Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4395–4404

http://www.ripublication.com/gjpam.htm

On free field realizations of affine Lie superalgebras of type $B(m, n)^{(1)}$

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Abstract

We study free field realizations for the affine Lie superalgebras of type $B(m, n)^{(1)}$ arising from vertex representations of the affine general linear Lie superalgebras.

AMS subject classification: 17B10, 14L30, 16G20.
Keywords: Affine Lie superalgebras, orthosymplectic Lie superalgebras, free fields, vertex representations.

1. Introduction

The representation theory of affine Lie algebras has played crucial roles in some areas of mathematics and physics. When we apply the representation theory of affine Lie algebras to mathematical physics, we used to need some concrete realizations of affine Lie algebras in terms of operators defined on a tangible space. This kind of realizations first obtained successfully for the type $A_1^{(1)}$ by Lepowsky and Wilson [10]. In their study, Lepowsky and Wilson introduced the vertex operators that arose in string theory. In later, this vertex operator realization of $\widehat{\mathfrak{sl}(2)}$ was generalized to the affine Lie algebras of types ADE using twisted vertex operators [6]. We refer to [1, 11] for the untwisted vertex operator realizations of the affine Lie algebras of types ADE. Recently, the study of vertex operator realizations of affine Lie algebras has been extended to the case of affine Lie superalgebras [3, 7, 8].

In this paper, we shall particularly focus on vertex operator realizations of the affine Lie superalgebras of type $B(m.n)^{(1)}$. Recall that the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m+1|2n)$ provides an explicit example of the simple Lie superalgebra of type $B(m, n)$. In addition, $\widehat{\mathfrak{osp}(2m+1|2n)}$ are naturally embedded into the affine general linear Lie superalgebra $\widehat{\mathfrak{gl}(2m+1|2n)}$. So, in our study it is important to understand vertex operator constructions of the affine general linear Lie superalgebras. For this reason, we first introduce a free field realization of $\widehat{\mathfrak{gl}(M|N)}$. Then, we shall analyze the structure of $\mathfrak{osp}(2m+1|2n)$ to obtain a free realization of $\widehat{\mathfrak{osp}(2m+1|2n)}$. Namely, we shall find explicitly a basis of $\mathfrak{osp}(2m+1|2n)$ through the decompositions of $\mathfrak{osp}(2m+1|2n)$ into block matrices. Using our construction of a basis for $\mathfrak{osp}(2m+1|2n)$, we shall construct a free field representation of $\widehat{\mathfrak{osp}(2m+1|2n)}$ from a vertex representation of $\widehat{\mathfrak{gl}(2m+1|2n)}$.

2. Preliminaries

Orthosymplectic Lie superalgebras. Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a parity decomposition of a superspace V, where $\bar{0}$ and $\bar{1}$ stand for the elements of $\mathbb{Z}/2\mathbb{Z}$. Then, we say that an element a of V has parity $p(a)$ if $a \in V_{p(a)}$.

Suppose that $(\,|\,)$ is a nondegenerate bilinear form on V such that

$$\begin{aligned} &\left(V_{\bar{0}}|V_{\bar{1}}\right) = 0, \\ &\text{the restriction of } (\,|\,) \text{ to } V_{\bar{0}} \text{ is symmetric,} \\ &\text{the restriction of } (\,|\,) \text{ to } V_{\bar{1}} \text{ is skew-symmetric.} \end{aligned} \tag{1}$$

The orthosymplectic Lie superalgebra $\mathfrak{osp}\,(2m+1|2n)$ is a Lie superalgebra that preserves the bilinear form (1), where $dim V_{\bar{0}} = 2m+1$ and $dim V_{\bar{1}} = 2n$. Namely, for $\alpha \in \mathbb{Z}/2\mathbb{Z}$ we have

$$\begin{aligned} &\mathfrak{osp}\,(2m+1|2n)_\alpha \\ &= \left\{T \in \mathfrak{gl}\,(2m+1|2n)_\alpha \mid (T(x)|y) + (-1)^{\alpha p(x)}\,(x|T(y)) = 0 \ \ (x,\ y \in V)\right\}. \end{aligned} \tag{2}$$

