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# On $T$-Equivalence of Some Families of Ladder-Type Graphs

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**Abstract**

We give recursive forms of the Tutte polynomial of some families of ladder-type graphs, which, in some sense, can be considered as geometric models of DNA of some species, and conclude that neither two of these are $T$-equivalent.

**AMS subject classification:** Primary 05C31; Secondary 57M27.
**Keywords:** Tutte polynomial, recursive relation, ladder graph.

## 1. Introduction

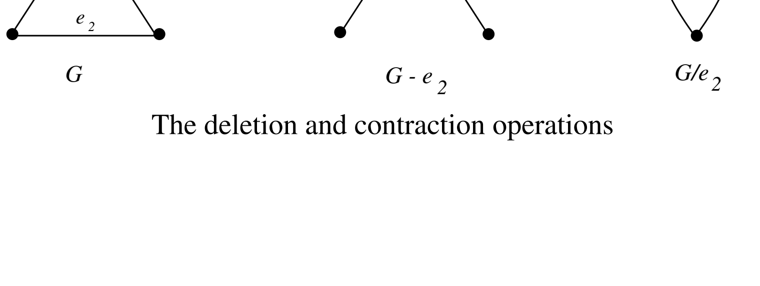
The Tutte polynomial was introduced by Tutte in 1954 in [5] as a generalization of chromatic polynomials studied by Birkhoff [1] and Whitney [8].

The Tutte polynomial became popular because of its universal property that any multiplicative graph invariant with a deletion/contraction reduction must be an evaluation of it, and because of its applications in other disciplines as computer science, engineering, optimization, physics, biology, and knot theory. This paper deals with ladder-type graphs whose combinatorial interpretations derived as different evaluations of Tutte polynomial can be of some interest for molecular biologists.

A *graph* $G$ is triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its endpoints. An edge is a *bridge* if its deletion increases the number of components of the graph. A *loop* is an edge whose endpoints are equal. The *path* graph $P_n$ is a tree with $n$ vertices $V = \{v_1, \ldots, v_n\}$ and edges $E = \{(v_i, v_{i+1}) \mid 1 \leq i \leq n-1\}$. In this case, $\deg(v_1) = \deg(v_n) = 1$ and $\deg(v_i) = 2$ for $1 < i < n$.

Two graphs $G$ and $G'$ are *isomorphic*, written $G \cong G'$, if there is a one-to-one correspondence between their vertex sets that preserves adjacency. In order to check whether two graphs are different (means non-isomorphic) or not one needs a graph invariant, which is a function $f$ on the collection of all graphs such that if $G_1 \cong G_2$ then $f(G_1) = f(G_2)$, and if $f(G_1) \neq f(G_2)$ then $G_1$ is not isomorphic to $G_2$. The present work is concerned with the Tutte polynomial, which is a graph invariant having values in $\mathbb{Z}[x, y]$.

The following two operations are essential to understand the Tutte polynomial definition for a graph $G$. These are: *edge deletion* denoted by $G' = G - e$, and *edge contraction* $G'' = G/e$.




The deletion and contraction operations

**Definition 1.1. [5, 6, 7]** The *Tutte polynomial* of a graph $G$ is a two-variable polynomial $T_G(x, y)$ defined as follows:

$$T_G(x, y) = \begin{cases} 1 & \text{if } E \text{ is empty,} \\ xT(G/e) & \text{if } e \text{ is a bridge,} \\ yT(G-e) & \text{if } e \text{ is a loop,} \\ T(G-e) + T(G/e) & \text{if } e \text{ is neither a bridge nor a loop.} \end{cases}$$

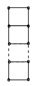
**Definition 1.2.** Two graphs $G_1$ and $G_2$ are *Tutte-equivalent* (simply, *T-equivalent*) if $T(G_1; x, y) = T(G_2; x, y)$.

**Theorem 1.3. [2]** If $G$ and $G'$ are graphs then

$$T(G \sqcup G') = T(G)T(G') \quad \text{and} \quad T(G * G') = T(G)T(G'),$$

where $G \sqcup G'$ is the disjoint union of $G$ and $G'$ and $G * G'$ is formed by identifying a vertex of $G$ and a vertex of $G'$ into a single vertex.

