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Well posedness and solitons stability for a 1D Benney-Luke Model of Higher order

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Abstract

We study the local and global well posedness associated a 1D Benney-Luke model of higher order that models long water with small amplitude. We show that local mild solutions are already global mild solutions, in the case we have an appropriate Hamiltonian structure. We establish orbital stability of solitons using strongly the variational characterization of the ground state solutions and the invariance of some special regions under the flow associated with the generalized Benney-Luke model considered.

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1. Introduction

In this paper we are interested in study the local and global well posedness, and the orbital stability of solitons (travelling wave solution of finite energy) associated with a Benney-Luke model of higher order (m-Benney-Luke equation),

$$M_{1,m}\Phi_{tt} - M_{2,m}\Phi_{xx} = \partial_x \left(G_{1,m}\right) + \partial_t \left(G_{2,m}\right) \tag{1}$$

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where $M_{1,m}$, $M_{2,m}$ are operators of degree $2m$ of the form

$$M_{i,m} = \sum_{j=0}^{m} (-1)^j a_{i,j} \partial_x^{2j}, \quad a_{i,j} \geq 0, \tag{2}$$

and the nonlinear terms $G_{i,m}$ for $i = 1, 2$ are homogeneous functions of degree $n+1$ such that $G_{1,m}$ could depend on the variables Φ_t, Φ_x, Φ_{xx}, Φ_{xxx} and $G_{2,m}$ could depend on the variables Φ_x, Φ_{xx}, Φ_{xxx} (see [2] for more details on the model). The motivation in looking to this model comes with the recent development for generalized KdV models and generalization of Benney-Luke models. For instance, L. Paumond in [3] established that the evolution of two-dimensional water waves with surface tension can be reduced to studying the solution $\Phi(x,t)$ of the dispersive equation

$$\begin{aligned}\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \epsilon(B\Phi_{xxxxtt} - A\Phi_{xxxxxx}) \\ + \epsilon\big(n\Phi_t\Phi_x^{n-1}\Phi_{xx} + 2\Phi_x^n\Phi_{xt}\big) = 0,\end{aligned} \tag{3}$$

where ϵ represents the amplitude parameter (nonlinear coefficient), μ represents the long-wave parameter (dispersion coefficient), and the real coefficients a, b, A, B are such that

$$a - b = \sigma - \frac{1}{3} \quad \text{and} \quad A - B = \frac{1}{45} + (a-b)\left(b - \frac{1}{3}\right) + \frac{1}{6}\left(\frac{1}{2} - b\right)$$

with $a, A > 0$ and σ^{-1} is named the Bond number. It is important to point out that the Benney-Luke-Paumond equation (3) corresponds to a generalization for values of $\sigma \sim \frac{1}{3}$ of the Benney-Luke model derived by J. Quintero and R. Pego ([9])

$$\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \epsilon\big(n\Phi_t\Phi_x^{n-1}\Phi_{xx} + 2\Phi_x^n\Phi_{xt}\big) = 0, \tag{4}$$

which in the case of $\sigma = \frac{1}{3}$ means that there is no dispersion effect in equation (4). As shown by J. Quintero and R. Pego in [9] for the Benney-Luke model (4) and the KdV equation, the Benney-Luke-Paumond equation (3) reduces in a suitable limit to the fifth order KdV type equation known as the Kawahara equation for $a - b = \theta\mu$ and $\epsilon = \mu^2$ (see Paumond ([3]). More concretely, if we set $\Phi(x,t) = f(X,\tau)$ where $X = x - t$ and $\tau = \frac{\epsilon}{2}t$, we find that $f = \eta_X$ satisfies the Kawahara equation (up to order μ)

$$\eta_\tau - \theta\partial_X^3\eta + \frac{1}{45}\partial_X^5\eta + (n+2)\eta^n\partial_X\eta = 0, \tag{5}$$

where η already represents the wave elevation. Regarding the local well posedness and stabilty/instability of solitons for generalized KdV equations, there are some works dealing with this topic (see for example [11], [13], [12], [21], [8] [15], [14], [22]). In particular, J. Albert in [11] analyzed the stability issue of certain solitary-wave solutions for a general model (like the generalized Kawahara (5)) of the form

$$u_t + u^n u_x - (N_m(u))_x = 0,$$

where N_m is a differential operator of order $2m$. For this model, solitary wave solutions have the form $\varphi(x) = (\operatorname{sech}(x))^r$, where $r = \frac{2n}{m}$. Under certain conditions, J. Albert in [11] showed orbital stability of this traveling wave solutions for n integer and $1 \le n < 4m$. This result for N_m associated with the Kawahara equation implies orbital stability for $1 \le n < 8$ ($m = 2$). On the other hand, J. Angulo showed in [12] a result of the instability of solitary traveling-wave solutions associated with the generalized fifth-order KdV equation of the form

$$u_t + u_{xxxxx} + bu_{xxx} = (G(u, u_x, u_{xx}))_x,$$

where

$$G(q, r, s) = F_q(q, r) - rF_{qr}(q, r) - sF_{rr}(q, r)$$

for some $F(q, r)$ which is homogeneous of degree $n+2$ for some $n \ge 1$. In this general case, the solitary wave was obtained by solving a constrained minimization problem in $H^2(\mathbb{R})$ which is based on results derived by S. Levandosky (see [13]). Note that taking $F(q, r) = dq^{n+2}$ for an appropriate constant d, we obtain a form of the generalized Kawahara equation (5). In this case, J. Angulo in [12] established orbital instability of solitary waves for $n > 8$.

We see that the Benney-Luke models (4) and (3) has a special structure since

$$\begin{aligned}
&\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) \\
&\quad = \partial_x \left(G_{1,m}(\Phi_t, \Phi_x)\right) + \partial_t \left(G_{2,m}(\Phi_x)\right) \\
&\Phi_{tt} - \Phi_{xx} + \mu(a\Phi_{xxxx} - b\Phi_{xxtt}) + \epsilon(B\Phi_{xxxxtt} - A\Phi_{xxxxxx}) \\
&\quad = \partial_x \left(G_{1,m}(\Phi_t, \Phi_x)\right) + \partial_t \left(G_{2,m}(\Phi_x)\right),
\end{aligned}$$

where $G_{1,m}$ and $G_{2,m}$ are homogeneous function of degree $n+1$ given by

$$G_{1,m}(p, q) = -\epsilon pq^n, \quad G_{2,m}(q) = -\frac{\epsilon}{n+1} q^{n+1}.$$

We note in this case that there are homogeneous functions $F(p, q)$ and $H(q)$ of degree $n+2$ such that

$$G_{1,m}(p, q) = F_p(p, q) + F_q(p, q), \quad G_{2,m} = H_q(q).$$

In fact, using characteristics we see that F and H must be

$$F(p, q) = -\frac{\epsilon pq^{n+1}}{n+1} + \frac{\epsilon q^{n+2}}{(n+1)(n+2)}, \quad H(q) = -\frac{\epsilon q^{n+2}}{(n+1)(n+2)}.$$

As done for KdV generalized model (see [11], [12], [13], [14], [21], [15], [22], among others), we will assume hereafter that $G_{1,m}$ and $G_{2,m}$ are homogeneous functions in one of the following classes:

[C1] For $m \geq 1$,

$$\begin{aligned} G_{1,m}(p,q) &= F_p(p,q) + F_q(p,q) \\ G_{2,m}(q) &= H_q(q). \end{aligned}$$

[C2] For $m \geq 3$,

$$\begin{aligned} G_{1,m}(p,q,r,s) &= F_p(p,q,r) + F_q(p,q,r) - (rF_{rp}(p,q,r) \\ &\quad + rF_{rq}(p,q,r) + sF_{rr}(p,q,r)) \\ G_{2,m}(q,r,s) &= H_q(q,r) - (rH_{rq}(q,r) + sH_{rr}(q,r)) \end{aligned}$$

where F and H are homogeneous function of degree $n+2$ such that

$$F(p,q,\lambda r) = \lambda^{n_1} F(p,q,r), \quad F(\lambda p, \lambda q, r) = \lambda^{n_2} F(p,q,r), \tag{6}$$

$$H(q,\lambda r) = \lambda^{n_1} H(q,r), \qquad H(\lambda q, r) = \lambda^{n_3} H(q,r), \tag{7}$$

with $n_1+n_2 = n+2$ and $n_1+n_3 = n+2$ (if F and H do not depend on r, then $n_1 = 0$). We can see directly that $G_{1,m}$ and $G_{2,m}$ are homogeneous functions of degree $n+1$.