Affine Lie superalgebras. Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$ with a non-degenerate even invariant bilinear form $(\,|\,)$. Then, the associated affine Lie superalgebra of $\mathfrak{g}$ is

$$\widehat{\mathfrak{g}} = \left(\mathfrak{g} \otimes \mathbb{C}\left[t, t^{-1}\right]\right) \oplus \mathbb{C}K$$

with the commutation relations: for $m, n \in \mathbb{Z}$,

$$[a(m), b(n)] = [a, b](m+n) + m\delta_{m+n,0}\,(a|b)\,K, \tag{3}$$

$$\left[K, \widehat{\mathfrak{g}}\right] = 0, \tag{4}$$

where $a, b \in \mathfrak{g}$ and $a(m)$ stands for $a \otimes t^m$.

Operator product expansions. Let V be a superspace. A *field* is a series of the form $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ with $a_{(n)} \in End(V)$ so that for all $v \in V$ one has $a_{(n)}(v) = 0$ for

$n >> 0$. We say that a field $a(z)$ has parity $p(a) \in \mathbb{Z}/2\mathbb{Z}$ if $a_{(n)} V_\alpha \subset V_{\alpha+p(a)}$ for all $\alpha \in \mathbb{Z}/2\mathbb{Z}$ and $n \in \mathbb{Z}$.

Given a Lie superalgebra $\mathfrak{g}$, we call a formal expression

$$a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathfrak{g}[[z, z^{-1}]]$$

a *formal distribution with values in* $\mathfrak{g}$. We shall always assume that all coefficients of a formal distribution $a(z)$ have the same parity, and this parity will be denoted by $p(a)$.

Let $a(z)$ and $b(w)$ be formal distributions with values in $\mathfrak{g}$. Then, we set

$$a(z)_- = \sum_{n\in\mathbb{Z}_{\geq 0}} a_{(n)} z^{-n-1}, \; a(z)_+ = \sum_{n\in\mathbb{Z}_{<0}} a_{(n)} z^{-n-1}$$

and

$$: a(z)b(w) := a(z)_+ b(w) + (-1)^{p(a)p(b)} b(w) a(z)_-.$$

For $n \in \mathbb{Z}_{\geq 0}$, we introduce the following n-th product $a(w)_{(n)} b(w)$ between two formal distributions:

$$\begin{aligned} a(w)_{(n)} b(w) &= Res_z \left[a(z), b(w)\right] (z-w)^n, \\ a(w)_{(-n-1)} b(w) &= : \left(\frac{1}{n!}\partial^n a(w)\right) b(w) : . \end{aligned} \tag{5}$$

For a given rational function $f(z, w)$ with poles only at $z = 0$, $w = 0$, and $|z| = |w|$, we denote by $i_{z,w} f(z, w)$ (resp. $i_{w,z} f(z, w)$) the power series expansion of $f(z, w)$ in the domain $|z| > |w|$ (resp. $|w| > |z|$).

The following theorem is well-known. We refer to [5] or [9] for the proofs.

Theorem 2.1. Let $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ and $b(z) = \sum_{n\in\mathbb{Z}} b_{(n)} z^{-n-1}$ be formal distributions with values in a Lie superalgebra $\mathfrak{g}$. Set $c^j(w) = a(w)_{(j)} b(w)$. Then, the following properties are equivalent.

1. $(z-w)^N \left[a(z), b(w)\right] = 0$ for $N >> 0$.
2. $a(z)b(w) = \sum_{j=0}^{N-1} \left(i_{z,w}\frac{1}{(z-w)^{j+1}}\right) c^j(w) + : a(z)b(w) :$ and
$$(-1)^{p(a)p(b)} b(w)a(z) = \sum_{j=0}^{N-1} \left(i_{w,z}\frac{1}{(z-w)^{j+1}}\right) c^j(w) + : a(z)b(w) : .$$
3. For $m, \; n \in \mathbb{Z}$, $\left[a_{(m)}, b_{(n)}\right] = \sum_{j=0}^{N-1} \binom{m}{j} c^j_{(m+n-j)}$, where $c^j(w) = \sum_{n\in\mathbb{Z}} c^j_{(n)} w^{-n-1}$.