**Definition 1.4.** A *ladder graph* $L_n$ is the Cartesian product of path graphs $p_n$ and $p_1$:

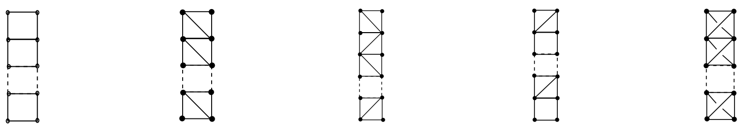
$$L_n = p_n \times p_1 =$$ 

We define a *ladder-type graph* as a graph $L_n$ with the addition of some edges, in some pattern.

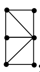
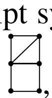
This paper is concerned with ladder and ladder-type graphs.

## 2. The Results

Here we are concerned with the following ladder and ladder-type graphs.



$L_n$ $L_{n_1}$ $L_{n_2}$ $L_{n_3}$ $L_{n_4}$

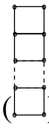
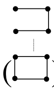
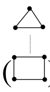
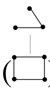
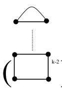
In our case, the subscript symbols $n$, $n_1$, $n_2$, $n_3$, and $n_4$ represent number of 'unit' boxes of types , , , , and , respectively in their corresponding ladders.

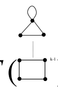
**Theorem 2.1.** The Tutte polynomial of $L_n$ $(n \geq 2)$ is

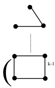
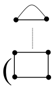
$$T(L_n) = PT(L_{n-1}) + (x+1)\sum_{i=1}^{n-2} y^{n-i-1}T(L_i) + y^{n-1}(x^2 + x + y),$$

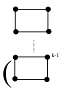
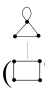
where $P(x, y) = x^2 + x + 1$.

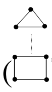
*Proof.* Using directly the definition of the Tutte polynomial, we get

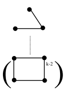
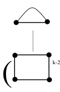
$T($$) = T($$) + T($$) = x^2T(L_{n-1}) + T($$) + T($$)$

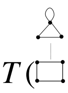
$= x^2T(L_{n-1}) + xT(L_{n-1}) + T(L_{n-1}) + T($$)$

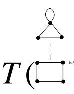
$= PT(L_{n-1}) + y\big[T($$) + T($$)\big]$

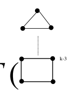
$= PT(L_{n-1}) + y\big[xT(L_{n-2}) + T($$) + T($$)\big]$

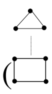
$= PT(L_{n-1}) + y\big[xT(L_{n-2}) + T(L_{n-2}) + yT($$)\big]$

$= PT(L_{n-1}) + y\big[(x+1)T(L_{n-2}) + y\big[T($$) + T($$)\big]\big]$

$= PT(L_{n-1}) + y\big[(x+1)T(L_{n-2}) + y\big[xT(L_{n-3}) + T(L_{n-3}) + T($$)\big]\big]$

$= PT(L_{n-1}) + y\big[(x+1)T(L_{n-2}) + y\big[(x+1)T(L_{n-3}) + yT($$)\big]\big]$

$= PT(L_{n-1}) + y\big[(x+1)T(L_{n-2})\big] + y^2(x+1)T(L_{n-3}) + y^3T($$)$

$= PT(L_{n-1}) + (x+1)\big[yT(L_{n-2}) + y^2[T(L_{n-3})\big] + y^3T($$)$.

Each iteration reduces the number of boxes; repeating the same process for finite times, we get, at the end, the cycle of three vertices $C_3$ with $T(C_3) = x^2 + x + y$. ■

