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The method to find a basis of Euler-Cauchy equation by transforms

Hwajoon Kim[1]
Faculty of General Education,
Kyungdong University,
11458, Kyungdong Univ. Rd. 27,
Yangju, Gyeonggi, S. Korea.

Abstract

Euler-Cauchy equation is a fundamental ODE with variable coefficients, and we have continued efforting to find the solution of it by transforms because we judge that this equation is the key to solve ODEs with variable coefficients by transforms. In this article, we have checked the method to find a basis of Euler-Cauchy equation by transforms.

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1. Introduction

The Euler-Cauchy equation is the typical example of linear ODEs, and it frequently appears in solving Laplace's equation in a polar coordinates, describing time-harmonic vibrations of a thin elastics rod and boundary value problems in spherical coordinates. In financial mathematics, it has to do with Black-Scholes terminal value problem which gives a theoretical estimate of the price of European call options. The most common form is the second-order equation and this equation is a fundamental ODEs with variable coefficients. In the article [6], we have checked a basis of Euler-Cauchy equation of the second order and the third one by transforms. As a progressed version, we would like to deal with the Euler-Cauchy equation of the n-th order and the general logic of it in this article.

[1]Corresponding author

This equations are ODEs of the form $t^2y'' + aty' + by = 0$ with given constants a and b and unknown $y(t)$. To extend this form to order n, the form would become

$$a_n t^n y^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_0 y(t) = 0$$

for $y^{(n)}(t)$ is the n-th derivative of the function $y(t)$.

On the other hand, there are lots of type of integral transforms but their similarity makes us approach Laplace transform only. Of course, the proposed method is valid even if the transform changes to Elzaki's or Sumudu one, and it can be applied to the other equation as well.

2. The method to find a basis of Euler-Cauchy equation by transforms

Let us see the form of Euler-Cauchy equation which has the form $t^2y'' + aty' + by = 0$ with given constants a and b and unknown $y(t)$ for the second order. Since the existing transforms are similar, we would just like to approach the method by Laplace transform only.

From the interaction formulas

$$\begin{aligned} £(t^2y'') &= -\frac{d}{ds}(-2sY - s^2\frac{dY}{ds} + y(0)) = s^2\frac{d^2Y}{ds} + 4s\frac{dY}{ds} + 2Y, \\ £(y''') &= s^3£(y) - s^2y(0) - sy'(0) - y''(0), \\ £(t^3y''') &= -s^3\frac{d^3Y}{ds^3} - 9s^2\frac{d^2Y}{ds^2} - 18s\frac{dY}{ds} - 6Y, \end{aligned}$$

we can obtain the results of the following lemmas.

Lemma 2.1. (The second order) Let $Y = £(y) = F(s)$. Then A basis of Euler-Cauchy equation $t^2y'' + aty' + by = 0$ can be represented by $y = £^{-1}(s^m)$ where,

$$m = \frac{a - 3 \pm \sqrt{(a-1)^2 - 4b}}{2} \tag{1}$$

for $Y = s^m$ [6].

Of course, the above m is the roots of the transformed auxiliary equation

$$m^2 + (3-a)m + b - a + 2 = 0. \tag{2}$$

Lemma 2.2. (The third order) The solution of Euler-Cauchy equation $t^3y''' + at^2y'' + bty' + cy = 0$ can be represented by $Y = s^m$, where m satisfies the equation

$$m^3 + (6-a)m^2 + (b - 3a + 11)m + (b - 2a - c + 6) = 0 \tag{3}$$

for $Y = \pounds(y) = F(s)$ [6].

In the above lemmas, the values of m determine a basis of the given equation. Putting $Y = s^m = \pounds(y) = F(s)$, we have $y = \pounds^{-1}(s^m)$. First, let us consider the Euler-Cauchy equation

$$t^2y'' + 1.5ty' - 0.5y = 0$$

as a simple example. From the equation (2), we have $m^2 + \frac{3}{2}m = 0$. Organizing the equality, we have $m = -3/2$. Since $\pounds\left(2\sqrt{\frac{t}{\pi}}\right) = s^{-\frac{3}{2}}$, we obtain a basis $\sqrt{t}$.

