Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4363–4383

http://www.ripublication.com/gjpam.htm

Identities of symmetry for degenerate Bernoulli polynomials and generalized falling factorial sums

Tae Kyun Kim

*Department of Mathematics,
Kwangwoon University, Seoul 139-701, Republic of Korea.*

Dae San Kim

*Hanrimwon, Kwangwoon University,
Seoul 139-701, Republic of Korea.
Institute of Natural Sciences,
Far Eastern Federal University,
Vladivostok 690950, Russia.*

Dmitry V. Dolgy

*Department of Mathematics, Sogang University,
Seoul 121-742, Republic of Korea.
E-mail: dskim@sogang.ac.kr*

Hyuck-In Kwon

*Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea.*

Jong-Jin Seo

*Department of Mathematics Applied Mathematics,
Pukyong National University,
Busan, Republic of Korea.
Department of Mathematics,
Kwangwoon University,
Seoul 139-701, Republic of Korea.*

Abstract

Eight basic identities of symmetry in three variables, which are related to degenerate Bernoulli polynomials and generalized falling factorial sums, are derived. These are the degenerate versions of the symmetric identities in three variables obtained in a previous paper. The derivations of identities are based on the p-adic integral expression of the generating function for the modified degenerate Bernoulli polynomials and the quotient of integrals that can be expressed as the exponential generating function for the generalized falling factorial sums. Those eight basic identities and most of their corollaries are new, since there have been results only about identities of symmetry in two variables.

AMS subject classification:
Keywords: degenerate Bernoulli polynomial, modified degenerate Bernoulli polynomial, generalized falling factorial sum, Volkenborn integral, identities of symmetry.

1. Introduction and preliminaries

Let p be a fixed prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ denotes the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. For a uniformly differentiable function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the Volkenborn integral of f is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(z)d\mu(z) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{j=0}^{p^N-1} f(j).$$

Then it is easy to see that

$$I(f_1) = I(f) + f'(0), \tag{1.1}$$

where $f_1(z) = f(z+1)$.

Let $|\cdot|_p$ be the normalized absolute value of $\mathbb{C}_p$, normalized by $|p|_p = \frac{1}{p}$. Throughout this paper, assume that $\lambda, t \in \mathbb{C}_p$ satisfy

$$0 < |\lambda|_p \leq 1, |t|_p < p^{-\frac{1}{p-1}}.$$

Then, as $|\lambda t|_p < p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p = |\lambda t|_p$ and hence $\left|\frac{1}{\lambda}\log(1+\lambda t)\right|_p = |t|_p < p^{-\frac{1}{p-1}}$. Thus $f(z) = (1+\lambda t)^{\frac{z}{\lambda}} = e^{\frac{z}{\lambda}\log(1+\lambda t)}$ is a well-defined analytic function on $\mathbb{Z}_p$. Applying (1.1) to this f, we get the p-adic integral representation of the generating function for this *modified degenerate Bernoulli numbers* $\tilde{\beta}_n(\lambda)$:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{z}{\lambda}} \, d\mu(t) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda) \frac{t^n}{n!}. \tag{1.2}$$

Thus from (1.2) we have the follwoing p-adic integral representation of the generating function for the *modified degenerate Bernoulli polynomais* $\tilde{\beta}_n(\lambda, x)$:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{x+z}{\lambda}} \, d\mu(z) = \frac{\log(1+\lambda t)^{\frac{1}{\lambda}}}{(1+\lambda t)^{\frac{1}{\lambda}}-1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda, x) \frac{t^n}{n!}. \tag{1.3}$$

Note here that $\tilde{\beta}_n(\lambda) = \tilde{\beta}_n(\lambda, 0)$.

At this ponit, we must remind the reader that the modified degenerate Bernoulli polynomials (resp. numbers) $\tilde{\beta}_n(\lambda, x)$ (resp. $\tilde{\beta}_n(\lambda)$) are different from the Carlitz degenerate Bernoulli polynomials (resp. numbers) $\beta_n(\lambda, x)$ (resp. $\beta_n(\lambda)$). Indeed,

$$\frac{t}{(1+\lambda t)^{\frac{1}{\lambda}}-1} (1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!} \tag{1.4}$$

with $\beta_n(\lambda) = \beta_n(\lambda, 0)$. However,

$$\lim_{\lambda \to 0} \tilde{\beta}_n(\lambda, x) = \lim_{\lambda \to 0} \beta_n(\lambda, x) = B_n(x), \tag{1.5}$$

where $B_n(x)$ is the ordinary Bernoulli polynomial.

In addition, comparing (1.3) and (1.4), we have

$$\sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda, x) \frac{t^n}{n!} = \frac{\log(1+\lambda t)}{\lambda t} \sum_{n=0}^{\infty} \beta_n(\lambda, x) \frac{t^n}{n!}. \tag{1.6}$$

The generalized falling factorial $(x \mid \lambda)_n$ is defined as

$$(x \mid \lambda)_n = x(x-\lambda)\cdots(x-(n-1)\lambda), \tag{1.7}$$

for $n > 0$ and $(x \mid \lambda)_0 = 1$.

Let $\sigma_k(\lambda, n)$ be the generalized falling factorial sum defined by

$$\sigma_k(\lambda, n) = \sum_{i=0}^{n} (i \mid \lambda)_k, \quad (n \geq 0). \tag{1.8}$$

In particular, we have

$$\sigma_0(\lambda, n) = n + 1, \qquad \sigma_n(\lambda, 0) = \begin{cases} 1, & n = 0, \\ 0, & n > 0. \end{cases} \tag{1.9}$$

Moreover,

$$\lim_{\lambda \to 0} \sigma_k(\lambda, n) = S_k(n), \tag{1.10}$$

where $S_k(n)$ denotes the k-th power sum of the first $n + 1$ nonnegative integers, namely

$$S_k(n) = \sum_{i=0}^{n} i^k = 0^k + 1^k + \cdots + n^k. \tag{1.11}$$

From (1.2) and (1.8), we can derive that

$$\frac{w \int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{x}{\lambda}} d\mu(x)}{\int_{\mathbb{Z}_p} (1 + \lambda t)^{\frac{wy}{\lambda}} d\mu(y)} = \sum_{k=0}^{\infty} \sigma_k(\lambda, w - 1) \frac{t^k}{k!} \tag{1.12}$$

$$= \sum_{i=0}^{w-1} (1 + \lambda t)^{\frac{i}{\lambda}}. \tag{1.13}$$

Throughout this paper, there will be many instances to be able to interchange integrals and infinite sums. That can be justified by proposition 5.5.4 in [16].