For convenience, we often write the first relation of Theorem 2.1(2) as

$$a(z)b(w) \sim \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{(z-w)^{j+1}}, \tag{6}$$

and call (6) the *operator product expansion* (OPE).

3. Free field realizations of $\widehat{\mathfrak{gl}(M|N)}$

Let $A = A_{\bar{0}} \oplus A_{\bar{1}}$ be a superspace. Let $\{\varphi^1, \ldots, \varphi^N, \varphi^{1*}, \ldots, \varphi^{N*}\}$ and $\{\psi^1, \ldots, \psi^M, \psi^{1*}, \ldots, \psi^{M*}\}$ be bases of $A_{\bar{0}}$ and $A_{\bar{1}}$, respectively. Define a nondegenerate skew-supersymmetric (i.e., $(x|y) = -(-1)^{p(x)}(y|x)$) bilinear form $(\ |\ )$ on A as follows:

$$\begin{aligned} &(\psi^i|\psi^{j*}) = \delta_{ij},\ (\varphi^i|\varphi^{j*}) = -\delta_{ij},\\ &(\psi^i|\varphi^{j*}) = (\varphi^i|\psi^{j*}) = 0,\\ &(\psi^i|\psi^j) = (\varphi^i|\varphi^j) = (\psi^i|\varphi^j) = 0,\\ &(\psi^{i*}|\psi^{j*}) = (\varphi^{i*}|\varphi^{j*}) = (\psi^{i*}|\varphi^{j*}) = 0. \end{aligned} \tag{7}$$

Recall that the Clifford affinization of the superspace $(A, (\ |\ ))$ is a Lie superalgebra $C_A = A[t, t^{-1}] \oplus \mathbb{C}K$ with the bracket

$$[x_p, y_q] = (x \mid y)\, \delta_{p+q,0} K \text{ for } p,\ q \in \frac{1}{2} + \mathbb{Z}, \tag{8}$$

$$[C_A, K] = 0, \tag{9}$$

where $x,\ y \in A$ and x_p stands for $xt^{p-\frac{1}{2}}$.

Consider $\mathbb{C}$ as an 1-dimensional $A[t] \oplus \mathbb{C}K$-module via the following actions:

1. $A[t]$ acts trivially on $\mathbb{C}$.
2. K acts as identity on $\mathbb{C}$.

Then, we have the induced C_A-module

$$F_A = U(C_A) \otimes_{U(A[t] \oplus \mathbb{C}K)} \mathbb{C}, \tag{10}$$

where $U(C_A)$ means the universal enveloping algebra of C_A.

It is well-known that the C_A-module F_A has a unique vertex algebra structure with vacuum vector $|0\rangle = 1 \otimes 1$ and generated by the fields $\phi(z) = \sum_{k \in \mathbb{Z}} \phi_{k+\frac{1}{2}} z^{-k-1}$, where $\phi \in A$ and $\phi_{n+\frac{1}{2}}\ (= \phi_{(n)})$ acts on F_A by left multiplication of ϕt^n. We refer to [2, 5, 9] for more details of the theory of vertex algebras.

For the bases $\{\psi^i, \psi^{i*} \mid 1 \le i \le M\}$ and $\{\varphi^j, \varphi^{j*} \mid 1 \le j \le N\}$ of $A_{\bar{1}}$ and $A_{\bar{0}}$ respectively, we introduce the following fields:

$$\begin{aligned} a^{ij+}(z) &=: \psi^i(z)\psi^{j*}(z):, \qquad a^{ij-}(z) = :\varphi^i(z)\varphi^{j*}(z):, \\ E^{ij+}(z) &=: \psi^i(z)\varphi^{j*}(z):, \qquad E^{ij-}(z) = :\varphi^i(z)\psi^{j*}(z):. \end{aligned} \tag{11}$$

Through the following theorem, we see that the vertex algebra F_A is equipped with $\widehat{\mathfrak{gl}(M|N)}$-module structure.