For this particular model, the local well posedness and uniqueness for the m-Benney-Luke model follow by standard fixed point argument in a Sobolev type space. The global well posedness result depends strongly in finding conditions on the nonlinear homogeneous functions $G_{i,m}$ for $i = 1, 2$ to guarantee that the Hamiltonian is conserved in time on mild solutions, which is natural when $G_1(p,q) = -pq^n$ and $G_2(q) = -\dfrac{1}{n+1}q^{n+1}$. Before we go further, we point out that the global well-posedness for generalized KdV model is difficult for general homogeneous functions G. For instance, in case of the Kawahara model

$$\begin{aligned} &u_t + u_{xxxxx} + bu_{xxx} \\ &\quad = (G(u,u_x,u_{xx}))_x, \quad G(q,r,s) = W_q(q,r) - rW_{rq}(q,r) - sW_{rr}(q,r), \end{aligned}$$

we know that this model has a Hamiltonian structure but the global well posedness is known only in special cases (for example $W(q,r) = W(q)$ (see ([12], [25]). Moreover, G. Ponce in [26] showed a well-posedness theory in $H^s(\mathbb{R})$ with $s \geq 4$ for $W(q,r) = qr^2 - q^3$ and $G(q,r,s) = -3q^2 - r^2 - 2qs$. On the other hand, C. Kenig, G. Ponce and L. Vega in ([27]) showed that the following higher-order nonlinear dispersive equation

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \cdots, \partial_x^{2j} u) = 0, \quad x, t \in \mathbb{R}$$

where P is a polynomial having no constant or linear terms, is locally well posed in the weighted Sobolev spaces $H^s(\mathbb{R}) \cap L^2(|x|^m\, dx)$ for some s sufficiently large and $m \in \mathbb{N}$.

Regarding stability issues, as shown by J. Albert in [11] for the generalized KdV equation of order m, we want to point out that in the special case $G_1(p,q) = -pq^n$ and $G_2(q) = -\dfrac{1}{n+1}q^{n+1}$, there exists some sort of orbital stability of travelling waves for $n < 4m$ and instability for $n > 4m$, when $c \to 1^-$. On the other hand, for the 3-Benney-Luke model, taking $a_{1,0} = a_{2,0} = 1$, $a_{2,1} = a$, $a_{2,2} = A$, $a_{2,3} = F$, $a_{1,1} = b$, $a_{1,2} = B$, and $a_{1,3} = G$, where a, b, A, B, F, G satisfy some conditions (see [1]), we have for $c^* = \sqrt{\min\left\{1, \frac{a}{b}, \frac{A}{B}\right\}}$, for $\frac{F}{G} > \max\left\{1, \frac{a}{b}, \frac{A}{B}\right\}$, and for $\frac{AF}{BG} > 1$ that

1. if $1 \le n \le 2$, traveling waves are orbitally stable for $|c| < c^*$.
2. if $2 < n \le 12$, $a > b$, and either $A > B$ or $A < B$, with $A - B$ small enough, there is $0 < c_0(n) < 1$ such that travelling waves are orbitally stable, for $c_0(n) < |c| < c^*$.

For analogous results in the case $m = 1$ see [4], and for $m = 2$ see [8].

The paper is organized as follows. In section 2, we show local well-posedness and uniqueness for the Cauchy problem associated with the m-Benney-Luke equation. In section 3, we establish global well-posedness and uniqueness for the m-Benney-Luke equation based on the energy functional and the conservation in time the Hamiltonian for a special class of $G_{i,m}$, which includes the case $G_1(p,q) = -pq^n$ and $G_2(q) = -\dfrac{1}{n+1}q^{n+1}$. In Section 4, we establish the Hamiltonian structure and give a variational characterization of solitons. We also obtain the convexity condition required in Grillakis *et al.* approach to prove orbital stability. In Section 5, we prove the orbital stability of the ground state solutions of the m-Benney-Luke model when the wave speed c is near 0^+ or 1^- and $1 \le n < 4$.

2. Local Well-Posedness and Uniqueness

In this section, we establish the local well posedness for the m-Benney-Luke equation. The notion of well posedness to be used here is in the sense of Kato: consider an abstract Cauchy problem

$$\frac{du}{dt} = f(u), \qquad u(0) = u_0. \tag{8}$$

Suppose that there are two Banach spaces $Y \hookrightarrow X$, with the embedding continuous, such that f is continuous from Y to X. We say that the problem (8) is *locally well-posed* in Y, if for each $u_0 \in Y$ there are a real number $T = T(u_0) > 0$ and a unique function $u \in C([0,T], Y)$ satisfying the integral equation associated to (8), depending continuously on the initial data in the sense that the solution map $u_0 \mapsto u$ is continuous: if $u_n \to u$ in Y and $T' \in (0,T)$, then for n large enough $u_n \in C\left([0,T'], Y\right)$ and,

$$\lim_{n\to\infty} \sup_{[0,T']} \|u_n(t) - u(t)\|_Y = 0.$$

We say that the problem is *globally well posed* in Y, if T can be taken arbitrarily large. We recall that if E is a Banach space then $C([0,T],E)$ denote the space of continuous functions defined in $[0,T]$ with values in E.

In order to discuss the local well posedness and uniqueness associated with the m-Benney-Luke model of higher order (1) we perform the following change of variables: $q=\Phi_x$ and $r=\Phi_t$. In the first class **[C1]**, we note that equation (1) formally becomes into the following system

$$q_t = r_x \tag{9}$$

$$r_t = M_{1,m}^{-1}\left(M_{2,m}(q_x) + \partial_x\left(G_{1,m}(r,q)\right) + DG_{2,m}(q).r_x\right), \tag{10}$$

and in the second class **[C2]**, r_t becomes

$$r_t = M_{1,m}^{-1}\left(M_{2,m}(q_x) + \partial_x\left(G_{1,m}(r,q,q_x,q_{xx})\right) + DG_{2,m}(q,q_x,q_{xx}).(r_x,r_{xx},r_{xxx})\right), \tag{11}$$

where the linear operators $M_{l,m}$ and $M_{l,m}^{-1}$ $(l=1,2)$ are defined using the Fourier transform:

$$\widehat{M_{l,m}} = \sum_{j=0}^{m} a_{l,j}\zeta^{2j}, \quad \widehat{M_{l,m}^{-1}} = \frac{1}{\sum_{j=0}^{m} a_{l,j}\zeta^{2j}} \tag{12}$$

Using this notation, we see that the system can be viewed as the first order system

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = M\begin{pmatrix} q \\ r \end{pmatrix} + G\begin{pmatrix} q \\ r \end{pmatrix}, \tag{13}$$

where

$$M = \begin{pmatrix} 0 & \partial_x \\ M_{1,m}^{-1}M_{2,m}\partial_x & 0 \end{pmatrix}$$

and with G in the first class **[C1]** as

$$G\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ M_{1,m}^{-1}\left(\partial_x\left(G_{1,m}(r,q)\right) + DG_{2,m}(q).r_x\right) \end{pmatrix}$$

and with G in the second class **[C2]** as

$$G\begin{pmatrix} q \\ r \end{pmatrix} = \begin{pmatrix} 0 \\ M_{1,m}^{-1}\left(\partial_x\left(G_{1,m}(r,q,q_x,q_{xx})\right) + DG_{2,m}(q,q_x,q_{xx})(r_x,r_{xx},r_{xxx})\right) \end{pmatrix}.$$