The other basis can be obtained by the method of reduction of order or an inspection. Let us check the above example by using the method of reduction of order. Let us put $y = u\sqrt{t}$. Then we have

$$y' = u'\sqrt{t} + 1/2\, ut^{-1/2},\ \ y'' = u''\sqrt{t} + u't^{-1/2} - 1/4\, ut^{-3/2},$$

and substitute these values into the given equation. This gives

$$u''t^{5/2} + \frac{5}{2}u't^{3/2} = 0.$$

Dividing by $t^{3/2}$ and putting $v = u'$, we have $v't + \frac{5}{2}v = 0$. Separating variable and integrating, we get

$$-\frac{2}{5}\frac{dv}{v} = \frac{dt}{t},\quad -\frac{2}{5}\ln v = \ln t.$$

Taking exponentials and integrating again, we have

$$v = t^{-5/2},\ \ u = \int v\, dt = -\frac{2}{3}\, t^{-3/2},$$

hence the other basis is

$$y = u\sqrt{t} = -\frac{2}{3}\, t^{-1}.$$

Thus, $1/t$ is the other basis.

The several examples are found in [6].

Next, let us see the case of the n-th order Euler-Cauchy equation.

Theorem 2.3. (The n-th order) The solution of Euler-Cauchy equation

$$a_nt^ny^{(n)}(t) + a_{n-1}t^{n-1}y^{(n-1)}(t) + \cdots + a_0y(t) = 0$$

can be represented by $Y = s^m$, where m satisfies the equation

$$\begin{aligned}&(-1)^na_n(m(m-1)\cdots(m-n+1)s^{m-n})\\ &\quad + (-1)^{n-1}a_{n-1}(m(m-1)\cdots(m-n+2)s^{m-n+1}) + \cdots + a_ns^m = 0\end{aligned}$$

for $y^{(n)}(t)$ is the n-th derivative of $y(t)$.

Proof. Let us use the equality

$$£(t^n y^{(n)}) = (-1)^n \frac{d^n}{ds^n} £(y^{(n)}),$$

and take Laplace transform on both sides. This gives

$$(-1)^n a_n \frac{d^n}{ds^n} £(y^{(n)}(t)) + (-1)^{n-1} a_{n-1} \frac{d^{n-1}}{ds^{(n-1)}} £(y^{(n-1)}(t)) + \cdots + a_0 £(y(t)) = 0.$$

Let us denote $£(y^{(n)}(t)) = Y^{(n)}$ and put $Y = s^m$. Then the above equality becomes

$$(-1)^n a_n (m(m-1)\cdots(m-n+1)s^{m-n}) + (-1)^{n-1} a_{n-1}(m(m-1)\cdots(m-n+2)s^{m-n+1}) + \cdots + a_n s^m = 0. \quad (4)$$

Organizing the equality, we can find the value of m which has values x_i $(i = 1, 2, \cdots n)$. Hence, $Y = s^{x_i}$ and so, $y = £^{-1}(s^{x_i})$ for $(i = 1, 2, \ldots, n)$. ■

Finally, we would like to check the general logic of Euler-Cauchy equation which has the form is $f(t, y) = 0$.

Theorem 2.4. (The general logic)The solution of Euler-Cauchy equation $f(t, y) = 0$ is represented by $Y = s^m$, where m satisfies the equation

$$\sum_{j=0}^{n} l_j \, m^{n-j} = 0$$

for l_j is a constant.

