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Some identities of symmetry for the degenerate q-Bernoulli polynomials under symmetry group of degree n

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Abstract

Recently, Kim-Kim Introduced some interesting identities of symmetry for q-Bernoulli polynomials under symmetry group of degree n. In this paper, we study the degenerate q-Euler polynomials and derive some identities of symmetry for these polynomials arising from the p-adic q-integral on $\mathbb{Z}_p$.

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1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. Let $q \in \mathbb{C}_p$ be an indeterminate such that $|1-q|_p < p^{-\frac{1}{p-1}}$. The q-analogue of the number x is defined by $[x]_q = \frac{1-q^x}{1-q}$. Let $f(x)$ be uniformly differentiable function on $\mathbb{Z}_p$. Then the p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim to be

$$\begin{aligned} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_q(x+p^N\mathbb{Z}_p) \\ &= \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad \text{(see [13])}. \end{aligned} \tag{1}$$

In [1], L. Carlitz considered q-analogue of Bernoulli numbers which are given by recurrence relation to be

$$\beta_{0,q} = 1, \quad q(q\beta_q+1)^n - \beta_{n,q} = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases} \tag{2}$$

with the usual convention about replacing β_q^n by $\beta_{n,q}$. He defined q-Bernoulli polynomials as

$$\beta_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} \beta_{l,q} [x]_q^{n-l}, \quad \text{(see [1, 13])}. \tag{3}$$

In [13], Kim proved that the Carlitz's q-Bernoulli polynomials are represented as the p-adic q-integral on $\mathbb{Z}_p$ which are given by

$$\int_{\mathbb{Z}_p} [x+y]_q^n d\mu_q(y) = \beta_{n,q}(x), \quad (n \geq 0). \tag{4}$$

When $x = 0$, $\beta_{n,q} = \beta_{n,q}(0)$ are the Carlitz q-Bernoulli numbers.

In [2], L. Carlitz also introduced the degenerate Bernoulli polynomials which are given by the generating function to be

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B^{*}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{5}$$

Note that $\lim B^{*}_{n,\lambda}(x) = B^{*}_{n}(x)$, where $B_n(x)$ are ordinary Bernoulli polynomials (see [1-10]). When $x = 0$, $B^{*}_{n,\lambda} = B^{*}_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers. Recently, Kim-Kim introduced (fully) degenerate Bernoulli polynomials which are derived from the p-adic invariant integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+y}{\lambda}}d\mu_q(x) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1}(1+\lambda t)^{\frac{x}{\lambda}} \quad \text{(see [7])}, \tag{6}$$

where $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \le 1$, $|t|_p < p^{-\frac{1}{p-1}}$, and

$$\lim_{q\to 1}\int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \lim_{N\to\infty}\frac{1}{p^N}\sum_{x=0}^{p^N-1} f(x).$$

The (fully) degenerate Bernoulli polynomials are defined by the generating function to be

$$\frac{\log(1+\lambda t)}{\lambda(1+\lambda t)^{\frac{1}{\lambda}}-\lambda}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} B_{n,\lambda}(x)\frac{t^n}{n!} \quad \text{(see [7])}. \tag{7}$$

Note that Kim's degenerate Bernoulli polynomials are slightly different from the Carlitz's degenerate Bernoulli polynomials.

From (6) and (7), we note that

$$\lambda^n \int_{\mathbb{Z}_p}\left(\frac{x+y}{\lambda}\right)_n d\mu_1(x) = B_{n,\lambda}(x) \quad (n \ge 0), \tag{8}$$

where $(x)_0 = 1$, $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \ge 1)$.

In [16], Kim considered degenerate q-Bernoulli polynomials which are given by the generating function to be

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q}d\mu_q(y) = \sum_{n=0}^{\infty}\beta_{n,\lambda,q}(x)\frac{t^n}{n!}. \tag{9}$$

When $x = 0$, $\beta_{n,\lambda,q} = \beta_{n,\lambda,q}(0)$ are called (fully) degenerate q-Bernoulli numbers. Note that $\lim_{\lambda\to 0}\beta_{n,\lambda,q}(x) = \beta_{n,q}(x)$, $(n \ge 0)$.