Theorem 3.1. Let e_{ij} $(1 \le i,\ j \le M+N)$ be the standard basis of $\mathfrak{gl}(M|N)$ whose (i, j)-entry is 1 and the other entries are 0. For $T \in \mathfrak{gl}(M|N)$, set $T(z) = \sum_{n\in\mathbb{Z}} T_{(n)} z^{-n-1}$ with $T_{(n)} = T \otimes t^n$. Then, the linear map given by

$$\begin{aligned} &e_{ij}(z) \longrightarrow a^{ij+}(z)\ (1 \le i,\ j \le M), \\ &e_{i+m,j+m}(z) \longrightarrow a^{ij-}(z)\ (1 \le i,\ j \le N), \\ &e_{i,j+m}(z) \longrightarrow E^{ij+}(z)\ (1 \le i \le M,\ 1 \le j \le N), \\ &e_{i+m,j}(z) \longrightarrow E^{ij-}(z)\ (1 \le i \le N,\ 1 \le j \le M), \\ &K \longrightarrow 1 \end{aligned}$$

defines a representation of $\widehat{\mathfrak{gl}(M|N)}$ in the vertex algebra F_A.

Proof. According to Wick's theorem (see [5] for the details of Wick's theorem), we obtain the following OPEs

$$\begin{aligned} E^{ij\pm}(z)E^{kl\mp}(w) &\sim \frac{\delta_{jk}a^{il\pm}(w) + \delta_{il}a^{kj\mp}(w)}{z-w} \pm \frac{\delta_{il}\delta_{jk}}{(z-w)^2}, \\ a^{ij\pm}(z)E^{kl\pm}(w) &\sim \frac{\delta_{jk}E^{il\pm}(w)}{z-w}, \\ a^{ij\pm}(z)E^{kl\mp}(w) &\sim \frac{-\delta_{li}E^{kj\mp}(w)}{z-w}, \\ a^{ij\pm}(z)a^{kl\pm}(w) &\sim \frac{\delta_{jk}a^{il\pm}(w) - \delta_{il}a^{kj\pm}(w)}{z-w} \pm \frac{\delta_{il}\delta_{jk}}{(z-w)^2}, \\ a^{ij\pm}(z)a^{kl\mp}(w) &\sim 0. \end{aligned}$$

The result is now immediate from Theorem 2.1. ■

4. Free realization of $\widehat{\mathfrak{osp}(2m+1|2n)}$

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a superspace. Let $\{v^1, \dots, v^m, v^{1*}, \dots v^{m*}, v\}$ and $\{w^1, \dots, w^n, w^{1*}, \dots w^{n*}\}$ be bases of $V_{\bar{0}}$ and $V_{\bar{1}}$, respectively. Define a bilinear form $(\ |\ )$ on V as follows:

(12)

- $(v^i|v^{j*}) = (v^{j*}|v^i) = \delta_{ij}$ for $1 \le i \le m$ and $1 \le j \le m$,
- $(w^i|w^{j*}) = -(w^{j*}|w^i) = -\delta_{ij}$ for $1 \le i \le n$ and $1 \le j \le n$,
- $(v|v^i) = (v|v^{i*}) = (v|w^j) = (v|w^{j*}) = 0$ for $1 \le i \le m$ and $1 \le j \le n$,
- $(v|v) = 1$,
- $(v^i|v^j) = (w^i|w^j) = (v^i|w^j) = (w^j|v^i) = 0$,
- $(v^{i*}|v^{j*}) = (w^{i*}|w^{j*}) = (v^{i*}|w^{j*}) = (w^{j*}|v^{i*}) = 0$.