There have been some results on identities of symmetry in two variables involving Bernoulli polynomials and power sums. The reader may refer to [4, 5, 10, 18, 19] for some of the previous works. Especially, [10] is the first paper where a p-adic approach is introduced. The identities of symmetry for two variables are for the first time extended to the case of three ones in [9]. These have been done also for the q-Bernoulli polynomials (cf. [11]) and for the higher-order Bernoulli polynomials (cf. [7, 8]). It turns out that this extension gives not only new identities for three variables but also those for two variables by specializing one of the variables as 1.

In this paper, we will produce eight basic identities of symmetry in three variables w_1, w_2, w_3 related to degenerate Bernoulli polynomials and generalized falling factorial sums (cf. (50), (51), (54), (57), (61), (63), (65), (67)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature (cf. [12, 20]). Also, one refers to [1, 6, 13–15, 17] for some related works. These abundance of symmetries shed new light even on the existing identities in two variables. For instance, it has been known that (14) and (15) are equal (cf. [20, Theor. 3.1]). In fact, (14)–(17) are all equal, as they can be derived from one and the same p-adic integral. Also, we have a bunch of new identities in (18)–(21). All of these were obtained as corollaries (cf. Cor. 9, 12, 15) to some of the basic identities by specializing the variable w_3 as 1. Those would not be unearthed if more symmetries had not been available. The degenerate Bernoulli polynomials were introduced by Carlitz in [2, 3]. In view of (5) and (10), all the symmetric identities in this paper approach to the corresponding ones in [9], as λ tends to 0 and hence our symmetric identities are the degenerate versions of the identities in [9].

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\sigma_{n-k}\left(\frac{\lambda}{w_1}, w_2-1\right)w_1^{n-k}w_2^{k-1} \tag{1.14}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\sigma_{n-k}\left(\frac{\lambda}{w_2}, w_1-1\right)w_2^{n-k}w_1^{k-1} \tag{1.15}$$

$$=w_1^{n-1}\sum_{i=0}^{w_1-1}\beta_n\left(\frac{\lambda}{w_1}, w_2y_1+\frac{w_2}{w_1}i\right) \tag{1.16}$$

$$=w_2^{n-1}\sum_{i=0}^{w_2-1}\beta_n\left(\frac{\lambda}{w_2}, w_1y_1+\frac{w_1}{w_2}i\right) \tag{1.17}$$

$$=\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\sigma_\ell\left(\frac{\lambda}{w_2}, w_1-1\right)$$

$$\times\,\sigma_m\left(\frac{\lambda}{w_1}, w_2-1\right)w_1^{k+m-1}w_2^{k+\ell-1} \tag{1.18}$$

$$=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1+\frac{i}{w_1}\right)\sigma_{n-k}\left(\frac{\lambda}{w_1}, w_2-1\right)w_2^{k-1} \tag{1.19}$$

$$=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1+\frac{i}{w_2}\right)\sigma_{n-k}\left(\frac{\lambda}{w_2}, w_1-1\right)w_1^{k-1} \tag{1.20}$$

$$=(w_1w_2)^{n-1}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\beta_n\left(\frac{\lambda}{w_1w_2}, y_1+\frac{i}{w_1}+\frac{j}{w_2}\right). \tag{1.21}$$

The derivations of identities are based on the p-adic integral expression of the generating function for the modified degenerate Bernoulli polynomials in (3) and the quotient of integrals in (12) that can be expressed as the exponential generating function for the generalized falling factorial sums.

2. Several types of quotients of Volkenborn integrals

Now, we will consider several types of quotients of Volkenborn integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^3$ from which some interesting symmetric identities follow owing to the invariance of the expressions under any permutation of w_1, w_2, w_3 and the explicit expressions for the following quotients of integrals are obtained from (1.2).

(a) Type Λ_{23}^{i} (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{23}^{i}) = \frac{\int_{\mathbb{Z}_p^3} (1+\lambda t)^{\frac{1}{\lambda}(w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(\sum_{j=1}^{3-i} y_j))} d\mu(x_1)d\mu(x_2)d\mu(x_3)}{(\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4} d\mu(x_4))^i} \tag{2.22}$$

$$= \frac{(w_1w_2w_3)^{2-i}(\log(1+\lambda t)^{\frac{1}{\lambda}})^{3-i}(1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}(\sum_{j=1}^{3-i} y_j)}((1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}}-1)^i}{((1+\lambda t)^{\frac{w_2w_3}{\lambda}}-1)((1+\lambda t)^{\frac{w_1w_3}{\lambda}}-1)((1+\lambda t)^{\frac{w_1w_2}{\lambda}}-1)}; \tag{2.23}$$

(b) Type Λ_{13}^{i} (for $i = 0, 1, 2, 3$)

$$I(\Lambda_{13}^{i}) = \frac{\int_{\mathbb{Z}_p^3} (1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3+w_1w_2w_3(\sum_{j=1}^{3-i} y_j))} d\mu(x_1)d\mu(x_2)d\mu(x_3)}{(\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4} d\mu(x_4))^i} \tag{2.24}$$

$$= \frac{(w_1w_2w_3)^{1-i}(\log(1+\lambda t)^{\frac{1}{\lambda}})^{3-i}(1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}(\sum_{j=1}^{3-i} y_j)}((1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}}-1)^i}{((1+\lambda t)^{\frac{w_1}{\lambda}-1})((1+\lambda t)^{\frac{w_2}{\lambda}}-1)((1+\lambda t)^{\frac{w_3}{\lambda}}-1)}; \tag{2.25}$$

($c-0$) Type Λ_{12}^{0}

$$I(\Lambda_{12}^{0}) = \int_{\mathbb{Z}_p^3} (1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3+w_2w_3y+w_1w_3y+w_1w_2y)} d\mu(x_1)d\mu(x_2)d\mu(x_3) \tag{2.26}$$

$$= \frac{w_1w_2w_3(\log(1+\lambda t)^{\frac{1}{\lambda}})^3(1+\lambda t)^{\frac{y}{\lambda}(w_2w_3+w_1w_3+w_1w_2)}}{((1+\lambda t)^{\frac{w_1}{\lambda}}-1)((1+\lambda t)^{\frac{w_2}{\lambda}}-1)((1+\lambda t)^{\frac{w_3}{\lambda}}-1)}; \tag{2.27}$$