Proof. Taking Laplace transform, we have $£[f(t, y)] = £(0) = 0$. Organizing this equality, we have

$$F\left[\sum_{i=0}^{n} f(k_i s^{-i}) Y^{(n-i)}\right] = 0 \quad (5)$$

for $k_0 = 0$ and for k_i is a constant. Putting $Y = s^m$, we get

$$Y^{(n)} = m(m-1)\cdots(m-n+1)s^{m-n}.$$

Substituting this value into (5), we have

$$F\left[\sum_{i=0}^{n} f(k_i s^{-i}) m(m-1)\cdots(m-n+i+1)s^{m-n+i}\right] = 0.$$

Dropping a common factor, we have

$$\sum_{j=0}^{n} l_j \, m^{n-j} = 0$$

for l_j is a constant. Solving this equation, we can find the values of m. Since $Y = s^m$, we obtain a basis $y = £^{-1}(s^m)$. ■

Of course, this method is valid even if we change the transform such as Elzaki's or Sumudu one. For example, let us change the transform to Elzaki's. Since the fundamental logic of this method is similar, we can find the solution as well. The followings are the fundamental interaction formulas of Elzaki transform.

$a)$ $E[f'(t)] = T(v)/v - vf(0)$

$b)$ $E[f''(t)] = T(v)/v^2 - f(0) - vf'(0)$

$c)$ $E[tf'(t)] = v^2 \dfrac{d}{dv}[T(v)/v - vf(0)] - v[T(v)/v - vf(0)]$

$d)$ $E[t^2 f'(t)] = v^4 \dfrac{d^2}{dv^2}[T(v)/v - vf(0)]$

$e)$ $E[tf''(t)] = v^2 \dfrac{d}{dv}[T(v)/v^2 - f(0) - vf'(0)] - v[T(v)/v^2 - f(0) - vf'(0)]$

$f)$ $E[t^2 f''(t)] = v^4 \dfrac{d^2}{dv^2}[T(v)/v^2 - f(0) - vf'(0)]$.

$g)$ $E\left[\dfrac{\partial f(x,t)}{\partial t}\right] = \dfrac{1}{v}T(x,v) - vf(x,0)$

$h)$ $E\left[\dfrac{\partial^2 f(x,t)}{\partial t^2}\right] = \dfrac{1}{v^2}T(x,v) - f(x,0) - v\dfrac{\partial f(x,0)}{\partial t}$

$i)$ $E\left[\dfrac{\partial f(x,t)}{\partial x}\right] = \dfrac{d}{dx}[T(x,v)]$

$j)$ $E\left[\dfrac{\partial^2 f(x,t)}{\partial x^2}\right] = \dfrac{d^2}{dx^2}[T(x,v)]$.

where, Elzaki transform defined by

$$T(v) = v\int_0^\infty e^{-t/v} f(t)dt$$

for $E[f(t)] = T(v)$ [4, 11, 14].

Additionally, the proposed method can be applied to another equation as well.

On the other hand, Euler-Cauchy equation is connected with Black-Scholes terminal value problem, and it play a role to represent the derivative of value of a European call option V with respect to the time t. The problem for the value $V(S,t)$ of a European call option on a security with price S at time t is given as

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \tag{6}$$

with $V(0,t) = 0$, $V(S,t) \sim S$ as $s \to \infty$ and $V(S,t) = \max(S-K, 0)$ for K is strike price. In the equation, $\partial V/\partial t = 0$ is Euler-Cauchy equation.

Example 2.5. (an application in the value of European call option) In Black-Scholes equation, $\partial V/\partial t = 0$ has the solution of the form of $V = S$ or $V = \pounds^{-1}(W^{\frac{2r}{\sigma^2}-1})$ where, $\pounds(V) = W$.

Solution. First, let us divide $\sigma^2/2$ on both sides. This gives

$$S^2\frac{\partial^2 V}{\partial S^2} + \frac{2r}{\sigma^2}S\frac{\partial V}{\partial S} - \frac{2r}{\sigma^2}V = 0$$

for σ is the volatility and for r is the interest rate on a risk-free bond. By the equation (1) of lemma 2.1, we have

$$m = \frac{2r}{\sigma^2} - 1$$

if $2r \geq -\sigma^2$ or $m = -2$ otherwise. Since $V = \pounds^{-1}(W^m)$, we obtain the above result.

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