Then, (12) yields a nondegenerate supersymmetric bilinear form on the superspace V.

We now describe a matrix realization of $\mathfrak{osp}(2m+1|2n)$ over the superspace V with the bilinear form (12).

Let $\mathbf{F}$ be a $(2m+1+2n) \times (2m+1+2n)$ matrix $\left(\begin{array}{c|c} \mathbf{I_{2m+1}} & \mathbf{O} \\ \hline \mathbf{O} & -\mathbf{I_{2n}} \end{array}\right)$.

Let $\mathbf{C}$ be a $(2m+1+2n) \times (2m+1+2n)$ matrix $\left(\begin{array}{c|c} \mathbf{C_1} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{C_2} \end{array}\right)$, where $\mathbf{C}_1 = \left(\begin{array}{cc|c} \mathbf{O} & \mathbf{I_m} & 0 \\ \mathbf{I_m} & \mathbf{O} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$ is a $(2m+1) \times (2m+1)$ matrix and $\mathbf{C}_2 = \left(\begin{array}{c|c} \mathbf{O} & -\mathbf{I}_n \\ \hline \mathbf{I}_n & \mathbf{O} \end{array}\right)$ is a $2n \times 2n$ matrix.

Then, we obtain from (2) that

$$\mathfrak{osp}(2m+1|2n)_{\bar{0}} = \left\{\mathbf{a} \in \mathfrak{gl}(2m+1|2n)_{\bar{0}} \mid \mathbf{a}^t\mathbf{C} + \mathbf{C}\mathbf{a} = \mathbf{O}\right\} \tag{13}$$

and

$$\mathfrak{osp}(2m+1|2n)_{\bar{1}} = \left\{\mathbf{a} \in \mathfrak{gl}(2m+1|2n)_{\bar{1}} \mid \mathbf{F}\mathbf{a}^t\mathbf{C} + \mathbf{C}\mathbf{a} = \mathbf{O}\right\}. \tag{14}$$

In the following lemma, we present explicitly a basis of $\mathfrak{osp}(2m+1|2n)_\alpha$ for $\alpha \in \mathbb{Z}/2\mathbb{Z}$.

Lemma 4.1. Let e_{ij} $(1 \le i,\ j \le 2m+2n+1)$ be the standard basis of $\mathfrak{gl}(2m+1|2n)$ whose (i, j)-entry is 1 and the other entries are 0. Then,

1. if we set

$$
\begin{aligned}
B_1^1 &= \left\{e_{i,j} - e_{m+j,m+i} \mid 1 \le i \le m,\ 1 \le j \le m\right\},\\
B_2^1 &= \left\{e_{i,m+j} - e_{j,m+i} \mid 1 \le i < j \le m\right\},\\
B_3^1 &= \left\{e_{m+i,j} - e_{m+j,i} \mid 1 \le i < j \le m\right\},\\
B_4^1 &= \left\{e_{2m+1+i,2m+1+j} - e_{2m+n+1+j,2m+n+1+i} \mid 1 \le i \le n,\ 1 \le j \le n\right\},\\
B_5^1 &= \left\{e_{2m+1+i,2m+n+1+j} + e_{2m+1+j,2m+n+1+i} \mid 1 \le i < j \le n\right\},\\
B_6^1 &= \left\{e_{2m+n+1+i,2m+1+j} + e_{2m+n+1+j,2m+1+i} \mid 1 \le i < j \le n\right\},\\
B_7^1 &= \left\{e_{2m+1+i,2m+n+1+i} \mid 1 \le i \le n\right\},\\
B_8^1 &= \left\{e_{2m+n+1+i,2m+1+i} \mid 1 \le i \le n\right\},\\
B_9^1 &= \left\{e_{i,2m+1} - e_{2m+1,m+i} \mid 1 \le i \le m\right\},\ \text{and}\\
B_{10}^1 &= \left\{e_{m+i,2m+1} - e_{2m+1,i} \mid 1 \le i \le m\right\},
\end{aligned}
$$

then $\cup_{i=1}^{10} B_i^1$ forms a basis of $\mathfrak{osp}\,(2m+1|2n)_{\bar{0}}$.