We want to point out that in the variables $q=\Phi_x$ and $r=\Phi_t$, we see that in these new variables the quantity

$$\mathcal{M}(q)(x,t) = \int_{\mathbb{R}} q(t,x)\,dx$$

is conserved in time for classical solutions and even for mild solutions. So, if we consider the Cauchy problem associated with the initial data in a appropriate Sobolev space such that

$$\widehat{q_0}(0)) = \int_{\mathbb{R}} q_0(x)\,dx, \tag{14}$$

then we have that

$$\widehat{q}(t,\xi) = \int_{\mathbb{R}} q(x,t)\,dx = 0,$$

as long as the solutions exists for t. Now, it is known that if

$$\dot{H}^r := H^r \bigcap \{f \in H^r : \widehat{f}(0) = 0\},$$

then there is an onto linear map $\partial_x^{-1} : \dot{H}^r \to \dot{H}^{r+1}$ defined via the Fourier transform by

$$\widehat{\partial_x^{-1}(f)}(\xi) = \frac{\widehat{f}(\xi)}{i\xi}.$$

Moreover, for a given function $q \in \dot{H}^r$, the function $\Phi = \partial_x^{-1}q \in \mathcal{V}^{r+1}$ is such that $q = \Phi_x$ and $r = \Phi_t$ where

$$\mathcal{V}^{r+1} = \{f \in \mathcal{S} : f_x \in H^r\}.$$

So, we are able to solve the Cauchy problem associated with the m-Benney-Luke model (1) by solving the Cauchy problem associated with systems (9)-(10) or (9)-(11), with initial condition satisfying (15). So, we focus in the local well posedness for the Cauchy problem associated with system (9)-(10) or (9)-(11).

As usual for evolution equations, the local well posedness follows after solving the linear homogeneous initial value problem, using Duhamel principle (constant variation formula) and nonlinear estimates to solve the nonlinear initial value problem. First, we describe the group $T(t)$ associated with the linear problem

$$\begin{pmatrix} q \\ r \end{pmatrix}_t + M\begin{pmatrix} q \\ r \end{pmatrix} = 0. \tag{15}$$

A simple calculation shows that the unique solution of the linear problem (15) with the initial condition

$$(q(0,\cdot), r(0,\cdot)) = (q_0, r_0) = \Psi_0 \in \mathcal{X}^s := H^s(\mathbb{R}) \times H^s(\mathbb{R}), \tag{16}$$

is given by

$$\Psi(t) = (q(t), r(t)) = T(t)\Psi_0,$$

where $T(t)$ is defined in terms of the Fourier transform as

$$\widehat{T}(t) = \begin{pmatrix} \cos(\xi\Lambda(\xi)t) & i\dfrac{\sin(\xi\Lambda(\xi)t)}{\Lambda(\xi)} \\ i\Lambda(\xi)\sin(\xi\Lambda(\xi)t) & \cos(\xi\Lambda(\xi)t) \end{pmatrix},$$

and $\Lambda^2(\xi) = \dfrac{\widehat{M_{2,m}}}{\widehat{M_{1,m}}}$. It is convenient to use the following notation

$$Q(t)(\widehat{q}, \widehat{r}) = \big(Q_1(t), Q_2(t)\big)(\widehat{q}, \widehat{r}),$$

where

$$Q_1(t)(\widehat{q}, \widehat{r}) = \cos(\xi\Lambda(\xi)t)\widehat{q} + i\frac{\sin(\xi\Lambda(\xi)t)}{\Lambda(\xi)}\widehat{r},$$
$$Q_2(t)(\widehat{q}, \widehat{r}) = i\Lambda(\xi)\sin(\xi\Lambda(\xi)t)\widehat{q} + \cos(\xi\Lambda(\xi)t)\widehat{r}.$$

Then we have that

$$T(t)(\Psi) = \left(\mathcal{F}^{-1}\big(Q_1(t)(\widehat{q}, \widehat{r})\big), \mathcal{F}^{-1}\big(Q_2(t)(\widehat{q}, \widehat{r})\big)\right).$$

On the other hand, it is known that the Duhamel's principle implies that if Ψ is a solution of (13) with the initial condition (16), then this solution satisfies the integral equation

$$\Psi(t) = T(t)\Psi_0 - \int_0^t T(t-\tau)\, G(\Psi)(\tau)\, d\tau =: \Theta(\Psi)(t). \tag{17}$$

Hereafter, we refer to a $\Psi \in C([0, T], \mathcal{X}^s)$ satisfying the integral equation (17) as a mild solution for the initial value problem (15)-(16). Now, we will establish the existence of mild solutions. For this, we use some linear and nonlinear estimates. Let us start with the following result on the group $(T(t))_{t\in\mathbb{R}}$.

Lemma 2.1. Suppose $s \in \mathbb{R}$. Then $T(t)$ is a bounded linear operator from $\mathcal{X}^s$ into $\mathcal{X}^s$. Moreover, there exists $K_1(m) > 0$ such that for all $t \in \mathbb{R}$,

$$\|T(t)(\Psi)\|_{\mathcal{X}^s} \le K_1 \|\Psi\|_{\mathcal{X}^s}.$$

Proof. We first note that Λ is bounded above and below. In fact, we have the following bounds for $\widehat{M_{l,m}}$,

$$\lambda_1(m)\sum_{j=0}^{m}\zeta^{2j} \le |\widehat{M_{l,m}}| \le \lambda_2(m)\sum_{j=0}^{m}\zeta^{2j}, \tag{18}$$

where

$$\lambda_1(m) = \min_{0\le j\le m}\{a_{l,j},\ l=1,2\},\quad \lambda_2(m) = \max_{0\le j\le m}\{a_{l,j},\ l=1,2\}.$$

These estimates imply that the Λ has the following bounds

$$L^{-1}(m) \le |\Lambda(\zeta)| \le L(m), \quad L(m) = \frac{\lambda_2(m)}{\lambda_1(m)}. \tag{19}$$

On the other hand, using previous fact and that the functions sine and cosine are bounded, we see directly for $l = 1, 2$ and for some positive constant $L_1(m)$ that

$$|Q_l(t)(\widehat{q}, \widehat{r})| \leq L_1(m) \left(|\widehat{q}|^2 + |\widehat{r}|^2\right)^{\frac{1}{2}}.$$

We conclude from this fact that for some positive constant $K_1(m)$,

$$\begin{aligned}\|T(t)(\Psi)\|_{\mathcal{X}^s} &\leq \|(1 + |\zeta|^2)^{\frac{s}{2}}|Q_1(t)(\widehat{q}, \widehat{r})|\|_{L^2} + \|(1 + |\zeta|^2)^{\frac{s}{2}}|Q_1(t)(\widehat{q}, \widehat{r})|\|_{L^2} \\ &\leq K_1(m)\|(1 + |\zeta|^2)^{\frac{s}{2}}|\widehat{q}|\|_{L^2} + \|(1 + |\zeta|^2)^{\frac{s}{2}}|\widehat{r}|\|_{L^2} \\ &\leq K_1(m)\|\Psi\|_{\mathcal{X}^s},\end{aligned}$$

as desired. ■

We now are ready to establish the nonlinear estimates, for which we will use a *Sobolev Multiplication Law* (see Corollary 3.16 in [10] by T. Tao).

Lemma 2.2. (*Sobolev Multiplication Law*) Let $d \geq 1$ and let s_1, s_2, t be such that either

1. $s_1 + s_2 \geq 0,\ \ t \leq s_1 + s_2,\ \ t < s_1 + s_2 - \dfrac{d}{2}$, or
2. $s_1 + s_2 > 0,\ \ t < s_1 + s_2,\ \ t \leq s_1 + s_2 - \dfrac{d}{2}$.