In this paper, we give some identities of symmetry for the degenerate q-Bernoulli polynomials under symmetry group of degree n arising from the p-adic q-integral on $\mathbb{Z}_p$.

2. Identities of symmetry for the degenerate q-Bernoulli polynomials

We assume that $\lambda, t \in \mathbb{C}_p$ with $|\lambda|_p \leq 1$, $|t|_p < p^{-\frac{1}{p-1}}$. In this section, let $w_1, w_2, \ldots, w_n$ be positive integers. For $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}\left[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q} d\mu_{q^{w_1 w_2 \cdots w_{n-1}}}(y) \\
&= \lim_{N\to\infty} \frac{1}{[w_n p^N]_{q^{w_1\cdots w_{n-1}}}} \\
&\quad \times \sum_{y=0}^{w_n p^N - 1} (1+\lambda t)^{\frac{1}{\lambda}\left[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q} q^{w_1 w_2 \cdots w_{n-1} y} \\
&= \lim_{N\to\infty} \frac{1}{[w_n p^N]_{q^{w_1\cdots w_{n-1}}}} \sum_{k_n=0}^{w_n-1} \sum_{y=0}^{p^N-1} q^{w_1 w_2 \cdots w_{n-1}(k_n + w_n y)} \\
&\quad \times (1+\lambda t)^{\frac{1}{\lambda}\left[\left(\sum_{j=1}^{n-1} w_j\right)(k_n + w_n y) + \sum_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q}.
\end{aligned} \tag{10}$$

From (10), we note that

$$\begin{aligned}
&\frac{1}{[w_1 \cdots w_{n-1}]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} q^{w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j} \\
&\quad \times \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}\left[w_1 w_2 \cdots w_{n-1} y + w_1 \cdots w_n x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q} d\mu_{q^{w_1 w_2 \cdots w_{n-1}}}(y) \\
&= \lim_{N\to\infty} \frac{1}{[w_1 \cdots w_n p^N]_q} \prod_{l=1}^{n-1} \sum_{k_l=0}^{w_l-1} \sum_{k_n=0}^{w_n-1} \sum_{y=0}^{p^N-1} q^{w_1 w_2 \cdots w_{n-1}(k_n + w_n y) + \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j w_n} \\
&\quad \times (1+\lambda t)^{\frac{1}{\lambda}\left[\left(\sum_{j=1}^{n-1} w_j\right)(k_n + w_n y) + \sum_{j=1}^{n} w_j x + w_n \sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right) k_j\right]_q}.
\end{aligned} \tag{11}$$

It is easy to show that (11) is invariant under any permutation in the symmetry group of degree n. Therefore, by (11), we obtain the following theorem.

Theorem 2.1. Let $w_1, w_2, \ldots, w_n$ be positive integers . Then, the following expressions

$$\frac{1}{[w_{\sigma(1)}\cdots w_{\sigma(n-1)}]_q}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1} q^{w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j}$$
$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}y+\sum_{j=1}^{n} w_j x+w_{\sigma(n)}\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q} d\mu_{q^{w_{\sigma(1)}w_{\sigma(2)}\cdots w_{\sigma(n-1)}}}(y)$$

are the same for any permutation σ in the symmetry group of order n.

It is not difficult to show that

$$\begin{aligned}&\left[w_1w_2\cdots w_{n-1}y+w_1w_2\cdots w_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q\\ &=[w_1w_2\cdots w_{n-1}]_q\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1w_2\cdots w_{n-1}}}.\end{aligned} \tag{12}$$

From (12), we note that

$$\begin{aligned}&\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}\left[w_1\cdots w_{n-1}y+w_1\cdots w_n x+w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j\right]_q} d\mu_{q^{w_1w_2\cdots w_{n-1}}}(y)\\
&=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{[w_1\cdots w_{n-1}]_q}{\lambda}\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1\cdots w_{n-1}}}} d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&=\int_{\mathbb{Z}_p}\left(1+\frac{\lambda}{[w_1\cdots w_{n-1}]_q}[w_1\cdots w_{n-1}]_q t\right)^{\frac{[w_1\cdots w_{n-1}]_q}{\lambda}\left[y+w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right]_{q^{w_1\cdots w_{n-1}}}}\\
&\times d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{m=0}^{\infty}[w_1\cdots w_{n-1}]_q^m\beta_{m,\frac{\lambda}{[w_1\cdots w_{n-1}]_q},q^{w_1\cdots w_{n-1}}}\left(w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right)\frac{t^n}{n!}.\end{aligned} \tag{13}$$

Therefore, by Theorem 2.1 and (13), we obtain the following theorem.