2. if we set

$$
\begin{aligned}
B_1^2 &= \left\{-e_{i,2m+1+j} + e_{2m+n+1+j,m+i} \mid 1 \le i \le m,\ 1 \le j \le n\right\},\\
B_2^2 &= \left\{e_{m+i,2m+n+1+j} + e_{2m+1+j,i} \mid 1 \le i \le m,\ 1 \le j \le n\right\},\\
B_3^2 &= \left\{e_{i,2m+n+1+j} + e_{2m+1+j,m+i} \mid 1 \le i \le m,\ 1 \le j \le n\right\},\\
B_4^2 &= \left\{-e_{m+i,2m+1+j} + e_{2m+n+1+j,i} \mid 1 \le i \le m,\ 1 \le j \le n\right\},\\
B_5^2 &= \left\{e_{2m+1,2m+n+1+i} + e_{2m+1+i,2m+1} \mid 1 \le i \le n\right\}\ \text{and}\\
B_6^2 &= \left\{-e_{2m+1,2m+1+i} + e_{2m+n+1+i,2m+1} \mid 1 \le i \le n\right\},
\end{aligned}
$$

then $\cup_{i=1}^{6} B_i^2$ forms a basis of $\mathfrak{osp}\,(2m+1|2n)_{\bar{1}}$.

Proof. Let $\left(\begin{array}{c|c} \mathbf{A} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{B} \end{array}\right) \in \mathfrak{osp}\,(2m+1|2n)_{\bar{0}}$. Then, we obtain from (13) that

$$\mathbf{A}^t\mathbf{C}_1 + \mathbf{C}_1\mathbf{A} = \mathbf{O} \tag{15}$$

and

$$\mathbf{B}^t\mathbf{C}_2 + \mathbf{C}_2\mathbf{B} = \mathbf{O}. \tag{16}$$

Write the matrix $\mathbf{A}$ as

$$\left(\begin{array}{c|c|c} \mathbf{k} & \mathbf{l} & \mathbf{e} \\ \hline \mathbf{s} & \mathbf{t} & \mathbf{h} \\ \hline \mathbf{f} & \mathbf{g} & \mathbf{i} \end{array}\right).$$

Here,

1. $\mathbf{k}$, $\mathbf{l}$, $\mathbf{s}$ and $\mathbf{t}$ are $m \times m$ matrices.

2. $\mathbf{e}$ and $\mathbf{h}$ are $m \times 1$ matrices.
3. $\mathbf{f}$ and $\mathbf{g}$ are $1 \times m$ matrices.
4. $\mathbf{i}$ an 1×1 matrix.

Then, (15) implies that the block matrices of $\mathbf{A}$ satisfy

$$\mathbf{s}^t = -\mathbf{s},\ \mathbf{h} = -\mathbf{f}^t,\ \mathbf{k}^t = -\mathbf{t},\ \mathbf{l}^t = -\mathbf{l},\ \mathbf{e} = -\mathbf{g}^t,\ \mathbf{i} = \mathbf{o}. \tag{17}$$

Thus, by (17) we get the corresponding basis elements

$$e_{i,j} - e_{m+j,m+i}\ (1 \le i \le m,\ 1 \le j \le m)\,,$$
$$e_{i,m+j} - e_{j,m+i}\ (1 \le i < j \le m)\,,$$
$$e_{m+i,j} - e_{m+j,i}\ (1 \le i < j \le m)\,,$$
$$e_{i,2m+1} - e_{2m+1,m+i}\ (1 \le i \le m)\,,$$

and

$$e_{m+i,2m+1} - e_{2m+1,i}\ (1 \le i \le m)\,.$$

Similarly, we set

$$B = \left(\begin{array}{c|c} \mathbf{k} & \mathbf{l} \\ \hline \mathbf{s} & \mathbf{t} \end{array}\right),$$

where $\mathbf{k}$, $\mathbf{l}$, $\mathbf{s}$ and $\mathbf{t}$ are $n \times n$ matrices.