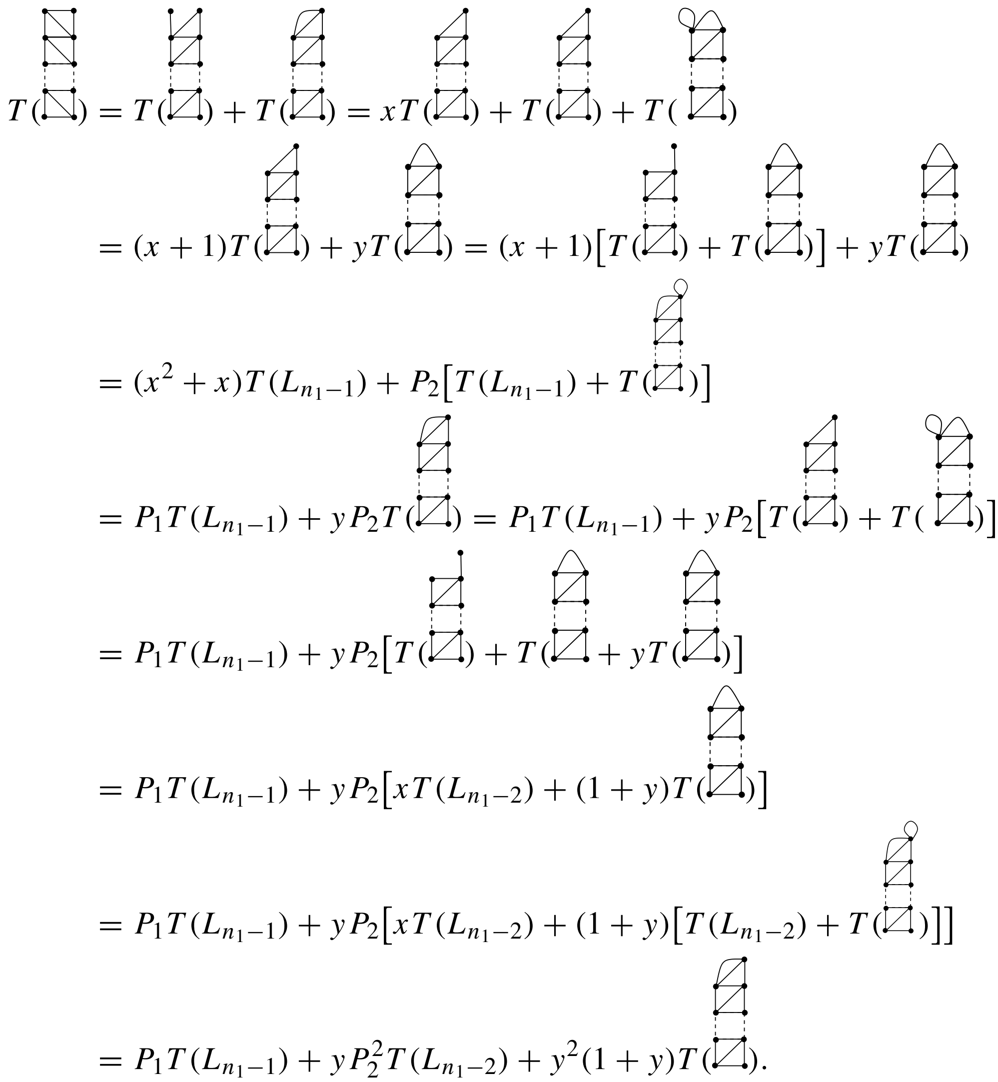
In the following result, we suppose for simplicity that

$$P_1(x, y) = x^2 + 2x + y + 1,$$
$$P_2(x, y) = x + y + 1.$$

**Theorem 2.2.** For $n_1 \geq 2$,

$$T(L_{n_1}) = P_1T(L_{n_1-1}) + P_2^2\sum_{i=1}^{n-2} y^{n-i-1}(y+1)^{n-i-2}T(L_{i_1}) + y^{n-1}(y+1)^{n-2}P_2(x^2 + x + y + xy + y^2).$$

*Proof.*



Here, at the end, we receive  whose Tutte polynomial is $x^2 + x + y + xy + y^2$. ■