($c-1$) Type Λ_{12}^{1}

$$I(\Lambda_{12}^{1}) = \frac{\int_{\mathbb{Z}_p^3} (1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3)} d\mu(x_1)d\mu(x_2)d\mu(x_3)}{\int_{\mathbb{Z}_p^3} (1+\lambda t)^{\frac{1}{\lambda}(w_2w_3z_1+w_1w_3z_2+w_1w_2z_3)} d\mu(z_1)d\mu(z_2)d\mu(z_3)} \tag{2.28}$$

$$= \frac{(w_1w_2w_3)^{-1}((1+\lambda t)^{\frac{w_2w_3}{\lambda}}-1)((1+\lambda t)^{\frac{w_1w_3}{\lambda}}-1)((1+\lambda t)^{\frac{w_1w_2}{\lambda}}-1)}{((1+\lambda t)^{\frac{w_1}{\lambda}}-1)((1+\lambda t)^{\frac{w_2}{\lambda}}-1)((1+\lambda t)^{\frac{w_3}{\lambda}}-1)}. \tag{2.29}$$

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3, as one can see either from p-adic integral representations in (22), (24), (26), and (28) or from their explicit evaluations in (23), (25), (27), and (29).

3. Proofs of main theorems

First, let's consider Type Λ_{23}^{i}, for each $i = 0, 1, 2, 3$. The following results can be easily obtained from (3), (6), (12) and (13).

$(a-0)$

$$
\begin{aligned}
&I(\Lambda_{23}^{0}) \qquad (3.30)\\
&= \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_2 w_3 (x_1 + w_1 y_1)} d\mu(x_1) \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_3 (x_2 + w_2 y_2)} d\mu(x_2)\\
&\times \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_2 (x_3 + w_3 y_3)} d\mu(x_3)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^3 \Big(\sum_{k=0}^{\infty} \frac{\beta_k(\frac{\lambda}{w_2 w_3}, w_1 y_1)}{k!} (w_2 w_3 t)^k\Big)\\
&\times \left(\sum_{\ell=0}^{\infty} \frac{\beta_\ell(\frac{\lambda}{w_1 w_3}, w_2 y_2)}{\ell!} (w_1 w_3 t)^\ell\right) \left(\sum_{m=0}^{\infty} \frac{\beta_m(\frac{\lambda}{w_1 w_2}, w_3 y_3)}{m!} (w_1 w_2 t)^m\right)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^3 \sum_{n=0}^{\infty} \Big(\sum_{k+\ell+m=n} \binom{n}{k, \ell, m} \beta_k(\frac{\lambda}{w_2 w_3}, w_1 y_1) \beta_\ell(\frac{\lambda}{w_1 w_3}, w_2 y_2)\\
&\times \beta_m(\frac{\lambda}{w_1 w_2}, w_3 y_3) w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell} \Big) \frac{t^n}{n!}, \qquad (3.31)
\end{aligned}
$$

where the inner sum is over all nonnegative integers k, ℓ, m,with $k + \ell + m = n$, and

$$
\binom{n}{k, \ell, m} = \frac{n!}{k!\ell!m!}.
$$

$(a-1)$ Here we write $I(\Lambda_{23}^{1})$ in two different ways:

$$
\begin{aligned}
(1)\ I(\Lambda_{23}^{1}) &= \frac{1}{w_3} \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_2 w_3 (x_1 + w_1 y_1)} d\mu(x_1)\\
&\times \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_3 (x_2 + w_2 y_2)} d\mu(x_2) \frac{w_3 \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_2 x_3} d\mu(x_3)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_2 w_3 x_4} d\mu(x_4)}
\end{aligned}
\qquad (3.32)
$$

$$
\begin{aligned}
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^2 \frac{1}{w_3}\Big(\sum_{k=0}^{\infty}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\Big)\\
&\times\Big(\sum_{\ell=0}^{\infty}\beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right)\frac{(w_1w_3t)^\ell}{\ell!}\Big)\Big(\sum_{m=0}^{\infty}\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^m}{m!}\Big)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^2 \sum_{n=0}^{\infty}\Big(\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\\
&\times\beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right)\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{\ell+m}w_2^{k+m}w_3^{k+\ell-1}\Big)\frac{t^n}{n!}. \qquad (3.33)
\end{aligned}
$$

(2) Invoking (13), (32) can also be written as

$$
\begin{aligned}
&I(\Lambda_{23}^1)\\
&= \frac{1}{w_3}\sum_{i=0}^{w_3-1}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu(x_1)\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3(x_2+w_2y_2+\frac{w_2}{w_3}i)}d\mu(x_2)\\
&= \frac{1}{w_3}\Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^2\sum_{i=0}^{w_3-1}\Big(\sum_{k=0}^{\infty}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\Big)\\
&\times\Big(\sum_{\ell=0}^{\infty}\beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2+\frac{w_2}{w_3}i\right)\frac{(w_1w_3t)^\ell}{\ell!}\Big)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)^2\sum_{n=0}^{\infty}\Big(w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\sum_{i=0}^{w_3-1}\\
&\times\beta_{n-k}\left(\frac{\lambda}{w_1w_3}, w_2y_2+\frac{w_2}{w_3}i\right)w_1^{n-k}w_2^k\Big)\frac{t^n}{n!}. \qquad (3.34)
\end{aligned}
$$

$(a-2)$ Here we write $I(\Lambda_{23}^2)$ in three different ways:

$$
\begin{aligned}
(1)\ I(\Lambda_{23}^2) &= \frac{1}{w_2w_3}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu(x_1)\\
&\times\frac{w_2\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3x_2}d\mu(x_2)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)}\frac{w_3\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}d\mu(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)} \qquad (3.35)
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)\frac{1}{w_2w_3}\Big(\sum_{k=0}^{\infty}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\Big)\\
&\times\Big(\sum_{\ell=0}^{\infty}\sigma_\ell\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\frac{(w_1w_3t)^\ell}{\ell!}\Big)\\
&\times\Big(\sum_{m=0}^{\infty}\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^m}{m!}\Big)\\
&= \sum_{n=0}^{\infty}\Big(\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\sigma_\ell\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\\
&\times\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{\ell+m}w_2^{k+m-1}w_3^{k+\ell-1}\Big)\frac{t^n}{n!}. \qquad (3.36)
\end{aligned}
$$