Then, (16) gives rise to $\mathbf{s} = \mathbf{s}^t$, $\mathbf{l} = \mathbf{l}^t$ and $\mathbf{t} = -\mathbf{k}^t$. So, we obtain the corresponding basis elements

$$e_{2m+n+1+i,2m+1+j} + e_{2m+n+1+j,2m+1+i}\ (1 \le i < j \le n)\,,$$
$$e_{2m+1+i,2m+n+1+j} + e_{2m+1+j,2m+n+1+i}\ (1 \le i < j \le n)\,,$$
$$e_{2m+1+i,2m+1+j} - e_{2m+n+1+j,2m+n+1+i}\ (1 \le i \le n,\ 1 \le j \le n)\,,$$
$$e_{2m+1+i,2m+n+1+i}\ (1 \le i \le n)\,,$$

and

$$e_{2m+n+1+i,2m+1+i}\ (1 \le i \le n)\,.$$

The first statement now follows.

Let us now assume that $\left(\begin{array}{c|c} \mathbf{O} & \mathbf{A} \\ \hline \mathbf{B} & \mathbf{O} \end{array}\right) \in \mathfrak{osp}\,(2m+1|2n)_{\bar{1}}$. Then, (14) implies

$$\mathbf{B} = \mathbf{C}_2^{-1}\mathbf{A}^t\mathbf{C}_1. \tag{18}$$

By (18), we see that $\mathbf{A}$ completely determines the block $\mathbf{B}$. Hence, it is enough to consider a basis of the space of all possible block matrices $\mathbf{A}$. We shall find the corresponding block matrix $\mathbf{B}$ from (18). In more details, if we fix a basis

$$e_{i,2m+1+j}\ (1 \le i \le m,\ 1 \le j \le n)\,,\ e_{i,2m+n+1+j}\ (1 \le i \le m,\ 1 \le j \le n)\,,$$

$$e_{m+i,2m+1+j}\ (1\le i\le m,\ 1\le j\le n)\,,\ e_{m+i,2m+n+1+j}\ (1\le i\le m,\ 1\le j\le n)$$

and

$$e_{2m+1,2m+n+1+i}\ (1\le i\le n)\,,\ e_{2m+1,2m+1+i}\ (1\le i\le n)$$

for the space of the block matrices **A**, then due to (18) we obtain the corresponding matrix respectively

$$-e_{2m+n+1+j,m+i}\ (1\le i\le m,\ 1\le j\le n)\,,$$

$$e_{2m+1+j,m+i}\ (1\le i\le m,\ 1\le j\le n)\,,$$

$$-e_{2m+n+1+j,i}\ (1\le i\le m,\ 1\le j\le n)\,,$$

$$e_{2m+1+j,i}\ (1\le i\le m,\ 1\le j\le n)\,,$$

$$e_{2m+1+i,2m+1}\ (1\le i\le n)\,,$$

and

$$-e_{2m+n+1+i,2m+1}\ (1\le i\le n)\,.$$

The second statement is now immediate. ■

Theorem 4.2. Let $A=A_{\bar 0}\oplus A_{\bar 1}$ be a superspace. Assume that $\{\varphi^1,\ldots,\varphi^{2n},\varphi^{1*},\cdots\varphi^{2n*}\}$ and $\{\psi^1,\ldots,\psi^{2m+1},\psi^{1*},\ldots\psi^{2m+1*}\}$ are bases of $A_{\bar 0}$ and $A_{\bar 1}$, respectively. Let F_A be the associated vertex algebra of A defined as in (10). Then, the linear map given by (19) defines a representation of $\widehat{\mathfrak{osp}(2m+1|2n)}$ in the vertex algebra F_A.

