Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4153–4158
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http://www.ripublication.com/gjpam.htm

The solution of Black-Scholes terminal value problem by means of Laplace transform

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Abstract

We have checked the solution of Black-Scholes terminal value problem which is closely related with call and put option of stock market by means of Laplace transform. The proposed method is simpler than the existing ones, and easier to apply it.

AMS subject classification: 35A22, 44A10.
Keywords: Black-Scholes equation, Laplace transform, call option.

1. Introduction

The Black-Scholes equation governs the price of call and put option, and it has the form of

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0$$

where v is the value of option as a function of stock price s and time t, r is the risk-free interest rate, and σ is the volatility of the stock. If the conditions

$$v(0,t) = 0, \quad \lim_{S\to\infty} v(S,t) = 0$$

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and $v(S, t) = \max(S - K,\ 0)$ are given for the strike price K, then we call the equation with the conditions as the Black-Scholes terminal value problem. Normally, this equation can be changed to heat equation by the transform as

$$t = T - \frac{2T}{\sigma^2},\ S = Ke^x$$

provided $v(S, t) = Kv(s, \tau)$ for the new time τ and price x. By the transform, we have

$$\tau = \frac{\sigma^2}{2}(T - t),\ x = \ln\left(\frac{S}{K}\right).$$

Substituting v_t, v_s and v_{ss} into the given equation, we easily get

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv.$$

In the above process, $v(S, t) = Kv(x, \tau)$ gives

$$Kv(x, 0) = v(S, T) = \max(S - K,\ 0) = \max(Ke^x - K,\ 0)$$

and so, we have $v(x, 0) = \max(e^x - 1,\ 0)$.

In the previous researches, the fractional Black-Scholes equation with boundary condition for a European option pricing dealt with Laplace homotopy perturbation method [9], and Lee [11] applied the modified Laplace transformation method proposed by Sheen, Sloan, and Thomee in [13] to solve the Black-Scholes equation. On the other hand, we have pursued the research on integral transforms to establish the exquisite theory with respect to general integral transform [2, 6–7, 12], and the general story of integral transforms can be found in [2] and [8].

In this article, we have used the original Laplace transform to find the solution of Black-Scholes terminal value problem. The proposed method has a strong point in a sense that it is easy to apply it, and still valid even if the transform is changed to another ones such as Sumudu's or Elzaki's. In case of Elzaki's, the needed formulas are found in [3–5].

2. The solution of Black-Scholes terminal value problem by means of Laplace transform

We would like to check the solution of Black-Scholes terminal value problem by means of integral transform, and the tool used is Laplace's.

Theorem 2.1. The Laplace transform of Black-Scholes terminal value problem

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 v}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv = 0 \tag{1}$$

can be represented by

$$u^2\pounds(v) - uv(x,0) - v_t(x,0) + \frac{1}{2}\sigma^2 us^2\pounds(v_{ss}\cdot t) + ru\pounds(v_s t) - r\pounds(v) = 0$$

for v is a polynomial function.

Proof. First, let us find the Laplace transform of each term.

$$\begin{aligned}
\pounds(v_t) &= u^2\pounds(v) - uv(x,0) - v_t(x,0) \\
\pounds(v_s) &= \int_0^\infty e^{-ut}\frac{\partial v}{\partial s}dt = \frac{\partial}{\partial s}\int_0^\infty e^{-ut}v(s,t)dt = \frac{\partial}{\partial s}\pounds(v) \\
\pounds(v_{ss}) &= \frac{\partial^2}{\partial s^2}\pounds(v) \\
\pounds(s\cdot v_s) &= \int_0^\infty e^{-ut}s\cdot v_s dt \\
&= e^{-ut}\cdot s\cdot v_s t]_0^\infty + u\int_0^\infty e^{-ut}v_s\, tdt = u\pounds(v_s t) \\
\pounds(s^2\cdot v_{ss}) &= \int_0^\infty e^{-ut}s^2\cdot v_{ss}dt \\
&= e^{-ut}\cdot s^2\cdot v_{ss}t]_0^\infty + us^2\int_0^\infty e^{-ut}v_{ss}\, tdt = us^2\pounds(v_{ss}t).
\end{aligned}$$