Then, we have that

$$\|\psi\varphi\|_{H^t(\mathbb{R}^d)} \leq \|\psi\|_{H^{s_1}(\mathbb{R}^d)}\|\varphi\|_{H^{s_2}(\mathbb{R}^d)}. \tag{20}$$

Hereafter, $s \in \mathbb{R}$ satisfies one of the following conditions:

(s_1) $s \geq \dfrac{1}{2}$, for $G_{l,m}$ in the first class **[C1]** for $m \geq 1$, ($l = 1, 2$) or

(s_2) $s \geq \dfrac{5}{2}$, for $G_{l,m}$ in the second class **[C2]** for $m \geq 2$, ($l = 1, 2$).

Theorem 2.3. Assume that $s \in \mathbb{R}$ satisfies either (s_1) or (s_2). If $\begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \in \mathcal{X}^s$ then there exist $-\infty \leq T_0 < 0 < T_1 \leq \infty$ such that the initial value problem (15) with inital condition (q_0, r_0) has a unique mild solution $(q, r) \in C((T_0, T_1) : \mathcal{X}^s(\mathbb{R}))$.

Proof. First we note that the operator M maps the Hilbert space $H^{s+1}(\mathbb{R}) \times H^{s+1}(\mathbb{R})$ into the space $H^s(\mathbb{R}) \times H^s(\mathbb{R})$. Moreover, we have that

$$M \in \mathcal{L}(\mathcal{X}^{s+1}(\mathbb{R}), \mathcal{X}^s(\mathbb{R})).$$

and that M is the infinitesimal generator of the bounded C_0-group $T(t)$ on $\mathcal{X}^s$. Now, we need to analyzed the nonlinear term. We must recall that $G_{1,m}$ and $G_{2,m}$ are homogeneous function of order $n+1$ because F and G are homogeneous of degree $n+2$. Assume that s satisfies condition (s_1), i.e., $G_{l,m}$ is in the first class for $m \geq 1$, then we have that $G_{1,m}(r,q), G_{2,m}(q) \in H^s$ for $s > \frac{1}{2}$, since $H^s(\mathbb{R})$ is an algebra with $s > \frac{1}{2}$. Then using the Sobolev Multiplication Law with $t = s-1$, $s_1 = s$ and $s_2 = s-1$, we have that

$$\partial_x G_{1,m}(r,q), \quad DG_{2,m}(q).r_x \in H^{s-1}(\mathbb{R}),$$

and so, we get that

$$M_{1,m}^{-1}\left[\partial_x G_{1,m}(r,q) + DG_{2,m}(q).r_x\right] \in H^{s-1+2m}(\mathbb{R}) \subset H^{s+1}(\mathbb{R}),$$

Now assume that s satisfies condition (s_2), i.e., $G_{l,m}$ is in the second class for $m \geq 2$, then we have that $G_{1,m}(r,q,q_x,q_{xx}) \in H^{s-2}$, since $H^{s-2}(\mathbb{R})$ is an algebra with $s > \frac{5}{2}$. Now, using the Sobolev Multiplication Law with $t = s-3$, $s_1 = s-3$ and $s_2 = s-2$, we have that $DG_{2,m}(q,q_x,q_{xx}).(r_x,r_{xx},r_{xxx}) \in H^{s-3}$ for $s > \frac{5}{2}$, which implies that

$$\begin{aligned} &M_{1,m}^{-1}\left[\partial_x\left(G_{1,m}(r,q,q_x,q_{xx})\right) + DG_{2,m}(q,q_x,q_{xx}).(r_x,r_{xx},r_{xxx})\right] \\ &\quad \in H^{s-3+2m}(\mathbb{R}) \subset H^{s+1}(\mathbb{R}) \end{aligned}$$

In other words, the nonlinear part G has a smoothing property since we have that G takes values in $\mathcal{X}^{s+1}$. We also have that G is locally Lipschitz on $\mathcal{X}^s$. Now, by Theorem 6.1.4 from A. Pazy [23], we have that for $(q_0,r_0) \in \mathcal{X}^s(\mathbb{R})$, there exist $-\infty \leq T_0 < 0 < T_1 \leq \infty$ such that the initial value problem associated with (15) has a unique local mild solution $\Psi = (q,r)^t$ on the interval (T_0,T_1) with values in $\mathcal{X}^s$ such that $q,r \in C([T_0,T_1]; H^s(\mathbb{R}))$. Using the homogeneity of $G_{i,m}$ for $i = 1,2$, we see directly for $t \in [T_0,T_1] = I_{max}$ that,

$$\begin{aligned} &\left\| -\int_0^t T(t-\tau)\, G(\Psi)(\tau)\, d\tau \right\|_{\mathcal{X}^s} \\ &\quad \leq T_1 K_1 K_2 \max_{t\in I_{max}} (\|\Psi(\tau)\|_{\mathcal{X}^s} + \|\Psi_1(\tau)\|_{\mathcal{X}^s})^n \max_{t\in I_{max}} \|\Psi - \Psi_1\|_{\mathcal{X}^s} \end{aligned}$$

which implies that

$$\begin{aligned} &\|\Theta(\Psi(t)) - \Theta(\Psi_1(t))\|_{\mathcal{X}^s} \leq K_1 \|\Psi(0) - \Psi_1(0)\|_{\mathcal{X}^s} \\ &\quad + T_1 K_1 K_2 \max_{t\in I_{max}} (\|\Psi(\tau)\|_{\mathcal{X}^s} + \|\Psi_1(\tau)\|_{\mathcal{X}^s})^n \max_{t\in I_{max}} \|\Psi - \Psi_1\|_{\mathcal{X}^s}, \end{aligned}$$

meaning that the solution map $\Theta : \mathcal{X}^s \to C((T_0,T_1) : \mathcal{X}^s)$ is continuous. ■

Moreover, we have the following result,

Corollary 2.4. Assume that $s \in \mathbb{R}$ satisfies either (s_1) or (s_2). If $\begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \in \mathcal{X}^s$ with $q \in \dot{H}^s$ then there exist $-\infty \leq T_0 < 0 < T_1 \leq \infty$ such that the initial value problem (15) with initial condition (q_0,r_0) has a unique mild solution $(q,r) \in C((T_0,T_1) : \dot{H}^s \times H^s(\mathbb{R}))$.

Moreover, if $\Phi_0 \in \mathcal{V}^{s+1}$ and $r \in H^s(\mathbb{R})$, then there exists $T = T(\Phi_0, r_0) > 0$ such that the integral equation associated with the m-Benney-Luke equation (1) has a unique solution $\Phi \in C^0([0, T_0], \mathcal{V}^{s+1})$ with

$$\Phi_t \in C^0([0, T_0], H^s(\mathbb{R})) \bigcap C^1([0, T_0], H^{s-1}(\mathbb{R}))$$

that satisfies the initial condition

$$\partial_x \Phi(0, \cdot) = \partial_x \Phi_0, \quad \partial_t \Phi(0, \cdot) = r_0.$$

3. Energy and Global Well-Posedness

As we saw in the discussion above, global well-posedness for generlized KdV models for general homogeneous functions G could be a difficult task, even though in the case of having a Hamiltonian structure. So, global well-posedness for the m-Benney-Luke model needs a non trivial generalization of the relationship between the homogeneous functions $G_{1,m}$ and $G_{2,m}$. As noted in the introduction, in the case of the Benney-Luke equation ($m = 1$) and Benney-Luke Paumond equation ($m = 2$), there is a special relationship between $G_{1,m}(p, q) = -\epsilon p q^n$ and $G_{2,m}(q) = -\dfrac{\epsilon}{n+1} q^{n+1}$ given by

$$G_{1,m}(p, q) = p G'_{2,m}(q), \tag{21}$$

which is clever to assure the existence of a Hamiltonian structure, and so global existence in the energy space for $m = 1, 2$. For instance, assume that $G_{1,m}$ and $G_{2,m}$ are in the first class **[C1]** satisfying condition (21), then we have that H is defined by integrating $G_{2,m}$ with respect to q. Moreover, to define F, we need to solve the differential equations

$$G_{1,m}(p, q) = F_p(p, q) + F_q(p, q),$$

whose solution via characteristics is given by

$$F(p, q) = \frac{1}{2} \int_0^{p+q} G_{1,m}\left(\frac{t - p + q}{2}, \frac{t + p - q}{2}\right) dt.$$

