t$ will be an increasing time function. That is solution $u_1 = 0$ will be unstable.

The solution of the first equation in (5), satisfying the boundary conditions (4), is represented as a trigonometric series

$$\delta u_1 = \sum_{k=1}^{\infty} A_k(t) \sin k \frac{\pi}{2} \frac{x}{l}\,.$$

Hence, expansion coefficients have to satisfy the equations

$$\frac{dA_k}{dt} = \left(1 - D_1 \left(\frac{k\pi}{2l}\right)^2\right) A_k \quad k = 1, 2, \dots.$$

It follows that performing the inequality $\left(2l/\pi\right)^2 < D_1$ all coefficients A_k are to be the decreasing time functions and, accordingly, solution $u_1 = 0$ is stable. This means that the smallest population in the studied model goes extinct at high preys' mobility. A similar result was obtained in [31] for a single population.

$u_1 = \gamma$, $u_2 = 1$ satisfy equations (2) in stationary case for the boundary conditions (3). In the neighborhood of this solution, a solution of equations (2) is represented in the form [30, 31]

$$u_1 = \gamma + \delta u_1,\ u_2 = \gamma + \delta u_2,$$

where δu_1 and δu_2 are small, compared to the unit value: $|\delta u_1| \ll 1$, $|\delta u_2| \ll 1$. Then the equations (2) with accuracy of the second order infinitesimal are brought to

$$\frac{\partial \delta u_1}{\partial t} = D_1 \frac{\partial^2 \delta u_1}{\partial x^2} - \gamma \delta u_2,$$
$$\frac{\partial \delta u_2}{\partial t} = D_2 \frac{\partial^2 \delta u_2}{\partial x^2} + \delta u_1.$$

The solution of these equations is represented as a trigonometric series [30, 31]

$$u_1 = \sum_{k=0}^{\infty} A_k(t) \cos k\pi x/l,\ u_2 = \sum_{k=0}^{\infty} B_k \cos k\pi x/l$$

The expansion coefficients A_k and B_k have to satisfy the system of ordinary differential equations ($k = 0,1,2,\ldots$)

$$\frac{dA_k}{dt} = -D_1\left(\frac{k\pi}{l}\right)^2 A_k - \gamma B_k,$$
$$\frac{dB_k}{dt} = -D_2\left(\frac{k\pi}{l}\right)^2 B_k + A_k.$$

The Jacobian matrix eigenvalues of right-hand side of these equations satisfy quadratic equation

$$\lambda_k^2 + \left(D_1 + D_2\right)\left(\frac{k\pi}{l}\right)^2 \lambda_k + D_1 D_2 \left(\frac{k\pi}{l}\right)^4 + \gamma = 0.$$

At $k = 1,2,\ldots$ λ_k will have negative real parts, and $\lambda_0 = \pm i\sqrt{\gamma}$. That is all the coefficients A_k and B_k at $k = 1,2,\ldots$ will be the decreasing time functions, and coefficients A_0 and B_0 will change harmonically. That is, at small deviations from fixed solution $u_1 = \gamma$, $u_2 = 1$ periodic fluctuations will appear, and the solution tends to homogeneity through time by the spatial variable.

NUMERICAL SOLUTION

An analytical solution of the nonlinear equations (2) is considered to be impossible. Therefore, various methods of approximations of equations (2) or their solutions are used. The finite-difference approximation of the equations and variational methods based on solutions as a linear combination of analytic functions are most distributed [34-40]. The numerical solution of the equations (2), satisfying the boundary conditions (4), at the interval is being sought as a sum of trigonometric functions

$$u_1 = \sum_{k=1}^{n} A_k(t)\sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l},\ u_2 = \sum_{k=1}^{n} B_k(t)\sin\left(k\pi - \frac{\pi}{2}\right)\frac{x}{l}. \qquad (6)$$

The system of functions $\sin(k\pi - \pi/2)x/l$ $(k = 1, 2, ...)$ satisfies the boundary conditions (4), is full and orthogonal at the segment $[0, l]$. After substitution of expressions (6) into equations (2), the multiplication of the latter by $\sin(k\pi - \pi/2)x/l$ $(k = 1, 2, ...)$ and subsequent integration by the interval $[0, l]$, the system of ordinary differential equations is received for coefficients A_k and B_k $(k = 1, 2, ...)$

$$\begin{aligned} \frac{dA_k}{dt} &= -D_1\left(\frac{k\pi - \pi/2}{l}\right)^2 A_k + \frac{2}{l}\int_0^l \left(u_1 - u_1 u_2\right)\sin(k\pi - \pi/2)x/l\,dx, \\ \frac{dB_k}{dt} &= -D_2\left(\frac{k\pi - \pi/2}{l}\right)^2 B_k + \frac{2}{l}\int_0^l \left(-\gamma u_2 + u_1 u_2\right)\sin(k\pi - \pi/2)x/l\,dx. \end{aligned} \qquad (7)$$