(2) Invoking (13), (35) can also be written as

$$
\begin{aligned}
&I(\Lambda_{23}^2)\\
&= \frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i)}d\mu(x_1)\\
&\times\frac{w_3\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}d\mu(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)} \qquad (3.37)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)\frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}\Big(\sum_{k=0}^{\infty}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i\right)\frac{(w_2w_3t)^k}{k!}\Big)\\
&\times\Big(\sum_{\ell=0}^{\infty}\sigma_\ell\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^\ell}{\ell!}\Big)\\
&= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)\sum_{n=0}^{\infty}\Big(w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i\right)\\
&\times\sigma_{n-k}\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{n-k}w_3^{k-1}\Big)\frac{t^n}{n!}. \qquad (3.38)
\end{aligned}
$$

(3) Invoking (13) once again, (37) can be written as

$$
\begin{aligned}
&I(\Lambda_{23}^2)\\
&= \frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j)}d\mu(x_1)
\end{aligned}
$$

$$= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)\frac{1}{w_2w_3}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}\Big(\sum_{n=0}^{\infty}\beta_n\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\frac{(w_2w_3t)^n}{n!}\Big)$$

$$= \Big(\frac{\log(1+\lambda t)}{\lambda t}\Big)\sum_{n=0}^{\infty}\Big((w_2w_3)^{n-1}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}\beta_n\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\Big)\frac{t^n}{n!}. \tag{3.39}$$

$(a-3)$

$$I(\Lambda_{23}^3) = \frac{1}{w_1w_2w_3}\times\frac{w_1\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3x_1}d\mu(x_1)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)}$$

$$\times\frac{w_2\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3x_2}d\mu(x_2)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)}\frac{w_3\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}d\mu(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu(x_4)}$$

$$= \frac{1}{w_1w_2w_3}\Big(\sum_{k=0}^{\infty}\sigma_k\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\frac{(w_2w_3t)^k}{k!}\Big)$$

$$\times\Big(\sum_{\ell=0}^{\infty}\sigma_\ell\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\frac{(w_1w_3t)^\ell}{\ell!}\Big)\Big(\sum_{m=0}^{\infty}\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^m}{m!}\Big)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\sigma_k\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\sigma_\ell\left(\frac{\lambda}{w_1w_3}, w_2-1\right)$$

$$\times\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{\ell+m-1}w_2^{k+m-1}w_3^{k+\ell-1}\Big)\frac{t^n}{n!}. \tag{3.40}$$

(b) For Type $\Lambda_{13}^i (i = 0, 1, 2, 3)$, we may consider the analogous things to the ones in $(a-0)$, $(a-1)$, $(a-2)$, and $(a-3)$. However, these do not lead us to new identities. Indeed, if we substitute w_2w_3, w_1w_3, w_1w_2 respectively for w_1, w_2, w_3 in (32), this amounts to replacing λ by $\dfrac{\lambda}{w_1w_2w_3}$ and t by $w_1w_2w_3t$ in (24). So, upon replacing w_1, w_2, w_3 respectively by w_2w_3, w_1w_3, w_1w_2, and then replacing λ by $w_1w_2w_3\lambda$ and dividing by $(w_1w_2w_3)^n$, in each of the expressions of (23), (26), (29), (32), (35), (38) and (39), we will get the corresponding symmetric identities for Type $\Lambda_{13}^i (i = 0, 1, 2, 3)$.

$(c-0)$

$$
\begin{aligned}
I(\Lambda_{12}^{0}) &= \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1(x_1+w_2 y)} d\mu(x_1) \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_2(x_2+w_3 y)} d\mu(x_2) \\
&\quad \times \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_3(x_3+w_1 y)} d\mu(x_3) \\
&= \Big(\sum_{k=0}^{\infty} \frac{\beta_k(\frac{\lambda}{w_1}, w_2 y)}{k!} (w_1 t)^k \Big) \Big(\sum_{\ell=0}^{\infty} \frac{\beta_\ell(\frac{\lambda}{w_2}, w_3 y)}{\ell!} (w_2 t)^\ell \Big) \\
&\quad \times \Big(\sum_{m=0}^{\infty} \frac{\beta_m(\frac{\lambda}{w_3}, w_1 y)}{m!} (w_3 t)^m \Big) \\
&= \sum_{n=0}^{\infty} \Big(\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k(\frac{\lambda}{w_1}, w_2 y) \beta_\ell(\frac{\lambda}{w_2}, w_3 y) \\
&\quad \times w_1^k w_2^\ell w_3^m \Big) \frac{t^n}{n!},
\end{aligned} \tag{3.41}
$$

$(c-1)$

$$
\begin{aligned}
& I(\Lambda_{12}^{1}) \\
&= \frac{1}{w_1 w_2 w_3} \frac{w_2 \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 x_1} d\mu(x_1)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_1 w_2 z_3} d\mu(z_3)} \\
&\times \frac{w_3 \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_2 x_2} d\mu(x_2)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_2 w_3 z_1} d\mu(z_1)} \frac{w_1 \int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_3 x_3} d\mu(x_3)}{\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{1}{\lambda} w_3 w_1 z_2} d\mu(z_2)} \\
&= \frac{1}{w_1 w_2 w_3} \Big(\sum_{k=0}^{\infty} \sigma_k \left(\frac{\lambda}{w_1}, w_2 - 1 \right) \frac{(w_1 t)^k}{k!} \Big) \Big(\sum_{\ell=0}^{\infty} \sigma_\ell \left(\frac{\lambda}{w_2}, w_3 - 1 \right) \frac{(w_2 t)^\ell}{\ell!} \Big) \\
&\times \Big(\sum_{m=0}^{\infty} \sigma_m \left(\frac{\lambda}{w_3}, w_1 - 1 \right) \frac{(w_3 t)^m}{m!} \Big) \\
&= \sum_{n=0}^{\infty} \Big(\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \sigma_k \left(\frac{\lambda}{w_1}, w_2 - 1 \right) \sigma_\ell \left(\frac{\lambda}{w_2}, w_3 - 1 \right) \sigma_m \left(\frac{\lambda}{w_3}, w_1 - 1 \right) \\
&\times w_1^{k-1} w_2^{\ell-1} w_3^{m-1} \Big) \frac{t^n}{n!}.
\end{aligned} \tag{3.42}
$$