Now, assume that $G_{1,m}$ and $G_{2,m}$ are in the second class **[C2]**, a simple argument shows that condition (21) holds only if $G_{1,m} = G_{2,m} = 0$. For instance, assume that $G_{1,m}$ and $G_{2,m}$ are in the second class **[C2]**, satisfying analogous condition as (21) given by

$$G_{1,m}(p, q, r, s) = p \partial_q G_{2,m}(q, r, s). \tag{22}$$

Then, it is straightforward to see that F and H must be independent of r. In fact, first note that for any homogeneous function $F_1(p, q)$ of degree $n + 1$, the homogeneous function of degree $n + 2$ given by $F(p, q, r) = r F_1(p, q)$ (homogeneous of degree 1 with respect to r) is such that

$$\begin{aligned} G_1(p,q,r,s) &= F_p(p,q,r) + F_q(p,q,r) \\ &\quad - \left[rF_{rp}(p,q,r) + rF_{rq}(p,q,r) + sF_{rr}(p,q,r) \right] \\ &= r\left(\partial_p F_1(p,q) + \partial_q F_1(p,q)\right) - r\left(\partial_p (F_1(p,q) + \partial_q F_1(p,q)\right) \\ &= 0, \end{aligned}$$

So, if F and H are homogeneous of degree $n+2$ with degree $n_1 = 1$ in the r variable, we have that $G_{1,m} = G_{2,m} = 0$. Thus we are allowed to assume that the homogeneity of F and H with respect to r must be $n_1 > 1$. So, using homogeneity in the (p, q) variable, we have that the equation (22) takes the form

$$(1-n_1)(F_p(p,q,r)+F_q(p,q,r))-sF_{rr}(p,q,r) = p((1-n_1)H_{qq}(q,r)-sH_{rrq}(q,r)). \tag{23}$$

Since F is independent of s, we conclude that $F_{rr}(p,q,r) = pH_{rrq}(q,r)$, which means that

$$F(p,q,r) = pH_q(q,r) + F_1(p,q)r + F_0(p,q).$$

Using homogeneity in the r variable and that $F_1(p,q)r$ is homogeneous of degree 1 with respect to r, we must have $F_1(p,q) = F_0(p,q) = 0$. So, plugging this in equation (23), we have that

$$pH_{qq}(q,r) + H_q(q,r) = pH_{qq}(q,r) \quad \Leftrightarrow \quad H_q(q,r) = 0 \quad \Leftrightarrow \quad F(p,q,r) = 0.$$

showing that $F(p,q,r) = H(q,r) = 0$, which implies that $G_{1,m} = G_{2,m} = 0$. This observation shows for $G_{1,m}$ and $G_{2,m}$ in the second class **[C2]** with $m \geq 2$ that to have a Hamiltonian structure, and so global well posedness for the Cauchy problem of the m-Benney-Luke equation, requires a more complicated relationship between $G_{1,m}$ and $G_{2,m}$ than condition (21), for $G_{1,m}$ and $G_{2,m}$ in the first class **[C1]**.

Hereafter, we assume that the relationship between $G_{1,m}$ and $G_{2,m}$ in the second class **[C2]** for $m \geq 3$ is given by

$$G_{1,m}(r,\widetilde{q}) = r\partial_1 G_{2,m}(\widetilde{q}) - \partial_x\left(r\partial_2 G_{2,m}(\widetilde{q})\right) + \partial_{xx}\left(r\partial_3 G_{2,m}(\widetilde{q})\right), \quad \widetilde{q} = (q, q_x, q_{xx}). \tag{24}$$

We note that this assumption corresponds to condition (21) for homogeneous functions $G_{1,m}$ and $G_{2,m}$ in the first class **[C1]** for $m \geq 1$.

Under the assumption (24), we see that the energy $\mathcal{E}_m$ associated with the m-Benney-Luke model is given by

$$\mathcal{E}_m\begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}} M_{1,m}(r)r + M_{2,m}(q)q\,dx. \tag{25}$$

is conserved in time along classical and mild solutions of the system (13). In fact, using that $q_t = r_x$ we have that

$$\frac{d}{dt}\mathcal{E}_m\begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{2}\int_{\mathbb{R}} \left(M_{1,m}(r_t)r + M_{1,m}(r)r_t + M_{2,m}(q_t)q + M_{2,m}(q)q_t\right)\, dx$$
$$= \int_{\mathbb{R}} \left(M_{1,m}(r_t)r + M_{2,m}(q)r_x\right)\, dx.$$

We only verify for the second class **[C2]** of homogenous functions $G_{l,m}$. Recall that

$$M_{1,m}(r_t) = M_{2,m}(q_x) + \partial_x\left(G_{1,m}\left(r, q, q_x, q_{xx}\right)\right) + DG_{2,m}\left(q, q_x, q_{xx}\right)\left(r_x, r_x, r_{xxx}\right).$$

So, using this in previous equation, we get that

$$\frac{d}{dt}\mathcal{E}_m\begin{pmatrix} q \\ r \end{pmatrix} = \int_{\mathbb{R}}(M_{2,m}(q)r_x + (M_{2,m}(q_x) + \partial_x(G_{1,m}(r, q, q_x, q_{xx})) + DG_{2,m}\left(q, q_x, q_{xx}\right)\left(r_x, r_{xx}, r_{xxx}\right))r)\, dx. \tag{26}$$

On the other hand, if $\widetilde{q} = (q, q_x, q_{xx})$, using the relationship between $G_{1,m}$ and $G_{2,m}$ given by (24), we have that

$$\int_{\mathbb{R}} \left(DG_{2,m}(\widetilde{q})(r_x, r_{xx}, r_{xxx})\right) r\, dx$$
$$= \int_{\mathbb{R}} \left(\partial_1 G_{2,m}(\widetilde{q})r_x r + \partial_2 G_{2,m}(\widetilde{q})r_{xx}r + \partial_3 G_{2,m}(\widetilde{q})r_{xxx}r\right)\, dx$$
$$= \int_{\mathbb{R}} \left(r\partial_1 G_{2,m}(\widetilde{q}) - \partial_x\left(r\partial_2 G_{2,m}(\widetilde{q})\right) + \partial_{xx}\left(r\partial_3 G_{2,m}(\widetilde{q})\right)\right) r_x\, dx$$
$$= \int_{\mathbb{R}} G_{1,m}(r, \widetilde{q})r_x\, dx,$$

after using integrating by part. From this fact, we have that equation (26) takes the form

$$\frac{d}{dt}\mathcal{E}_m\begin{pmatrix} q \\ r \end{pmatrix} = \int_{\mathbb{R}} \left(M_{2,m}(q)r_x + M_{2,m}(q_x)r + r\partial_x\left(G_{1,m}(r, q, q_x, q_{xx})\right) + G_{1,m}(r, q, q_x, q_{xx})r_x\right)\, dx$$
$$= \int_{\mathbb{R}} \left(\left(M_{2,m}(q)r\right)_x + r\partial_x\left(G_{1,m}\left(r, q, q_x, q_{xx}\right)\right) + G_{1,m}(r, q, q_x, q_{xx})r_x\right)\, dx$$
$$= \int_{\mathbb{R}} \left(\left(M_{2,m}(q)r\right)_x + \left(rG_{1,m}\left(r, q, q_x, q_{xx}\right)\right)_x\right)\, dx$$
$$= \int_{\mathbb{R}} \partial_x\left(M_{2,m}(q)r + rG_{1,m}\left(r, q, q_x, q_{xx}\right)\right)\, dx = 0.$$