For one expansion term $(n = 1)$ in (5) for the segment of single length $(l = 1)$ coefficients $A_1(t)$ and $B_1(t)$ satisfy the equations (7)

$$\begin{aligned} \frac{dA_1}{dt} &= \left(-D_1\left(\frac{\pi}{2}\right)^2 + 1 - \frac{8}{3\pi}B_1\right)A_1, \\ \frac{dB_1}{dt} &= \left(-D_2\left(\frac{\pi}{2}\right)^2 - \gamma + \frac{8}{3\pi}A_1\right)B_1. \end{aligned}$$

Non-trivial fixed point of these equations with values

$$A_1 = \frac{8}{3\pi}\left(\gamma + D_2\left(\frac{\pi}{2}\right)^2\right) \text{ and } B_1 = \frac{3\pi}{8}\left(1 - D_1\frac{\pi^2}{4}\right) \tag{8}$$

is implemented, if the inequality $D_1 < 4/\pi^2$ is performed.

The eigenvalues of the Jacobian matrix

$$J = \begin{pmatrix} 0 & -\left(\gamma + D_2\left(\frac{\pi}{2}\right)^2\right) \\ 1 - D_1\frac{\pi^2}{4} & 0 \end{pmatrix}$$

in this fixed point will be

$$\lambda_{1,2} = \pm i\omega,\ \omega = \sqrt{\left(D_2\left(\frac{\pi}{2}\right)^2 + \gamma\right)\left(1 - D_1\left(\frac{\pi}{2}\right)^2\right)}.$$

It follows from these relations in the first approximation, fluctuations occur in the system with frequency ω. The fluctuations frequency grows with the increased mobility of a predator (parameter D_2) and decreases with the increased mobility of a prey (parameter D_1). In this case, as it follows from (8), increased predators' mobility leads to an increase in the amplitude of preys' fluctuations. The amplitude of predator's fluctuations does not depend on its mobility in the first approximation.

Figure shows the dependence of the functions $M_1(t)$ and $M_2(t)$ from time to time. This result obtained for the case $n = 5$ in (2) with $\gamma = 0.7$, $D_1 = 0.01$ и $D_2 = 1$. In contrast to the model of Volterra (1) the solution of equations (2) tends to the stationary solution, the oscillations are damped.

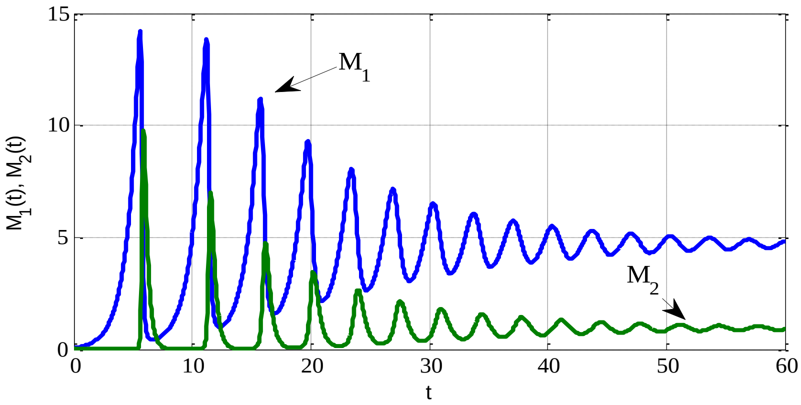

Figure: Changing functions $M_1(t)$ and $M_2(t)$ in time.

CONCLUSION

Consideration of the environment inhomogeneity in the Voltaire predator-prey mathematical model leads to results unavailable in the point model. The main of them: the total population depends on species' mobility both of predators and preys. Preys' population may go extinct in high species' mobility. The growth of predators' mobility decreases the period of fluctuations and increases the preys' number.

REFERENCES

[1] Almanza-Vasquez, E., Ortiz-Ortiz, R., Marin-Ramirez, A., 2015, "Dynamic Consequences of Lotka-Volterra Predation Model when the Area Inhabited by the Preys and Predators Decreases," Applied Mathematical Sciences, 9(40), pp. 1961-1970. http://dx.doi.org/10.12988/ams.2015.5172.

[2] Hofbauer, J., Kon, R., Saito, Y., 2008, "Qualitative permanence of Lotka-Volterra equations," J. Mathematical Biology, 57, pp. 863-881. DOI 10.1007/s00285-008-0192-0.