As we noted earlier in the last paragraph of Section 2, the various types of quotients of Volkenborn integrals are invariant under any permutation of w_1, w_2, w_3. So the corresponding expressions in the above are also invariant under any permutation of w_1, w_2, w_3. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in the above yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single equation labeled by them, there is a natural transitive action of the group S_3 on those set of expressions and hence they are in bijective correspondence with a quotient of S_3. In particular, the number of possible distinct expressions are 1,2,3, or 6. Indeed, (a-0),(a-1(1)),(a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 8 and 17, leaving the others as easy exercises for the reader. As for the case of Theorem 8, in addition to (56)–(58), we get the following three ones:

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\sigma_\ell\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\sigma_m\left(\frac{\lambda}{w_1w_3}, w_2-1\right) \times w_1^{\ell+m}w_3^{k+m-1}w_2^{k+\ell-1}, \tag{3.43}$$

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right)\sigma_\ell\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right) \times w_2^{\ell+m}w_1^{k+m-1}w_3^{k+\ell-1}, \tag{3.44}$$

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right)\sigma_\ell\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\sigma_m\left(\frac{\lambda}{w_2w_3}, w_1-1\right) \times w_3^{\ell+m}w_2^{k+m-1}w_1^{k+\ell-1}. \tag{3.45}$$

But, by interchanging ℓ and m, we see that (43), (44), and (45) are respectively equal to (56), (57) and (58).

As to Theorem 17, in addition to (67) and (68), we have:

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \sigma_k\left(\frac{\lambda}{w_1}, w_2-1\right)\sigma_\ell\left(\frac{\lambda}{w_2}, w_3-1\right)\sigma_m\left(\frac{\lambda}{w_3}, w_1-1\right) w_1^{k-1}w_2^{\ell-1}w_3^{m-1}, \tag{3.46}$$

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \sigma_k\left(\frac{\lambda}{w_2}, w_3-1\right)\sigma_\ell\left(\frac{\lambda}{w_3}, w_1-1\right)\sigma_m\left(\frac{\lambda}{w_1}, w_2-1\right) w_2^{k-1}w_3^{\ell-1}w_1^{m-1}, \tag{3.47}$$

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \sigma_k\left(\frac{\lambda}{w_1}, w_3-1\right)\sigma_\ell\left(\frac{\lambda}{w_3}, w_2-1\right)\sigma_m\left(\frac{\lambda}{w_2}, w_1-1\right) w_1^{k-1}w_3^{\ell-1}w_2^{m-1}, \tag{3.48}$$

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \sigma_k\left(\frac{\lambda}{w_3}, w_2-1\right)\sigma_\ell\left(\frac{\lambda}{w_2}, w_1-1\right)\sigma_m\left(\frac{\lambda}{w_1}, w_3-1\right) w_3^{k-1}w_2^{\ell-1}w_1^{m-1}. \tag{3.49}$$

However, (46) and (47) are equal to (67), as we can see by applying the permutations $k \to \ell, \ell \to m, m \to k$ for (46) and $k \to m, \ell \to k, m \to \ell$ for (47). Similarly, we see that (48) and (49) are equal to (68), by applying permutations $k \to \ell, \ell \to m, m \to k$ for (48) and $k \to m, \ell \to k, m \to \ell$ for (49).

4. Main theorems

Theorems 1, 2, 5, 8, 11, 14, 16 and 17 follow respectively from the considerations in (a-0), (a-1(1)), (a-1(2)), (a-2(1)), (a-2(2)), (a-2(3)), (c-0), and (c-1) in Section 3.

Theorem 4.1. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.

$$\begin{aligned}
&\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right) \beta_m\left(\frac{\lambda}{w_1w_2}, w_3y_3\right) \\
&\times w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_2}, w_3y_2\right) \beta_m\left(\frac{\lambda}{w_1w_3}, w_2y_3\right) \\
&\times w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right) \beta_\ell\left(\frac{\lambda}{w_2w_3}, w_1y_2\right) \beta_m\left(\frac{\lambda}{w_1w_2}, w_3y_3\right) \\
&\times w_2^{\ell+m} w_1^{k+m} w_3^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_2}, w_3y_2\right) \beta_m\left(\frac{\lambda}{w_2w_3}, w_1y_3\right) \\
&\times w_2^{\ell+m} w_3^{k+m} w_1^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right) \beta_\ell\left(\frac{\lambda}{w_2w_3}, w_1y_2\right) \beta_m\left(\frac{\lambda}{w_1w_3}, w_2y_3\right) \\
&\times w_3^{\ell+m} w_1^{k+m} w_2^{k+\ell} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right) \beta_m\left(\frac{\lambda}{w_2w_3}, w_1y_3\right) \\
&\times w_3^{\ell+m} w_2^{k+m} w_1^{k+\ell}.
\end{aligned} \tag{4.50}$$

Theorem 4.2. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.

$$\begin{aligned}
&\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right) \sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right) \\
&\times w_1^{\ell+m} w_2^{k+m} w_3^{k+\ell-1} \\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right) \beta_\ell\left(\frac{\lambda}{w_1w_2}, w_3y_2\right) \sigma_m\left(\frac{\lambda}{w_1w_3}, w_2-1\right) \\
&\times w_1^{\ell+m} w_3^{k+m} w_2^{k+\ell-1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right)\beta_\ell\left(\frac{\lambda}{w_2w_3}, w_1y_2\right)\sigma_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\\
&\times w_2^{\ell+m}w_1^{k+m}w_3^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right)\beta_\ell\left(\frac{\lambda}{w_1w_2}, w_3y_2\right)\sigma_m\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\\
&\times w_2^{\ell+m}w_3^{k+m}w_1^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right)\beta_\ell\left(\frac{\lambda}{w_1w_3}, w_2y_2\right)\sigma_m\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\\
&\times w_3^{\ell+m}w_2^{k+m}w_1^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right)\beta_\ell\left(\frac{\lambda}{w_2w_3}, w_1y_2\right)\sigma_m\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\\
&\times w_3^{\ell+m}w_1^{k+m}w_2^{k+\ell-1}.
\end{aligned} \tag{4.51}
$$

Putting $w_3 = 1$ in (51), we get the following corollary.

Corollary 4.3. Let w_1, w_2 be any positive integers.