This fact implies that $\mathcal{E}_m$ is conserved in time along classical solutions ($\mathcal{E}_m$ is also conserved in time along mild solutions). Now, we observe that $\mathcal{E}_m$ can be compared with the norm in $\mathcal{X}^m$. In fact, integrating by part, we have that

$$\mathcal{E}_m \begin{pmatrix} q \\ r \end{pmatrix} = \frac{1}{2} \int_R \left(\sum_{j=0}^{m} a_{1,j} \left(\partial_x^j r\right)^2 + \sum_{j=0}^{m} a_{2,j} \left(\partial_x^j q\right)^2 \right) dx.$$

Now, if we set

$$C_1 = \min \left\{a_{1,j}, a_{2,j} : \ 1 \le j \le m\right\}, \quad C_2 = \max \left\{a_{1,j}, a_{2,j} : \ 1 \le j \le m\right\},$$

we have that

$$C_1 \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|^2_{\mathcal{X}^m(\mathbb{R})} \le \mathcal{E}_m \begin{pmatrix} q \\ r \end{pmatrix} \le C_2 \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|^2_{\mathcal{X}^m(\mathbb{R})}, \tag{27}$$

which implies that

$$\sqrt{\mathcal{E}_m \begin{pmatrix} q \\ r \end{pmatrix}} \approx \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|_{\mathcal{X}^m(\mathbb{R})}.$$

Now we are in position to establish global well posedness in the energy space $\mathcal{X}^m$. Here we assume that homogeneous functions $G_{1,m}$ and $G_{2,m}$ in the class (**[C1]**) satisfy the condition (21) or in the class (**[C2]**) satisfy the condition (24).

Theorem 3.1. Let $m \ge 1$ and assume that $\begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \in \dot{H}^m \times H^m(\mathbb{R})$, then the initial value problem (15) has a unique global in time mild solution $\begin{pmatrix} q \\ r \end{pmatrix} \in C^0(\mathbb{R} : \dot{H}^m \times H^m(\mathbb{R}))$. Moreover, if $\Phi_0 \in \mathcal{V}^{m+1}$ and $r_0 \in H^m(\mathbb{R})$, there is $T = T(\Phi_0, r_0)$ such that the integral equation associated with the m-Benney-Luke equation (1) has a unique solution $\Phi \in C^0([0, T_0], \mathcal{V}^{m+1})$ with

$$\Phi_t \in C^0([0, T_0], H^s(\mathbb{R})) \bigcap C^1([0, T_0], H^{s-1}(\mathbb{R}))$$

that satisfies the initial condition

$$\partial_x \Phi(0, \cdot) = \partial_x \Phi_0, \quad \partial_t \Phi(0, \cdot) = r_0.$$

Proof. We will use the Hamiltonian energy and a density argument to prove that local mild solutions are already global mild solutions, since the nonlinear part G has a regularizing effect. Let first assume that $\begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \in \dot{H}^{m+1} \times H^{m+1}(\mathbb{R})$. Then, from the local existence result (Theorem (2.4), there are $-\infty < T_0 < 0 < T_1 < \infty$ and a unique local mild solutions (q, r) such that $q, r \in C((T_0, T_1) : H^{m+1}(\mathbb{R}))$. Moreover, if $T_1 < \infty$ or $-\infty < T_0$, then we must have that,

$$\lim_{|t| \uparrow |T^*|} \left\| \begin{pmatrix} q \\ r \end{pmatrix} \right\|_{H^m(\mathbb{R}) \times H^m(\mathbb{R})} = \infty, \text{ where } T^* = T_0 \text{ or } T^* = T_0. \tag{28}$$

Now, the energy $\mathcal{E}_m$ is conserved in time on mild solutions, meaning that

$$\mathcal{E}_m \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} = \mathcal{E} \begin{pmatrix} q(t,\cdot), \\ r(t,\cdot), \end{pmatrix}, \quad \text{for all } t \in (T_0, T_1).$$

Using this fact and inequality (27), we get a contradiction with the limit (28), implying that $(q, r)^t$ must be a global mild solution of the initial value problem associated with (15) in $\mathcal{X}^m(\mathbb{R})$. Moreover, from the variation constant formula and using (27), we have for these type of mild solutions the following a priori estimate

$$\left\| \begin{pmatrix} q(t,\cdot) \\ r(t,\cdot) \end{pmatrix} \right\|_{\mathcal{X}^{m+1}} \leq L \left(\left\| \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \right\|_{\mathcal{X}^{m+1}} + t \left\| \begin{pmatrix} q(t,\cdot) \\ r(t,\cdot) \end{pmatrix} \right\|_{\mathcal{X}^m}^{n+1} \right). \tag{29}$$

Now suppose that $(q_0, r_0)^t \in \mathcal{X}^m$. For $\epsilon > 0$ given, take $\left(q_0^\epsilon, r_0^\epsilon\right)^t \in \mathcal{X}^{m+1}$ such that

$$\left\| \begin{pmatrix} q_0^\epsilon \\ r_0^\epsilon \end{pmatrix} - \begin{pmatrix} q_0 \\ r_0 \end{pmatrix} \right\|_{\mathcal{X}^m} < \epsilon. \tag{30}$$

From previous remark, for each $\epsilon > 0$, there exists a unique global mild solution $(q^\epsilon, r^\epsilon)^t$ of (15) with initial condition $(q_0^\epsilon, r_0^\epsilon)^t$. On the other hand, we also have the existence of a unique local mild solution $(q, r)^t$ of the initial value problem associated with (15)). Using the variation of constant formula for the solution and the fact that the C_0-group $T(t)$ associated with the operator M is bounded, we have for $q_\epsilon = q^\epsilon - q$, $r_\epsilon = r^\epsilon - r$, $q^{0,\epsilon} = q_0^\epsilon - q_0$ and $r^{0,\epsilon} = r_0^\epsilon - r_0$ that

$$\left\| \begin{pmatrix} q_\epsilon(t,\cdot) \\ r_\epsilon(t,\cdot) \end{pmatrix} \right\|_{\mathcal{X}^m} \leq L \left\| \begin{pmatrix} q^{0,\epsilon} \\ r^{0,\epsilon} \end{pmatrix} \right\|_{\mathcal{X}^m} + L(n) \int_0^t \left\| \left(q_\epsilon(s,\cdot), r_\epsilon(s,\cdot)\right) \right\|_{\mathcal{X}^m} \times$$
$$\left(\|r^\epsilon(s,\cdot)\|_{H^2(\mathbb{R})} + \|q^\epsilon(s,\cdot)\|_{H^m(\mathbb{R})} + \|q(s,\cdot)\|_{H^m(\mathbb{R})} + \|r(s,\cdot)\|_{H^m(\mathbb{R})} \right)^n \, ds.$$

Now note that (27) and (30) imply that the family $\left\{\mathcal{E}(q_0^\epsilon, r_0^\epsilon)\right\}_\epsilon$ is bounded. Using again inequality (27), that $\mathcal{E}(\cdot,\cdot)$ is conserved in time along mild solutions and the estimate (29). We conclude, as long as the functions $q(s,\cdot)$, $r(s,\cdot)$ exist, that

$$\|r^\epsilon(s,\cdot)\|_{H^2(\mathbb{R})} + \|q^\epsilon(s,\cdot)\|_{H^2(\mathbb{R})} + \|q(s,\cdot)\|_{H^2(\mathbb{R})} + \|r(s,\cdot)\|_{H^m(\mathbb{R})}$$

is uniformly bounded in s and ϵ. So, the Gronwall inequality implies that any given fixed $T_* > 0$ and any sequence $\epsilon_j \to 0^+$, we have on $[0, T_*]$ that

$$\begin{pmatrix} q^{\epsilon_j}(t,\cdot) \\ r^{\epsilon_j}(t,\cdot) \end{pmatrix} \to \begin{pmatrix} q(t,\cdot) \\ r(t,\cdot) \end{pmatrix}$$

strongly in $\mathcal{X}^m$, as $\epsilon_j \to 0^+$. This implies that the energy $\mathcal{E}$ is also conserved locally in time for mild solutions because as $\epsilon_j \to 0^+$

$$\mathcal{E}\left(q^{\epsilon_j}(t,\cdot), r^{\epsilon_j}(t,\cdot)\right) \to \mathcal{E}\left(q(t,\cdot), r(t,\cdot)\right) \quad \text{for any } t \in [0, T_*].$$

From this fact, we conclude that local mild solutions are global mild solutions, as it was shown for classical solutions above. ■

4. Hamiltonian Structure and Variational Characterization of Solitons

In this section we construct a Hamiltonian structure for the m-Benney-Luke model and use the variational approach to characterize travelling wave solutions for the model.