Theorem 2.2. For $m\geq 0$, $w_1, w_2, \cdots, w_n \in \mathbb{N}$, the following expressions

$$[w_{\sigma(1)}\cdots w_{\sigma(n-1)}]_q^{m-1}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_{\sigma(l)}-1} q^{\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_{\sigma(i)}\right)k_j w_{\sigma(n)}}$$
$$\times\beta_{m,\frac{\lambda}{[w_{\sigma(1)}\cdots w_{\sigma(n-1)}]_q},q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x+\frac{w_{\sigma(n)}}{w_{\sigma(1)}}k_1+\cdots+\frac{w_{\sigma(n)}}{w_{\sigma(n-1)}}k_{n-1}\right)$$

are the same for any permutation σ in the symmetry group of order n.

From (9), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}\beta_{n,\lambda,q}(x)\frac{t^n}{n!} &= \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}[x+y]_q}d\mu_q(y)\\ &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{[x+y]_q}{\lambda}}{n}d\mu_q(x)\lambda^n t^n \\ &= \sum_{n=0}^{\infty}\lambda^n\int_{\mathbb{Z}_p}\left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_q(x)\frac{t^n}{n!}.\end{aligned}\tag{14}$$

By comparing the coefficients on the both sides of (14), we get

$$\begin{aligned}\beta_{n,\lambda,q} &= \lambda^n\int_{\mathbb{Z}_p}\left(\frac{[x+y]_q}{\lambda}\right)_n d\mu_q(x)\\ &= \lambda^n\sum_{m=0}^{n}S_1(n,m)\lambda^{-m}\int_{\mathbb{Z}_p}[x+y]_q^m d\mu_q(y)\\ &= \sum_{m=0}^{n}S_1(n,m)\lambda^{n-m}\beta_{m,q}(x).\end{aligned}\tag{15}$$

where $\beta_{m,q}(x)$ are called Carlitz's q-Bernoulli polynomials.

Now, we observe that

$$\begin{aligned}&\Big[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\Big]_{q^{w_1\cdots w_{n-1}}}\\ &= \frac{[w_n]_q}{[w_1\cdots w_{n-1}]_q}\Big[\sum_{j=1}^{n-1}\big(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\big)k_j\Big]_{q^{w_n}} + q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1}w_i\right)k_j}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}.\end{aligned}\tag{16}$$

By (15), we get

$$\begin{aligned}
&\beta_{m,\frac{\lambda}{[w_1\cdots w_{n-1}]_q},q^{w_1\cdots w_{n-1}}}\left(w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right)\\
&=\left(\frac{\lambda}{[w_1\cdots w_{n-1}]_q}\right)^m\int_{\mathbb{Z}_p}\left(\left(\frac{\lambda}{[w_1\cdots w_{n-1}]_q}\right)^{-1}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_q^l\right)d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&=\left(\frac{\lambda}{[w_1\cdots w_{n-1}]_q}\right)^m\sum_{l=0}^{m}S_1(m,l)[w_1\cdots w_{n-1}]_q^l\lambda^{-l}\\
&\quad\times\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_q^l d\mu_{q^{w_1\cdots w_{n-1}}}(y).
\end{aligned}\tag{17}$$

From (16), we can derive the following equation:

$$\begin{aligned}
&\int_{\mathbb{Z}_p}\left[y+w_nx+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right]_q^l d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{s=0}^{l}\binom{l}{s}\left(\frac{[w_n]_q}{[w_1\cdots w_{n-1}]_q}\right)^{l-s}\left[\sum_{j=1}^{n-1}\Big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\Big)k_j\right]_{q^{w_n}}^{l-s}q^{w_ns\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\quad\times\int_{\mathbb{Z}_p}[y+w_nx]_{q^{w_1\cdots w_{n-1}}}^s d\mu_{q^{w_1\cdots w_{n-1}}}(y)\\
&=\sum_{s=0}^{l}\binom{l}{s}\left(\frac{[w_n]_q}{[w_1\cdots w_{n-1}]_q}\right)^{l-s}\left[\sum_{j=1}^{n-1}\Big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\Big)k_j\right]_{q^{w_n}}^{l-s}q^{w_ns\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\\
&\times\beta_{s,q^{w_1\cdots w_{n-1}}}\big(w_nx\big).
\end{aligned}\tag{18}$$