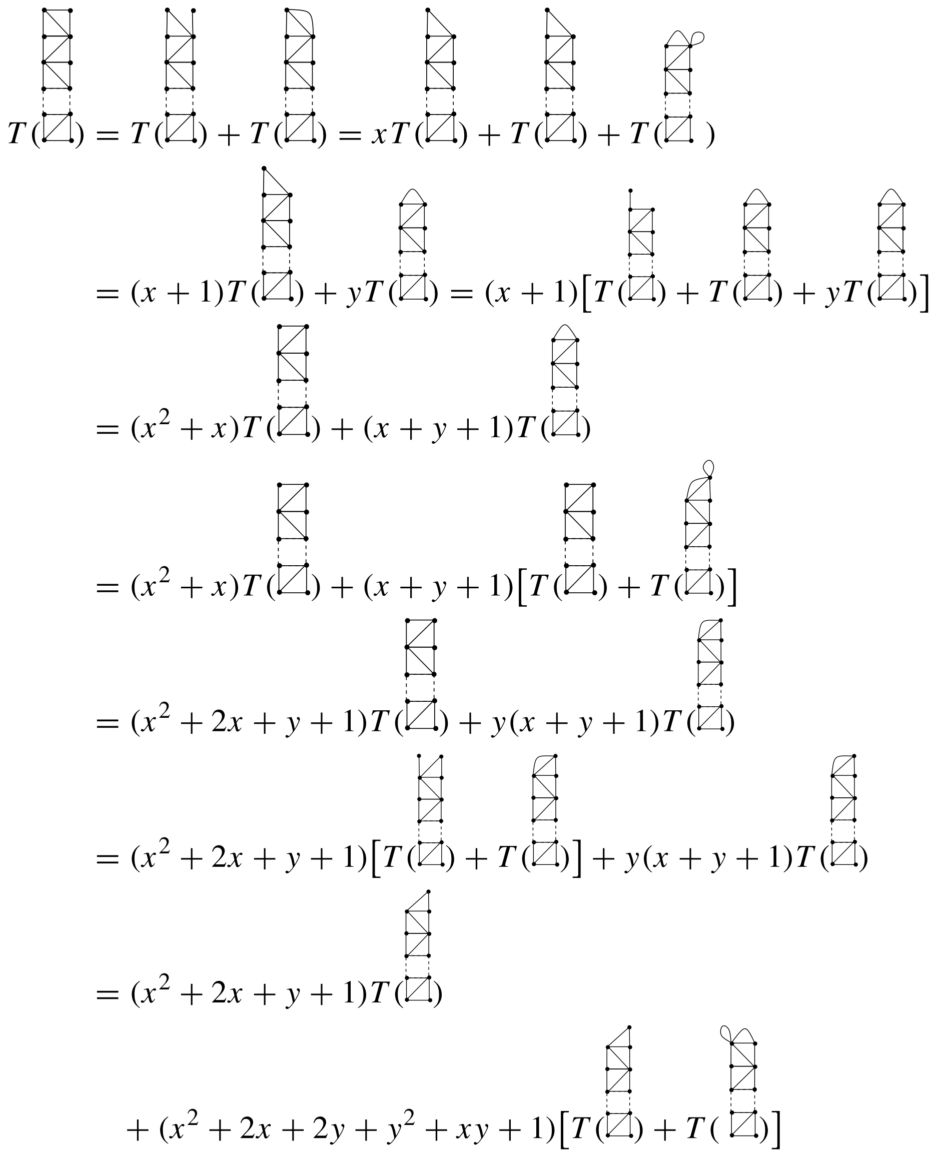
To avoid long expressions in the following result, we use the notations:

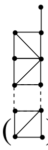
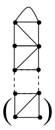
$$Q_1(x, y) = x^4 + 4x^3 + 6x^2 + 4x + 6xy + y^3 + 3y^2 + 3y + 3x^2y + 2xy^2 + 1,$$
$$Q_2(x, y) = x^3 + 3x^2 + 3x + 4xy + 3y + 3y^2 + y^3 + x^2y + xy^2 + 1,$$
$$Q_3(x, y) = x^2 + 2x + 3y + 2xy + 3y^2 + y^3 + 1,$$
$$Q_4(x, y) = x^2 + x + 3xy + xy^2 + y + 2y^2 + y^3.$$

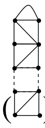
**Theorem 2.3.** For $n_2 \geq 2$,

$$T(L_{n_2}) = Q_1 T L_{n_2-1} + Q_2^2 \sum_{i=1}^{n-2} y^{n-i-1} Q_3^{n-i-2} T(L_{i_2})$$
$$+ y^{n-1} Q_2 Q_3 (x^2 + 2x + 2y + xy + y^2 + 1)(x^3 + x^2 + x + y) + y Q_4).$$

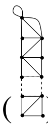
*Proof.*

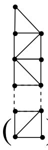
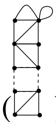


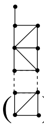
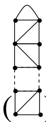
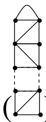
$= (x^3 + 3x^2 + 3x + 2xy + 2y + y^2 + 1)\big[T($$) + T($$)\big]$