4.1. Hamiltonian Structure

We only consider the case $G_{i,m}$ belonging to the class **[C2]**, since the computations are similar for $G_{i,m}$ in the class **[C1]**. As we saw above, the m-Benney-Luke equation (1) can be expressed in the variables $r = \Phi_t$ and $q = \Phi_x$ as

$$M_{1,m} r_t - M_{2,m} q_x = \partial_x \left(G_{1,m}(r, q, q_x, q_{xx})\right) + \partial_t \left(G_{2,m}(q, q_x, q_{xx})\right),$$

or equivalently in

$$M_{1,m}\left(r - M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})\right)_t = \partial_x \left(M_{2,m} q + G_{1,m}(r, q, q_x, q_{xx})\right).$$

If we take a conjugate momentum variable $p = r - M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})$, we obtain the equation

$$p_t = \partial_x \left(M_{1,m}^{-1} M_{2,m} q + M_{1,m}^{-1} G_{1,m}(r, q, q_x, q_{xx})\right).$$

So, in the variable (q, p), we see that the m-Benney-Luke equation takes the form

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \begin{pmatrix} 0 & \partial_x M_{1,m}^{-1} \\ \partial_x M_{1,m}^{-1} & 0 \end{pmatrix} \begin{pmatrix} M_{2,m} q + G_{1,m}(r, q, q_x, q_{xx}) \\ M_{1,m} p + G_{2,m}(q, q_x, q_{xx}) \end{pmatrix}.$$

Now, if we define the Hamiltonian $\mathcal{H}_m$ in the variable (q, p) as the energy $\mathcal{E}_m(q, r)$,

$$\begin{aligned}\mathcal{H}_m \begin{pmatrix} q \\ p \end{pmatrix} = \frac{1}{2} \int_{\mathbb{R}} \Big\{ &\left[p + M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})\right] \\ &\times \left[M_{1,m}\left(p + M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})\right)\right] + q M_{2,m}(q) \Big\}\, dx,\end{aligned}$$

Using the relationship (24) between $G_{1,m}$ and $G_{2,m}$, we see directly that

$$\mathcal{H}_m' \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} M_{2,m} q + G_{1,m}(r, q, q_x, q_{xx}) \\ M_{1,m} p + G_{2,m}(q, q_x, q_{xx}) \end{pmatrix}.$$

First, we see that $\partial_p \mathcal{H}_m$ is given by

$$\begin{aligned}\partial_p \mathcal{H}_m \begin{pmatrix} q \\ p \end{pmatrix}(P) = \frac{1}{2} \int_{\mathbb{R}} \Big\{ &P M_{1,m}\left[p + M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})\right] \\ &+ \left[p + M_{1,m}^{-1} G_{2,m}(q, q_x, q_{xx})\right] M_{1,m} P \Big\}\, dx.\end{aligned}$$

Since $M_{1,m}^{-1}$ is a self adjoint operator, we have that

$$\partial_p \mathcal{H}_m \begin{pmatrix} q \\ p \end{pmatrix} = M_{1,m}\left[p + M_{1,m}^{-1} G_{2,m}\left(q, q_x, q_{xx}\right)\right]. \tag{31}$$

Now, we have to compute $\partial_q \mathcal{H}_m$. If we set $r = p + M_{1,m}^{-1} G_{2,m}\left(q, q_x, q_{xx}\right)$, $\tilde{q} = (q, q_x, q_{xx})$ and $\tilde{Q} = (Q, Q_x, Q_{xx})$, then

$$\begin{aligned} \partial_q \mathcal{H}_m(q,p)(Q) &= \frac{1}{2}\int_{\mathbb{R}} \Big\{ M_{1,m}^{-1}\left(\nabla G_{2,m}(\tilde{q})\tilde{Q}\right) M_{1,m}\left(p + M_{1,m}^{-1} G_{2,m}(\tilde{q})\right) \\ &\quad + \left[p + M_{1,m}^{-1} G_{2,m}(\tilde{q})\right]\left[M_{1,m}\left(M_{1,m}^{-1}(\nabla G_{2,m}(\tilde{q})\tilde{Q})\right)\right] + 2M_{2,m}(q)Q \Big\}\, dx \\ &= \frac{1}{2}\int_{\mathbb{R}} \left\{ 2r\left(\nabla G_{2,m}(\tilde{q})\tilde{Q}\right) + 2M_{2,m}(q)Q \right\} dx \end{aligned}$$

Using integrating by part and the relationship between $G_{1,m}$ and $G_{2,m}$ given by (24), then

$$\begin{aligned} \partial_q \mathcal{H}_m(q,p)(Q) &= \int_{\mathbb{R}} \left\{ r\left[\partial_1 G_{2,m}(\tilde{q})Q + \partial_2 G_{2,m}(\tilde{q})Q_x + \partial_3 G_{2,m}(\tilde{q})Q_{xx}\right] + M_{2,m}(q)Q \right\} dx \\ &= \int_{\mathbb{R}} \left\{ \left[r\partial_1 G_{2,m}(\tilde{q}) - (r\partial_2 G_{2,m}(\tilde{q}))_x + (r\partial_3 G_{2,m}(\tilde{q}))_{xx} Q\right] + M_{2,m}(q)Q \right\} dx \\ &= \int_{\mathbb{R}} \left(G_{1,m}\left(p + M_{1,m}^{-1} G_{2,m}\left(q, q_x, q_{xx}\right), q, q_x, q_{xx}\right) + M_{2,m}(q) \right) Q\, dx. \end{aligned}$$

So, we conclude that

$$\partial_q \mathcal{H}\begin{pmatrix} q \\ p \end{pmatrix} = G_{1,m}\left(p + M_{1,m}^{-1} G_{2,m}\left(q, q_x, q_{xx}\right), q, q_x, q_{xx}\right) + M_{2,m}(q).$$

We finally see that

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \begin{pmatrix} \partial_x M_{1,m}^{-1}\left(\partial_p \mathcal{H}_m\right) \\ \partial_x M_{1,m}^{-1}\left(\partial_q \mathcal{H}_m\right) \end{pmatrix} = \begin{pmatrix} 0 & \partial_x M_{1,m}^{-1} \\ \partial_x M_{1,m}^{-1} & 0 \end{pmatrix}\begin{pmatrix} \partial_q \mathcal{H}_m \\ \partial_p \mathcal{H}_m \end{pmatrix}.$$

Hence, in the variable (q, p), the m-Benney-Luke model has the Hamiltonian structure

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \partial_x \mathcal{J}_m \mathcal{H}_m' \begin{pmatrix} q \\ p \end{pmatrix}, \quad \mathcal{J}_m = \begin{pmatrix} 0 & M_{1,m}^{-1} \\ M_{1,m}^{-1} & 0 \end{pmatrix}, \tag{32}$$

where the operator $\partial_x \mathcal{J}_m$ is a skew-symmetric in $\mathcal{X}^m$. From this Hamiltonian structure, we know that travelling wave solutions of wave speed c for the Benney-Luke model of higher order in the form (1) corresponds to stationary solution of the system

$$\mathcal{F}_m = \mathcal{H}_m + c\mathcal{Q}_m, \tag{33}$$

where $\mathcal{Q}_m$ (known as the charge) is defined as

$$\mathcal{Q}_m(\Psi) = \frac{1}{2}\left\langle \mathcal{J}_m^{-1}\Psi, \Psi \right\rangle_{((\mathcal{X}^m)^*, \mathcal{X}^m)} = \int_{\mathbb{R}} M_{1,m}(p)q\, dx. \tag{34}$$