[3] Svirezhev, Y. M., 2008, "Nonlinearities in mathematical ecology: Phenomena and models. Would we live in Volterra's world?" Ecological Modelling, 216, pp. 89-101.

[4] Markov, A.V., Korotayev, A.V., 2007, "The dynamics of Phanerozoic marine animal diversity agrees with the hyperbolic growth model," Zhurnal Obshchei Biologii, 68, pp. 3-18.

[5] Andayani, P., Kusumawinahyu, W.N., 2015, "Global Stability Analysis on a Predator-Prey Model with Omnivores," Applied Mathematical Sciences, 9(36), pp. 1771-1782. http://dx.doi.org/10.12988/ams.2015.5120.

[6] Grigorieva, X., Malafeev, O., 2014, "A competitive many-period postman problem with varying parameters," Applied Mathematical Sciences, 8, pp. 7249-7258. doi: 10.12988/ams.2014.47591.

[7] Kolpak, E.P., Kabrits, S.A., Bubalo, V., 2015, "The follicle function and thyroid gland cancer," Biology and Medicine, 7, pp. 1-6, Article ID: BM060.15.

[8] Tewa, J.J., Djeumen, V.Y., Bowong, S., 2013, "Predator-Prey model with Holling response function of type II and SIS infectious disease," Applied Mathematical Modelling, 37, pp. 4825-4841.

[9] Balykina, Y.E., Kolpak, E.P., Kotina, E.D., 2014, "Mathematical model of thyroid function," Middle East Journal of Scientific Research, 19, pp. 429-433.

[10] Kolpak, E.P., Gorynya, E.V., 2016, "Mathematical models of ecological niches search," Applied Mathematical Sciences, 10(38), pp. 1907-1921. http://dx.doi.org/10.12988/ams.2016.64139.

[11] Pusawidjayanti, K., Suryanto1, A. and Wibowo, R.B.E., 2015, "Dynamics of a Predator-Prey Model Incorporating Prey Refuge, Predator Infection and Harvesting," Applied Mathematical Sciences, 9(76), pp. 3751 – 3760. http://dx.doi.org/10.12988/ams.2015.54340.

[12] Railkin, A.I., Dobretsov, S.V., 1994. "Effect of bacterial repellents and narcotizing substances on marine macrofouling," Biologiya Morya (Vladivostok), 20 (1), pp. 20-27.

[13] Sharapova, T.A., 2010, "Spatial structure of zooperiphyton in a small river (West Siberia)," Inland Water Biology, 3, pp. 149-154.

[14] Ivanter, E.V., Korosov, A.V., Yakimova, A.E., 2015, "Ecological and statistical analysis of long-term changes in the abundance of small mammals at the northern limit of the range (Northeastern Ladoga region)," Russian Journal of Ecology, 46, pp. 89-95.

[15] Okulova, N.M., Kataev, G.D., 2007, "Interrelationships in the system predator-prey in vertebrate animal communities at the lapland reserve," Zoologicheskii Zhurnal, 86(8), pp. 989-998.

[16] Mamontov, S.N., 2009, "Distribution of the bark beetle (Ips typographus) and its entomophages along tree trunk," Zoologicheskii Zhurnal, 88(9), pp. 1139-1145.

[17] Gilev, A.V., 2010, "Spatial distribution and scientific bases of conservation of red wood ants," Zoologicheskii Zhurnal, 89(12), pp. 1413-1420.

[18] McLeod, P., Martin, A.P., Richards, K.J., 2002, "Minimum length scale for growth – limited oceanic plankton distributions," Ecological Modeling, 158(1-2), pp. 111-120.

[19] Mindlin, Y.B., Kolpak, E.P., Gasratova, N.A., 2016, "Clusters in system of instruments of territorial development of the Russian Federation," International Review of Management and Marketing, 6(1), pp. 245-249.

[20] Shiryaev, D.V., Litvinenko, I.L., Rubtsova, N.,V., Kolpak, E.P., Blaginin, V.A., Zakharova, E.N., 2016, "Economic Clusters as a form of Self-organization of the Economic System," International Journal of Economics and Financial Issues, 6(1S), pp. 284-288.

[21] Grigorenko, O.V., Klyuchnikov, D.A., Gridchina, A.V., Litvinenko, I.L., Kolpak, E.P., 2016, "The Development of Russian-Chinese Relations: Prospects for Cooperation in Crisis," International Journal of Economics and Financial Issues, 6(1S), pp. 256-260.