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\beta_{n-k}\left(\frac{\lambda}{w_1}, w_2y_2\right)w_1^{n-k}w_2^{k}\\
&= \sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\beta_{n-k}\left(\frac{\lambda}{w_2}, w_1y_2\right)w_2^{n-k}w_1^{k}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\beta_\ell\left(\frac{\lambda}{w_1}, w_2y_2\right)\sigma_m\left(\frac{\lambda}{w_2}, w_1-1\right)w_2^{k+m}w_1^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\beta_\ell\left(\frac{\lambda}{w_1w_2}, y_2\right)\sigma_m\left(\frac{\lambda}{w_2}, w_1-1\right)w_2^{\ell+m}w_1^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\beta_\ell\left(\frac{\lambda}{w_2}, w_1y_2\right)\sigma_m\left(\frac{\lambda}{w_1}, w_2-1\right)w_1^{k+m}w_2^{k+\ell-1}\\
&= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\beta_\ell\left(\frac{\lambda}{w_1w_2}, y_2\right)\sigma_m\left(\frac{\lambda}{w_1}, w_2-1\right)w_1^{\ell+m}w_2^{k+\ell-1}.
\end{aligned} \tag{4.52}
$$

Letting further $w_2 = 1$ in (52), we have the following corollary.

Corollary 4.4. Let w_1 be any positive integer.

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\beta_k(\lambda, w_1 y_1)\,\beta_{n-k}\left(\frac{\lambda}{w_1}, y_2\right) w_1^{n-k}\\
&=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, y_1\right)\beta_{n-k}(\lambda, w_1 y_2) w_1^{k}\\
&=\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_1}, y_1\right)\beta_\ell\left(\frac{\lambda}{w_1}, y_2\right)\sigma_m(\lambda, w_1-1) w_1^{k+\ell-1}.
\end{aligned} \tag{4.53}$$

Theorem 4.5. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.

$$\begin{aligned}
&w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1 w_2}, w_3 y_1\right)\sum_{i=0}^{w_1-1}\beta_{n-k}\left(\frac{\lambda}{w_1 w_3}, w_2 y_2+\frac{w_2}{w_1}i\right) w_3^{n-k} w_2^{k}\\
&= w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1 w_3}, w_2 y_1\right)\sum_{i=0}^{w_1-1}\beta_{n-k}\left(\frac{\lambda}{w_1 w_2}, w_3 y_2+\frac{w_3}{w_1}i\right) w_2^{n-k} w_3^{k}\\
&= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1 w_2}, w_3 y_1\right)\sum_{i=0}^{w_2-1}\beta_{n-k}\left(\frac{\lambda}{w_2 w_3}, w_1 y_2+\frac{w_1}{w_2}i\right) w_3^{n-k} w_1^{k}\\
&= w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2 w_3}, w_1 y_1\right)\sum_{i=0}^{w_2-1}\beta_{n-k}\left(\frac{\lambda}{w_1 w_2}, w_3 y_2+\frac{w_3}{w_2}i\right) w_1^{n-k} w_3^{k}\\
&= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1 w_3}, w_2 y_1\right)\sum_{i=0}^{w_3-1}\beta_{n-k}\left(\frac{\lambda}{w_2 w_3}, w_1 y_2+\frac{w_1}{w_3}i\right) w_2^{n-k} w_1^{k}\\
&= w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2 w_3}, w_1 y_1\right)\sum_{i=0}^{w_3-1}\beta_{n-k}\left(\frac{\lambda}{w_1 w_3}, w_2 y_2+\frac{w_2}{w_3}i\right) w_1^{n-k} w_2^{k}.
\end{aligned} \tag{4.54}$$

Letting $w_3 = 1$ in (54), we obtain alternative expressions for the identities in (34).

Corollary 4.6. Let w_1, w_2 be any positive integers.

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\beta_{n-k}\left(\frac{\lambda}{w_1}, w_2y_2\right)w_1^{n-k}w_2^{k}\\
&=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\beta_{n-k}\left(\frac{\lambda}{w_2}, w_1y_2\right)w_2^{n-k}w_1^{k}\\
&=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\sum_{i=0}^{w_1-1}\beta_{n-k}\left(\frac{\lambda}{w_1}, w_2y_2+\frac{w_2}{w_1}i\right)w_2^{k}\\
&=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\sum_{i=0}^{w_1-1}\beta_{n-k}\left(\frac{\lambda}{w_1w_2}, y_2+\frac{i}{w_1}\right)w_2^{n-k}\\
&=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\sum_{i=0}^{w_2-1}\beta_{n-k}\left(\frac{\lambda}{w_2}, w_1y_2+\frac{w_1}{w_2}i\right)w_1^{k}\\
&=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\sum_{i=0}^{w_2-1}\beta_{n-k}\left(\frac{\lambda}{w_1w_2}, y_2+\frac{i}{w_2}\right)w_1^{n-k}. \qquad (4.55)
\end{aligned}$$

Putting further $w_2 = 1$ in (55), we have the alternative expressions for the identities for (53).

Corollary 4.7. Let w_1 be any positive integer.

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, y_1\right)\beta_{n-k}(\lambda, w_1y_2)w_1^{k}\\
&=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, y_2\right)\beta_{n-k}(\lambda, w_1y_1)w_1^{k}\\
&=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, y_1\right)\sum_{i=0}^{w_1-1}\beta_{n-k}\left(\frac{\lambda}{w_1}, y_2+\frac{i}{w_1}\right).
\end{aligned}$$

Theorem 4.8. Let w_1, w_2, w_3 be any positive integers. Then we have the following three symmetries in w_1, w_2, w_3:

$$\sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k \left(\frac{\lambda}{w_2 w_3}, w_1 y_1\right) \sigma_\ell \left(\frac{\lambda}{w_1 w_3}, w_2 - 1\right) \sigma_m \left(\frac{\lambda}{w_1 w_2}, w_3 - 1\right) \times w_1^{\ell+m} w_2^{k+m-1} w_3^{k+\ell-1} \tag{4.56}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k \left(\frac{\lambda}{w_1 w_3}, w_2 y_1\right) \sigma_\ell \left(\frac{\lambda}{w_1 w_2}, w_3 - 1\right) \sigma_m \left(\frac{\lambda}{w_2 w_3}, w_1 - 1\right) \times w_2^{\ell+m} w_3^{k+m-1} w_1^{k+\ell-1} \tag{4.57}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k \left(\frac{\lambda}{w_1 w_2}, w_3 y_1\right) \sigma_\ell \left(\frac{\lambda}{w_2 w_3}, w_1 - 1\right) \sigma_m \left(\frac{\lambda}{w_1 w_3}, w_2 - 1\right) \times w_3^{\ell+m} w_1^{k+m-1} w_2^{k+\ell-1}. \tag{4.58}$$

Putting $w_3 = 1$ in (56)-(58), we get the following corollary.