By (17) and (18), we get

$$\begin{aligned}
&\beta_{m,\frac{\lambda}{[w_1\cdots w_{n-1}]_q},q^{w_1\cdots w_{n-1}}}\left(w_nx+\frac{w_n}{w_1}k_1+\cdots+\frac{w_n}{w_{n-1}}k_{n-1}\right)\\
&=\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[w_1\cdots w_{n-1}]_q^{s-m}[w_n]_q^{p-s}\left[\sum_{j=1}^{n-1}\Big(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\Big)k_j\right]_{q^{w_n}}^{p-s}\\
&\quad\times q^{w_ns\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\i\neq j}}^{n-1}w_i\right)k_j}\beta_{s,q^{w_1\cdots w_{n-1}}}\big(w_nx\big).
\end{aligned}\tag{19}$$

From (19), we note that

$$
\begin{aligned}
&[w_1\cdots w_{n-1}]_q^{m-1}\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\\
&\quad\times \beta_{m,\frac{\lambda}{[w_1\cdots w_{n-1}]_q},q^{w_1\cdots w_{n-1}}}\left(w_n x+w_n\sum_{j=1}^{n-1}\frac{k_j}{w_j}\right)\\
&=\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1}\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[w_1\cdots w_{n-1}]_q^{s-1}[w_n]_q^{p-s}\left[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\right]_{q^{w_n}}^{p-s}\\
&\quad\times q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\beta_{s,q^{w_1\cdots w_{n-1}}}\left(w_n x\right)\\
&=\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[w_1\cdots w_{n-1}]_q^{s-1}[w_n]_q^{p-s}\beta_{s,q^{w_1\cdots w_{n-1}}}\left(w_n x\right)\\
&\quad\times\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{(s+1)w_n\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\left[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\right]_{q^{w_n}}^{p-s}\\
&=\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[w_1\cdots w_{n-1}]_q^{s-1}[w_n]_q^{p-s}\beta_{s,q^{w_1\cdots w_{n-1}}}\left(w_n x\right)\\
&\quad\times K_{n,q^{w_n}}(w_1,\cdots w_{n-1}|p-s,s),
\end{aligned}
\tag{20}
$$

where

$$
K_{n,q}(w_1,\cdots w_{n-1}|i,t)=\prod_{l=1}^{n-1}\sum_{k_l=0}^{w_l-1} q^{(t+1)\sum_{j=1}^{n-1}\left(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i\right)k_j}\left[\sum_{j=1}^{n-1}(\prod_{\substack{i=1\\ i\neq j}}^{n-1} w_i)k_j\right]_q^{i}. \tag{21}
$$

Therefore, by (20) and (21), we obtain the following theorem.

Theorem 2.3. Let $m \geq 0$ and $w_1, w_2, \ldots, w_n\mathbb{N}$, Then the following expressions

$$
\begin{aligned}
&\sum_{p=0}^{m}\sum_{s=0}^{p}\binom{p}{s}S_1(m,p)\lambda^{m-p}[w_{\sigma(1)}\cdots w_{\sigma(n-1)}]_q^{s-1}[w_{\sigma(n)}]_q^{p-s}\beta_{s,q^{w_{\sigma(1)}\cdots w_{\sigma(n-1)}}}\left(w_{\sigma(n)}x\right)\\
&\times K_{n,q^{w_{\sigma(n)}}}(w_{\sigma(1)},\ldots,w_{\sigma(n-1)}|p-s,s)
\end{aligned}
$$

are the same for any permutation σ in the symmetry group of order n.

Note that some identities of Bernoulli and Euler polynomials are studied by several authors (see [1-19]).

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