[22] Chekunova, E.M., Yaronskaya, E.B., Yartseva, N.V., Averina, N.G., 2014, “New factors regulating magnesium chelatase in the green alga Chlamydomonas reinhardtii,” Russian Journal of Plant Physiology, 61(2), pp. 169-177. DOI: 10.1134/S1021443714020034.

[23] Nikulina, K.V., Chekunova, E.M., Rüdiger, W., Chunaev, A.S., 1997, “Genetic Analysis of Revertants of Chlorophyll b-Deficient Mutants of Chlamydomonas reinhardtii,” Russian Journal of Genetics, 33(5), pp. 474-479.

[24] Murray, J.D., 2002, Mathematical biology, Springer-Verlag, New York.

[25] Kareiva, P.M., 1983, “Local movement in herbivorous insects: applying a passive diffusion model to mark-recapture field experiments,” Oecologia, 57, pp. 322-327.

[26] Kot, M., 2001, Elements of mathematical ecology, Cambridge University Press.

[27] Chakraborty, M., Singh, D., Lucy, P., Ridland, 2007, “Predator-prey model with prey-taxis and diffusion,” Mathematical and Computer Modelling, 46, pp. 482-498.

[28] Balasuriya, S., Gottwald, G.A., 2010, ‘Wavespeed in reaction-diffusion systems, with applications to chemotaxis and population pressure,” Journal Mathematical Biology, 61, pp. 377-399, DOI 10.1007/s00285-009-0305-4.

[29] Dubey, B., Das, B., Hussain, J., 2001, “A predator-prey interaction model with self and cross-diffusion,” Ecological Modelling, 141, pp. 67-76.

[30] Kolpak, E.P., Frantsuzova, I.S., Alexandrovich, K.S., 2016, “A mathematical model of thyroid tumor,” Global Journal of Pure and Applied Mathematics, 12(1), pp. 55-66.

[31] Zhukova, I.V., Kolpak, E.P., Balykina, Yu.E., 2014, “Mathematical Model of Growing Tumor,” Applied Mathematical Sciences, 8(30), pp. 1455-1466. http://dx.doi.org/10.12988/ams.2014.4135.

[32] Kolpak, E.P., Ivanov, S.E., 2016, “On the three-dimensional Klein-Gordon equation with a cubic nonlinearity,” International Journal of Mathematical Analysis, 10(13), pp. 611-622. http://dx.doi.org/10.12988/ijma.2016.611.

[33] Andayani, P., Kusumawinahyu, W.M., 2015, “Global Stability Analysis on a Predator-Prey Model with Omnivores,” Applied Mathematical Sciences, 9(36), pp. 1771–1782. http://dx.doi.org/10.12988/ams.2015.5120.

[34] Zhabko, A.P., Ekimov, A.V., Smirnov, N.V., 2000, “Analysis of asymptotics of the solution of the integral system of convolution with a normed kernel,” Vestnik Sankt-Peterburgskogo Universiteta. Ser 1. Matematika Mekhanika Astronomiya, 1, pp. 27-34.

[35] Kolpak, E.P., Maltseva, L.S., Ivanov, S.E., 2015, "On the Stability of Compressed Plate," Contemporary Engineering Sciences, 8(20), pp. 933 – 942. http://dx.doi.org/10.12988/ces.2015.57213.

[36] Kabrits, S.A., Terent'ev, V.F., 1984, "Numerical solution of one-dimensional nonlinear statics problems for elastic rods and shells in the presence of rigid constraits," Soviet Applied Mechanics, 20(7), pp. 672-675. DOI: 10.1007/BF00891729.

[37] Kolpak, E.P., Maltseva, L.S., 2015, "Rubberlike Membranes at Inner Pressure," Contemporary Engineering Sciences, 8(36), pp. 1731 – 1742. http://dx.doi.org/10.12988/ces.2015.510289.

[38] Mickens, R.E., Washington, T.M., 2013, "NSFD discretizations of interacting population models satisfying conservation laws," Computers and Mathematics with Applications, 66, pp. 2307-2316. DOI: 10.1016/j.camwa.2013.06.011.

[39] Matrosov, A.V., 2014, "A numerical-analytical decomposition method in analyses of complex structures," 2014 International Conference on Computer Technologies in Physical and Engineering Applications, ICCTPEA 2014 - Proceedings, art. no. 6893305, pp. 104-105. doi: 10.1109/ICCTPEA. 2014.6893305.

[40] Goloskokov, D.P., Matrosov, A.V., 2016, "A superposition method in the analysis of an isotropic rectangle," Applied Mathematical Sciences, 10(54), pp. 2647-2660. http://dx.doi.org/10.12988/ams.2016.67211.