Corollary 4.9. Let w_1, w_2 be any positive integers.

$$\sum_{k=0}^{n} \binom{n}{k} \beta_k \left(\frac{\lambda}{w_2}, w_1 y_1\right) \sigma_{n-k} \left(\frac{\lambda}{w_1}, w_2 - 1\right) w_1^{n-k} w_2^{k-1}$$

$$= \sum_{k=0}^{n} \binom{n}{k} \beta_k \left(\frac{\lambda}{w_1}, w_2 y_1\right) \sigma_{n-k} \left(\frac{\lambda}{w_2}, w_1 - 1\right) w_2^{n-k} w_1^{k-1}$$

$$= \sum_{k+\ell+m=n} \binom{n}{k,\ell,m} \beta_k \left(\frac{\lambda}{w_1 w_2}, y_1\right) \sigma_\ell \left(\frac{\lambda}{w_2}, w_1 - 1\right) \sigma_m \left(\frac{\lambda}{w_1}, w_2 - 1\right) \times w_1^{k+m-1} w_2^{k+\ell-1}. \tag{4.59}$$

Letting further $w_2 = 1$ in (59), we get the following corollary.

Corollary 4.10. Let w_1 be any positive integer.

$$\beta_n(\lambda, w_1 y_1) = \sum_{k=0}^{n} \binom{n}{k} \beta_k \left(\frac{\lambda}{w_1}, y_1\right) \sigma_{n-k}(\lambda, w_1 - 1) w_1^{k-1}. \tag{4.60}$$

Theorem 4.11. Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.

$$\begin{aligned}
&w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}\beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1+\frac{w_2}{w_1}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_2^{n-k}w_3^{k-1}\\
&=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}\beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1+\frac{w_3}{w_1}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_1w_3}, w_2-1\right)w_3^{n-k}w_2^{k-1}\\
&=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{n-k}w_3^{k-1}\\
&=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}\beta_k\left(\frac{\lambda}{w_1w_2}, w_3y_1+\frac{w_3}{w_2}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_2w_3}, w_1-1\right)w_3^{n-k}w_1^{k-1}\\
&=w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_3-1}\beta_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_3}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_1w_3}, w_2-1\right)w_1^{n-k}w_2^{k-1}\\
&=w_3^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_3-1}\beta_k\left(\frac{\lambda}{w_1w_3}, w_2y_1+\frac{w_2}{w_3}i\right)\sigma_{n-k}\left(\frac{\lambda}{w_2w_3}, w_1-1\right)w_2^{n-k}w_1^{k-1}.
\end{aligned}\tag{4.61}$$

Putting $w_3 = 1$ in (61), we obtain the following corollary. In Section 1, the identities in (59), (62), and (64) are combined to give those in (14)-(21).

Corollary 4.12. Let w_1, w_2 be any positive integers.

$$\begin{aligned}
&w_1^{n-1}\sum_{i=0}^{w_1-1}\beta_n\left(\frac{\lambda}{w_1}, w_2y_1+\frac{w_2}{w_1}i\right)\\
&=w_2^{n-1}\sum_{i=0}^{w_2-1}\beta_n\left(\frac{\lambda}{w_2}, w_1y_1+\frac{w_1}{w_2}i\right)\\
&=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\sigma_{n-k}\left(\frac{\lambda}{w_2}, w_1-1\right)w_2^{n-k}w_1^{k-1}\\
&=\sum_{k=0}^{n}\binom{n}{k}\beta_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\sigma_{n-k}\left(\frac{\lambda}{w_1}, w_2-1\right)w_1^{n-k}w_2^{k-1}\\
&=w_1^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1+\frac{i}{w_1}\right)\sigma_{n-k}\left(\frac{\lambda}{w_1}, w_2-1\right)w_2^{k-1}\\
&=w_2^{n-1}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}\beta_k\left(\frac{\lambda}{w_1w_2}, y_1+\frac{i}{w_2}\right)\sigma_{n-k}\left(\frac{\lambda}{w_2}, w_1-1\right)w_1^{k-1}.
\end{aligned}\tag{4.62}$$

Letting further $w_2 = 1$ in (62), we get the following corollary. This is the multiplication formula for the degenerate Bernoulli polynomials (cf. [20, (3.11)]) together with the new identity mentioned in (60).

Corollary 4.13. Let w_1 be any positive integer.

$$\begin{aligned}\beta_n(\lambda, w_1 y_1) &= w_1^{n-1} \sum_{i=0}^{w_1-1} \beta_n\left(\frac{\lambda}{w_1}, y_1 + \frac{i}{w_1}\right) \\ &= \sum_{k=0}^{n} \binom{n}{k} \beta_k\left(\frac{\lambda}{w_1}, y_1\right) \sigma_{n-k}(\lambda, w_1 - 1) w_1^{k-1}.\end{aligned}$$

Theorem 4.14. Let w_1, w_2, w_3 be any positive integers. Then we have the following three symmetries in w_1, w_2, w_3:

$$\begin{aligned}&(w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \beta_n\left(\frac{\lambda}{w_1 w_2}, w_3 y_1 + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j\right) \\ &= (w_2 w_3)^{n-1} \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} \beta_n\left(\frac{\lambda}{w_2 w_3}, w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right) \\ &= (w_3 w_1)^{n-1} \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} \beta_n\left(\frac{\lambda}{w_3 w_1}, w_2 y_1 + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j\right). \qquad (4.63)\end{aligned}$$

Letting $w_3 = 1$ in (63), we have the following corollary.