$$\begin{aligned}
\left(e_{ij}-e_{m+j,m+i}\right)(z) &\longrightarrow a^{i,j+}(z)-a^{m+j,m+i+}(z),\\
\left(e_{i,m+j}-e_{j,m+i}\right)(z) &\longrightarrow a^{i,m+j+}(z)-a^{j,m+i+}(z),\\
\left(e_{m+i,j}-e_{m+j,i}\right)(z) &\longrightarrow a^{m+i,j+}(z)-a^{m+j,i+}(z),\\
\left(e_{2m+1+i,2m+1+j}-e_{2m+n+1+j,2m+n+1+i}\right)(z) &\longrightarrow a^{2m+1+i,2m+1+j-}(z)-a^{2m+n+1+j,2m+n+1+i-}(z),\\
\left(e_{2m+1+i,2m+n+1+j}+e_{2m+1+j,2m+n+1+i}\right)(z) &\longrightarrow a^{2m+1+i,2m+n+1+j-}(z)+a^{2m+1+j,2m+n+1+i-}(z),\\
\left(e_{2m+n+1+i,2m+1+j}+e_{2m+n+1+j,2m+1+i}\right)(z) &\longrightarrow a^{2m+n+1+i,2m+1+j-}(z)+a^{2m+n+1+j,2m+1+i-}(z),\\
e_{2m+1+i,2m+n+1+i}(z) &\longrightarrow a^{2m+1+i,2m+n+1+i-}(z),\\
e_{2m+n+1+i,2m+1+i}(z) &\longrightarrow a^{2m+n+1+i,2m+1+i-}(z),\\
\left(e_{i,2m+1}-e_{2m+i,m+i}\right)(z) &\longrightarrow a^{i,2m+1+}(z)-a^{2m+i,m+i}(z),\\
\left(e_{m+i,2m+1}-e_{2m+1,i}\right)(z) &\longrightarrow a^{m+i,2m+1+}(z)-a^{2m+1,i+}(z),\\
\left(-e_{i,2m+1+j}+e_{2m+n+1+j,m+i}\right)(z) &\longrightarrow -E^{i,2m+1+j+}(z)+E^{2m+n+1+j,m+i-}(z),\\
\left(e_{m+i,2m+n+1+j}+e_{2m+1+j,i}\right)(z) &\longrightarrow E^{m+i,2m+n+1+j+}(z)+E^{2m+1+j,i-}(z),\\
\left(e_{i,2m+n+1+j}+e_{2m+1+j,m+i}\right)(z) &\longrightarrow E^{i,2m+n+1+j+}(z)+E^{2m+1+j,m+i-}(z),\\
\left(-e_{m+i,2m+1+j}+e_{2m+n+1+j,i}\right)(z) &\longrightarrow -E^{m+i,2m+1+j+}(z)+E^{2m+n+1+j,i-}(z),\\
\left(e_{2m+1,2m+n+1+i}+e_{2m+1+i,2m+1}\right)(z) &\longrightarrow E^{2m+1,2m+n+1+i+}(z)+E^{2m+1+i,2m+1-}(z),\\
\left(-e_{2m+1,2m+1+i}+e_{2m+n+1+i,2m+1}\right)(z) &\longrightarrow -E^{2m+1,2m+1+i+}(z)+E^{2m+n+1+i,2m+1-}(z).
\end{aligned}\tag{19}$$

Proof. By Lemma 4.1, we can extend (19) to the linear map defined over the whole of $\widehat{\mathfrak{osp}\,(2m+1|2n)}$.

Let $\psi : \widehat{\mathfrak{gl}\,(2m+1|2n)} \longrightarrow End\,(F_A)$ be the representation obtained from Theorem 3.1. Then, due to the construction of (19), we see that the linear map from $\widehat{\mathfrak{osp}\,(2m+1|2n)}$ to $End\,(F_A)$ arising from (19) is just the restriction of ψ to the subspace $\widehat{\mathfrak{osp}\,(2m+1|2n)}$. The result now follows. ■

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