In the above equations, we can interchange integration and differentiation because of monotone convergence theorem of measure theory [10]. Substituting these values into the equation (1) and organizing the equality, we have

$$u^2\pounds(v) - uv(x,0) - v_t(x,0) + \frac{1}{2}\sigma^2 us^2\pounds(v_{ss}t) + ru\pounds(v_s t) - r\pounds(v) = 0.$$

Although the above v is not a constant, this result is still valid if we endure the difference as much as constant coefficient because v is a polynomial function. The related contents can be found in [5]. ∎

Theorem 2.2. The solution of the transformed equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \qquad (2)$$

can be represented as

$$v = \pounds^{-1}\left[A(\tau)e^{m_1 x} + B(\tau)e^{m_2 x} - \frac{1}{\tau^2+k} + \frac{1}{\tau^2}e^x\right]$$

for all x, and m_i is the roots of $m^2 + (k-1)m - (\tau^2+k) = 0$.

Proof. We would like to approach it by using the method of undetermined coefficients, and it can be replaced with the method of variation of parameter of Lagrange. Since $v(x,0) = \max(e^x - 1,\ 0) = e^x - 1$ on $x \geq 0$, we have

$$\begin{aligned}
£(v_\tau) &= \tau^2 £(v) - v(x,0) = \tau^2 £(v) - (e^x - 1) \\
£(v_{xx}) &= \int_0^\infty e^{-ut} \frac{\partial^2}{\partial x^2} dt = \frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x^2} £(v) \\
£(v_{xx}) &= \frac{\partial}{\partial x} £(v).
\end{aligned}$$

Hence, Laplace transform of (2) is

$$\tau^2 £(v) - (e^x - 1) = \frac{\partial^2}{\partial x^2} £(v) + (k-1)\frac{\partial}{\partial x} £(v) - k£(v)$$

for the strike price k. Organizing this equality, we have

$$\frac{\partial^2}{\partial x^2} £(v) + (k-1)\frac{\partial}{\partial x} £(v) - (\tau^2 + k)£(v) = 1 - e^x. \tag{3}$$

First, let us find a solution of homogeneous equation of (3). Since this is an ODE with constant coefficients, the auxiliary equation has the form of $m^2 + (k-1)m - (\tau^2 + k) = 0$ for $£(v) = e^{mx}$. Let us put the roots of this equation as m_1 and m_2. Then, a general solution is

$$£_h(v) = A(\tau)e^{m_1 x} + B(\tau)e^{m_2 x}.$$

Next, by the method of undetermined coefficients or an inspection, we can choose a particular solution $£_p(v)$ as $£_p(v) = c_1 + c_2 e^x$. Since

$$\frac{\partial}{\partial x} £_p(v) = c_2 e^x = \frac{\partial^2}{\partial x^2} £_p(v),$$

substituting $£_p(v)$ into the equation (3), we have

$$c_2 e^x + (k-1)c_2 e^x - (\tau^2 + k)(c_1 + c_2 e^x) = 1 - e^x.$$

Organizing this equality, we have

$$-c_1(\tau^2 + k) - c_2 \tau^2 e^x = 1 - e^x$$

for all x. Hence,

$$c_1 = -\frac{1}{\tau^2 + k},\ c_2 = \frac{1}{\tau^2}.$$

This gives the solution

$$£(v) = A(\tau)e^{m_1 x} + B(\tau)e^{m_2 x} - \frac{1}{\tau^2 + k} + \frac{1}{\tau^2} e^x. \tag{4}$$

It is clear that (4) is a solution of (3) because of m_i is a roots of

$$m^2 + (k-1)m - (\tau^2 + k) = 0.$$

On the other hand, Black-Scholes terminal value problem has to do with Euler-Cauchy equation, and it represents the derivative of value of a call and put option v with respect to the time t. When $\partial v/\partial t = 0$, the v has relative extremum, and this means that a change of the option value appears at the point. ■

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