Corollary 4.15. Let w_1, w_2 be any positive integers.

$$\begin{aligned}&w_1^{n-1} \sum_{j=0}^{w_1-1} \beta_n\left(\frac{\lambda}{w_1}, w_2 y_1 + \frac{w_2}{w_1} j\right) \\ &= w_2^{n-1} \sum_{i=0}^{w_2-1} \beta_n\left(\frac{\lambda}{w_2}, w_1 y_1 + \frac{w_1}{w_2} i\right) \\ &= (w_1 w_2)^{n-1} \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} \beta_n\left(\frac{\lambda}{w_1 w_2}, y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right). \qquad (4.64)\end{aligned}$$

Theorem 4.16. Let w_1, w_2, w_3 be any positive integers. Then we have the following two symmetries in w_1, w_2, w_3:

$$\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_3},w_1y\right)\beta_\ell\left(\frac{\lambda}{w_1},w_2y\right)\beta_m\left(\frac{\lambda}{w_2},w_3y\right)w_3^kw_1^\ell w_2^m \tag{4.65}$$

$$=\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\beta_k\left(\frac{\lambda}{w_2},w_1y\right)\beta_\ell\left(\frac{\lambda}{w_1},w_3y\right)\beta_m\left(\frac{\lambda}{w_3},w_2y\right)w_2^kw_1^\ell w_3^m. \tag{4.66}$$

Theorem 4.17. Let w_1, w_2, w_3 be any positive integers. Then we have the following two symmetries in w_1, w_2, w_3:

$$\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\sigma_k\left(\frac{\lambda}{w_3},w_1-1\right)\sigma_\ell\left(\frac{\lambda}{w_1},w_2-1\right)\sigma_m\left(\frac{\lambda}{w_2},w_3-1\right)w_3^{k-1}w_1^{\ell-1}w_2^{m-1} \tag{4.67}$$

$$=\sum_{k+\ell+m=n}\binom{n}{k,\ell,m}\sigma_k\left(\frac{\lambda}{w_2},w_1-1\right)\sigma_\ell\left(\frac{\lambda}{w_1},w_3-1\right)\sigma_m\left(\frac{\lambda}{w_3},w_2-1\right)w_2^{k-1}w_1^{\ell-1}w_3^{m-1}. \tag{4.68}$$

Putting $w_3 = 1$ in (67) and (68) and multiplying the resulting identity by w_1w_2, we get the following corollary.

Corollary 4.18. Let w_1, w_2 be any positive integers.

$$\sum_{k=0}^{n}\binom{n}{k}\sigma_k\left(\frac{\lambda}{w_1},w_2-1\right)\sigma_{n-k}(\lambda,w_1-1)w_1^k$$
$$=\sum_{k=0}^{n}\binom{n}{k}\sigma_k\left(\frac{\lambda}{w_2},w_1-1\right)\sigma_{n-k}(\lambda,w_2-1)w_2^k.$$

References

[1] A. Bayad, T. Kim, J. Choi, Y. H. Kim, B. Lee, *On the symmetry properties of the generalized higher-order Euler polynomials*, J. Appl. Math. Inform., 29(2011), no. 1-2, 511–516.

[2] L. Carlitz, *A degenerate Staudt-Clausen theorem*, Arch. Math. (Basel) 7(1956), 28–33.

[3] L. Carlitz, *Degenerate Stirling, Bernoulli and Eulerian numbers*, Utilitas Math., 15(1979), 51–88.

[4] E. Deeba, D. Rodriguez, *Stirling's and Bernoulli numbers*, Amer. Math. Monthly, 98(1991), 423–426.

[5] F. T. Howard, *Applications of a recurrence for the Bernoulli numbers*, J. Number Theory, 52(1995), 157–172.

[6] L.-C. Jang, H. Yi, K. Shivashankara, T. Kim, Y.-H. Kim, B. Lee, *A note on symmetric properties of the twisted q-Bernoulli polynomials and the twisted generalized q-Bernoulli polynomials*, Adv. Difference Equ. 2010, Art. ID 801580, 13 pp.

[7] D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (I)*, Adv. Stud. Contemp. Math. (Kyungshang) 23(2013), no. 3, 461–482.

[8] D. S. Kim, N. Lee, J. Na, K. H. Park, *Abundant symmetry for higher-order Bernoulli polynomials (II)*, Proc. Jangjeon Math. Soc., 16(2013), no. 3, 359–378.

[9] D. S. Kim, K. H. Park, *Identities of symmetry for Bernoulli polynomials arising from quotients of Volkenborn integrals invariant under* S_3, Appl. Math. Comput., 219(2013), 5096–5104.

[10] T. Kim, *Symmetry p-adic invariant integral on* $\mathbb{Z}_p$ *for Bernoulli and Euler polynomials*, J. Adv. Difference Equ. Appl., 14(2008), 1267–1277.

[11] T. Kim, D. V. Dolgy, D. S. Kim, *Some identities of q-Bernoulli polynomials under symmetry group* S_3, J. Nonlinear Convex Anal., 9(2015), 1869–1880.

[12] T. Kim, D. V. Dolgy, D. S. Kim, *Symmetric identities for degenerate generalized Bernoulli polynomials*, J. Nonlinear Sci. Appl., 9(2016), 677–683.

[13] T. Kim, S.-H. Rim, B. Lee, *Symmetry of power sum polynomials and multivariate fermionic p -adic invariant integral on* $\mathbb{Z}_p$, Russ. J. Math. Phys., 16(2009), no. 1, 93–96.

[14] V. Kurt, *A further symmetric relation on the analogue of the Apostol-Bernoulli and the analogue of the Apostol-Genocchi polynomials*, Appl. Math. Sci. (Ruse) 3(2009), no. 53–56, 2757–2764.

[15] S.-H. Rim, Y.-H. Kim, B. Lee, T. Kim, *A new proof of the symmetric properties for the twisted generalized Euler polynomials of higher order*, J. Comput. Anal. Appl., 13(2011), no. 7, 1258–1262.

[16] W. H. Schikhof, *"Ultrametric calculus: An introduction to p-adic analysis,"* Cambridge University Press, 2006.

[17] Y. Simsek, *Complete sum of products of* (h, q)*-extension of Euler polynomials and numbers*, J. Difference Equ. Appl., 16(2010), no. 11, 1331–1348.

[18] H. Tuenter, *A symmetry of power sum polynomials and Bernoulli numbers*, Amer. Math. Monthly, 108(2001), 258–261.

[19] S. Yang, *An identity of symmetry for the Bernoulli polynomials*, Discrete Math., 308(2008), 550–554.

[20] Z. Zhang, J. Yang, *On sums of products of the degenerate Bernoulli numbers*, Integral Transforms Spec. Funct., 20(2009), 751–755.