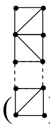
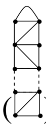
$+\, y(x^2 + 2x + 2y + y^2 + xy + 1)T($$)$

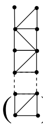
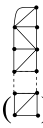
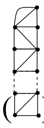
$= (x^4 + 3x^3 + 3x^2 + 2x^2y + 2xy + xy^2 + x)T(L_{n_2-1})$

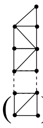
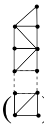
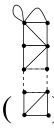
$+\, (x^3 + 3x^2 + 3x + 4xy + 3y + 3y^2 + y^3 + x^2y + xy^2 + 1)\big[T(L_{n_2-1}) + T($$)\big]$

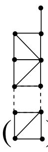
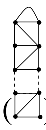
$= Q_1T(L_{n_2-1}) + yQ_2\big[T($$) + T($$)\big]$

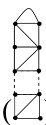
$= Q_1T(L_{n_2-1}) + yQ_2\big[T($$) + T($$) + T($$)\big]$

$= Q_1T(L_{n_2-1}) + yQ_2\big[xT($$) + (1 + y)T($$)\big]$

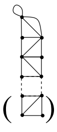
$= Q_1T(L_{n_2-1}) + yQ_2\big[(x + y + 1)\big[T($$) + T($$)\big] + y(1 + y)T($$)\big]$

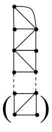
$= Q_1T(L_{n_2-1}) + yQ_2\big[(x^2 + xy + x)T($$) + (x + 2y + y^2 + 1)\big[T($$) + T($$)\big]\big]$

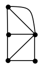
$= Q_1T(L_{n_2-1}) + yQ_2\big[(x^2 + 2x + 2y + xy + y^2 + 1)\big[T($$) + T($$)\big]$

$+\, y(x + 2y + y^2 + 1)T($$)\big]$

$= Q_1T(L_{n_2-1}) + yQ_2\big[(x^3 + 2x^2 + 2xy + x^2y + xy^2 + x)T(L_{n_2-2})$

$+\, (x^2 + 2x + 3y + 2xy + 3y^2 + y^3 + 1)\big[T(L_{n_2-2}) + T($$)\big]$

$= Q_1T(L_{n_2-1}) + yQ_2^2T(L_{n_2-2}) + y^2Q_2Q_3T($$).$

At the end we receive  whose Tutte polynomial is $(x^2+x+y)(x^2+xy+x)+y(x^2+x+y+xy+y^2)$. ■

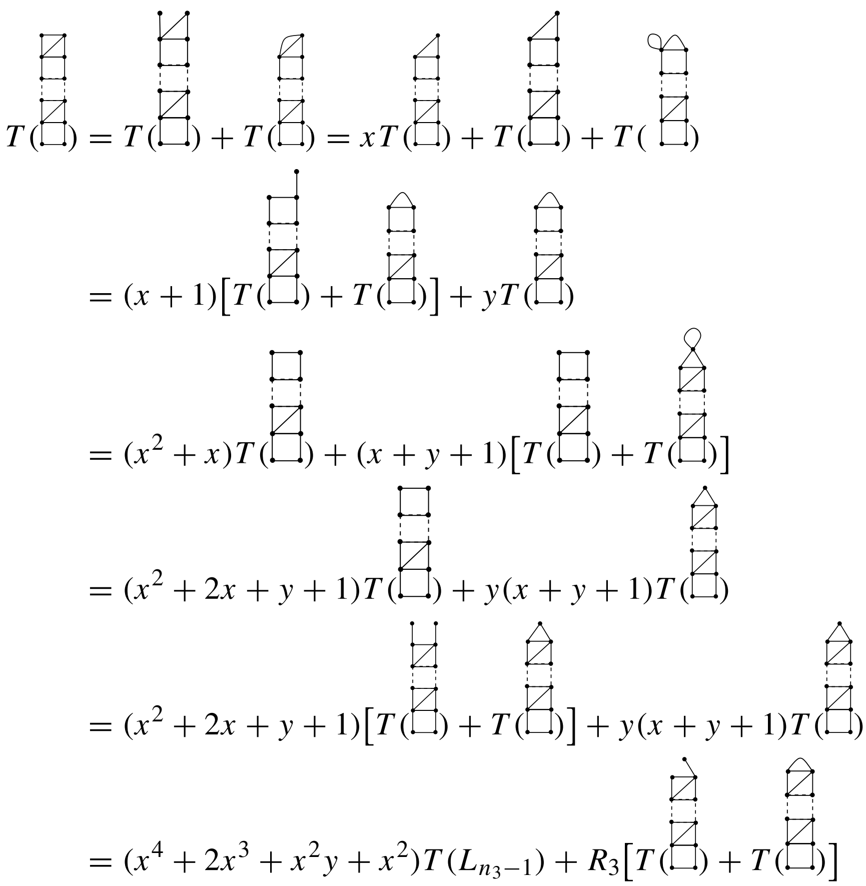
In the following result, we use the notations:

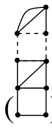
$$R_1(x, y) = x^4 + 3x^3 + 4x^2 + 3x + 3xy + 2x^2y + xy^2 + 2y + y^2 + 1,$$
$$R_2(x, y) = x^3 + 2x^2 + 2x + 2xy + x^2y + xy^2 + 2y + y^2 + 1,$$
$$R_3(x, y) = x^2 + 2x + 2y + xy + y^2 + 1,$$
$$R_4(x, y) = (x + y + 1)(x^3 + x^2 + x + y) + y(x^2 + x + y + y^2 + xy + x^2y).$$

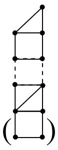
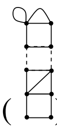
**Theorem 2.4.** For $n_3 \geq 2$,

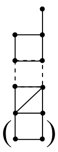
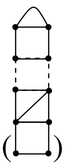
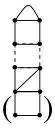
$$T(L_{n_3}) = R_1T(L_{n_3-1}) + R_2\sum_{i=1}^{n-2} y^{n-i-1}(x + 2y + y^2 + 1)^{n-i-2}T(L_{i_3}) + y^{n-1}(x + 2y + y^2 + 1)^{n-2}R_3R_4.$$

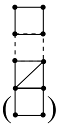
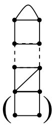
*Proof.*

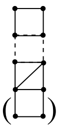
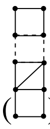
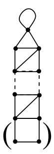


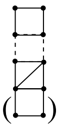
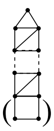
$= (x^4 + 2x^3 + x^2y + x^2)T(L_{n_3-1}) + R_3\big[(x+1)T(L_{n_3-1}) + yT($  $)\big]$

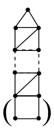
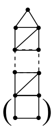
$= R_1T(L_{n_3-1}) + yR_3\big[T($  $) + T($  $)\big]$

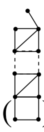
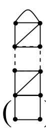
$= R_1T(L_{n_3-1}) + yR_3\big[T($  $) + T($  $) + yT($  $)\big]$

$= R_1T(L_{n_3-1}) + yR_3\big[xT($  $) + (1+y)T($  $)\big]$

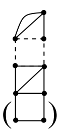
$= R_1T(_{Ln_3-1}) + yR_3\big[xT($  $) + (1+y)\big[T($  $) + T($  $)\big]\big]$

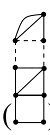
$= R_1T(L_{n_3-1}) + yR_3\big[(x+y+1)T($  $) + y(1+y)T($  $)\big]$

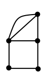
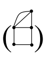
$= R_1T(L_{n_3-1}) + yR_3\big[(x+y+1)\big[x^2T(Ln_3-2) + T($  $)\big] + y(1+y)T($  $)\big]$

$= R_1T(L_{n_3-1}) + yR_3\big[x^2(x+y+1)T(L_{n_3-2}) + (x+2y+y^2+1)\big[T($  $) + T($  $)\big]\big]$

$= R_1T(L_{n_3-1}) + yR_3\big[x^2(x+y+1)T(L_{n_3-2})$

$+ (x+2y+y^2+1)\big[xT_{G_{k-2}} + T(L_{n_3-2}) + yT($  $)\big]\big]$

$= R_1T(L_{n_3-1}) + yR_2T(L_{n_3-2}) + y^2R_3(x+2y+y^2+1)T($  $).$

Here, we receive  at the end with $T($  $) = (x+y+1)(x^3+x^2+x+y) + y(x^2+x+y).$

∎

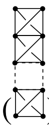
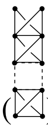
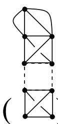
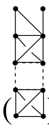
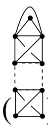
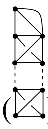
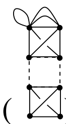
Here, again, we suppose:

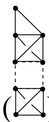
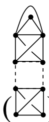
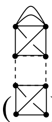
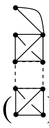
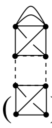
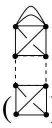
$$\begin{aligned} S_1(x, y) &= x^2 + 2x + 2y + 1, \\ S_2 &= (x + 3y + y^2 + 3)(x + 2y + 1), \\ S_3 &= x + 3y + y^2 + 3, \\ S_4 &= x^2 + x + 2xy + y + 2y^2 + y^3. \end{aligned}$$

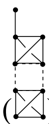
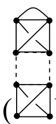
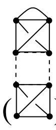
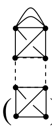
**Theorem 2.5.** The Tutte polynomial of $L_{n_4}, n_4 \geq 2$, is

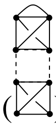
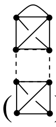
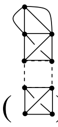
$$T(L_{n_4}) = S_1 T(L_{n_4-1}) + S_2 \sum_{i=1}^{n-2} y^{n-i-1}(1+y)^{2k-1} T(L_{i_4}) + y^{n-1}(y+1)^{2k} S_3 S_4.$$

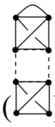
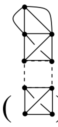
*Proof.*

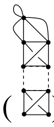
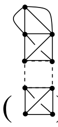
$T($$) = T($$) + T($$) = T($$) + T($$) + T($$) + T($$)$

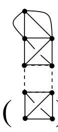
$= xT($$) + T($$) + T($$) + T($$) + T($$) + yT($$)$

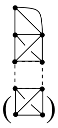
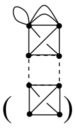
$= xT($$) + T($$) + xT($$) + (x+y)T(L_{n_4-1}) + (y+2)T($$)$

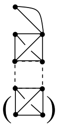
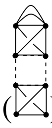
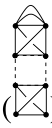
$= x^2(L_{n_4-1}) + (x+1)T($$) + (x+y)T(L_{n_4-1}) + (y+2)\big[T($$) + y^2 T($$)\big]$

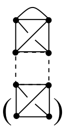
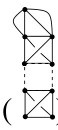
$= (x^2 + x + y)T(L_{n_4-1}) + (x+y+3)T($$) + y^2(y+2)T($$)$

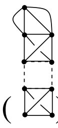
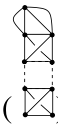
$= (x^2 + x + y)T(L_{n_4-1}) + (x+y+3)\big[T(L_{n_4-1}) + T($$)\big] + y^2(y+2)T($$)$

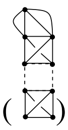
$= S_1 T(L_{n_4-1}) + \big[y(x+y+3) + y^2(y+2)\big]T($$)$

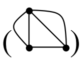
$= S_1T(L_{n_4-1}) + yS_3\big[T($$) + T($$)\big]$

$= S_1T(L_{n_4-1}) + yS_3\big[T($$) + T($$) + yT($$)\big]$

$= S_1(T(L_{n_4-1}) + yS_3\big[(x+y)T(L_{n_4-2}) + (y+1)\big[T($$) + y^2T($$)\big]\big]$

$= S_1T(L_{n_4-1}) + yS_3\big[(x+y)T(L_{n_4-2}) + (y+1)\big[T(L_{n_4-2}) + yT($$) + y^2T($$)\big]\big]$

$= S_1T(L_{n_4-1}) + yS_2T(L_{n_4-2}) + y^2(y+1)^2S_3T($$).$

The last expression in this case is $T($$) = x^2 + x + y + 2xy + 2y^2 + y^3.$ ■

## 3. Conclusions

Above results reveal, that the families obtained above, are not $T$-equivalent. Also the recursive form of these polynomials give us fruitful combinatorial information about these families like the number of spanning trees specialized at $(1, 1)$, $T(2, 1)$ as spanning forests, $T(1, 2)$ spanning connected subgraphs and $T(2, 2)$ all subgraphs, see [2, 5]. On another node, chromatic polynomials, flow, reliability and jones polynomials can easily be derived from above results, see also [1, 3, 5].

## Acknowledgments

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