In other words, a travelling wave Ψ_c of wave speed c satisfies the equation

$$\mathcal{H}'_m(\Psi_c) + c\mathcal{Q}'_m(\Psi_c) = 0, \tag{35}$$

which implies that q satisfies the equation (in the class **[C2]**)

$$\sum_{j=0}^{m}(-1)^j(a_{2,j} - c^2 a_{1,j})\partial_x^{2j} q - G_{1,m}(-cq, q, q_x, q_{xx}) + cG_{1,m}(q, q_x, q_{xx}) = 0. \tag{36}$$

and the equation (in the class **[C1]**)

$$\sum_{j=0}^{m}(-1)^j(a_{2,j} - c^2 a_{1,j})\partial_x^{2j} q - G_{1,m}(-cq, q) + cG_{1,m}(q) = 0. \tag{37}$$

We must remember that M. Grillakis, J. Shatah and W. Strauss ([24]) gave a general result that has been used in a great variety of models to establish orbital stability of solitary waves for a class of abstract Hamiltonian system of the form

$$u_t = \mathcal{J}\mathcal{H}'(u),$$

where the operator $\mathcal{J}$ is a skew-symmetric in a Hilbert space X. In our particular case, solitary waves of least energy Ψ_c are minimum of the action functional $\mathcal{F}_m = \mathcal{H}_m + c\mathcal{Q}_m$ under a constrain, and the stability analysis depends on the positiveness of the symmetric operator $\mathcal{F}''_m$ in a neighborhood of the solitary wave Ψ_c, except possibly in two directions, and also on the strict convexity of the real function $d(c) = \mathcal{F}_c(\Psi_c)$, which means to determine sign of function

$$d''(c) = \frac{d}{dc}\left((\mathcal{H}'_m(\Psi_c) + c\mathcal{Q}'_m(\Psi_c))\left(\frac{d}{dc}\Psi_c\right) + \mathcal{Q}_m(\Psi_c)\right) = \frac{d}{dc}\mathcal{Q}_m(\Psi_c).$$

The first difficulty in the Grillakis, J. Shatah and W. Strauss approach appears when computing $\mathcal{F}''_m$ around the travelling wave Ψ_c, since we do not know explicitly Ψ_c for the m-Benney-Luke model for general homogeneous functions $G_{i,m}$ for $i = 1, 2$. In other words, it would be almost impossible to establish the spectral hypotheses on second variation of the action functional on the travelling wave. To overcome this difficult task, we appeals to the variational characterization of travelling wave solutions and establish the convexity of d by those properties.

4.2. Variational Preliminaries for Stability

In this subsection, we only consider $G_{i,m}$ in the class **[C2]** since computation are similar for $G_{i,m}$ in the class **[C1]**. An important issue is that travelling wave solutions of (1) are characterized as critical points of the functional J_c given by

$$J_{m,c}(u) = I_{m,c}(u) - K_c(u), \tag{38}$$

where functional I and $K_c = K_{1,c} + cK_2$ are defined on the space $\mathcal{V}^{m+1}$ as follows

$$I_{m,c}(u) = \frac{1}{2}\int_{\mathbb{R}} \left[M_{2,m}(u_x)u_x - c^2 M_{1,m}(u_x)u_x\right] dx \tag{39}$$

$$K_{1,c}(u) = -\int_{\mathbb{R}} F(-cu_x, u_x, u_{xx})\, dx \tag{40}$$

$$K_2(u) = \int_{\mathbb{R}} H(u_x, u_{xx})\, dx. \tag{41}$$

First we note that setting

$$c_m = \min\left\{\sqrt{\frac{a_{2,j}}{a_{1,j}}} \;:\; 0 \le j \le m\right\},$$

we have that $\sqrt{I_c}$ is like a norm in $\mathcal{V}^{m+1}$ for $0 < |c| \le c_m$. More exactly, there are constants

$$C_1 = \min_{0\le j\le m}\left\{(a_{2,j} - c^2 a_{1,j})\right\} \le \max_{0\le j\le m}\left\{(a_{2,j} - c^2 a_{1,j})\right\} = C_2 \tag{42}$$

such that for $u \in \mathcal{V}^{m+1}$,

$$C_1||u||^2_{\mathcal{V}^{m+1}} \le I_{m,c}(u) \le C_2||u||^2_{\mathcal{V}^{m+1}}. \tag{43}$$

As done in many cases (see among many, [2], [4], [9], [13], [12], [16], [17], for example), existence of critical point for J_c follows as a consequence of the Concentration-Compactness principle by L. Lions in [18]-[19] (see also [20]), by considering an appropriate minimization problem with restrictions.

Theorem 4.1. (Existence of traveling wave solutions [2]) Let $m \ge 1$ and $0 < |c| \le c_m$. Given a minimizing sequence $(\psi_n)_n$ of $\mathcal{I}_{m,c}$, there exist a subsequence $\left\{\psi_{n_k}\right\}_k$, a sequence of points $y_k \in \mathbb{R}$ and $\psi \in \mathcal{V}^{m+1}(\mathbb{R})$ such that $\psi_{n_k}(. + y_k) \to \psi$ in $\mathcal{V}^{m+1}(\mathbb{R})$, where

$$\mathcal{I}_{m,c} = \inf\left\{I_{m,c}(u) : u \in \mathcal{V}^{m+1},\; K_c(u) = (1)^{n+1}\right\}. \tag{44}$$

The function $\psi \in \mathcal{V}^{m+1}$ found in Theorem (4.1) is a minimizer of $\mathcal{I}_{m,c}$ subject to the constraint $K_c = 1$ and is therefore a weak solution of the Euler-Lagrange equation

$$\begin{aligned} c^2 M_{1,m,\mu}(\partial_x\psi) - M_{2,m,\mu}(\partial_x\psi) = -\beta\big(&G_{1,m}(-c\psi_x, \psi_x, \psi_{xx}, \psi_{xxx}) \\ &-cG_{2,m}(\psi_x, \psi_{xx}, \psi_{xxx})\big). \end{aligned}$$

for some Lagrange multiplier $\beta \ne 0$. By the homogeneity of the functions G_1 and G_2 we have that $\beta = \dfrac{2\mathcal{I}_c}{n+2}$ and also that $v_0 = \left(\dfrac{2\mathcal{I}_c}{n+2}\right)^{\frac{1}{n}} \psi$ is a weak solution of equation (36) (which is known as a ground state solution) and achieves the minimum

$$M(m,c) = \inf\left\{\frac{I_{m,c}(v)}{(K_c(v))^{\frac{2}{n+2}}} : v \in \mathcal{V}^{m+1},\; v \ne 0\right\}. \tag{45}$$

We see directly that

$$\begin{aligned}
\left\langle I'_{m,c}(u), u\right\rangle &= 2I_{m,c}(u)\\
\left\langle K'_c(u), u\right\rangle &= \int_{\mathbb{R}} \left(-G_{1,m}\left(-cu_x, u_x, u_{xx}, u_{xxx}\right) + cG_{2,m}\left(u_x, u_{xx}, u_{xxx}\right)\right) u_x\, dx\\
&= (n+2)K_c(u) \qquad (46)\\
\left\langle J'_{m,c}(u), u\right\rangle &= 2I_{m,c}(u) - (n+2)K_c(u)\\
&= 2J_{m,c}(u) - nK_c(u).
\end{aligned}$$

Thus, if u_0 is any critical point for J_c, then

$$\begin{aligned}
J_{m,c}(u_0) &= \left(\frac{n}{n+2}\right) I_{m,c}(u_0),\\
J_{m,c}(u_0) &= \left(\frac{n}{2}\right) K_c(u_0),\\
I_{m,c}(u_0) &= \left(\frac{n+2}{2}\right) K_c(u_0).
\end{aligned}$$

We will see below that the analysis of the orbital stability of ground states solutions depend upon some properties of the function d defined by

$$d(m, c) := \inf\{J_{m,c}(u) : \ u \in \mathcal{M}_c\}.$$

where $\mathcal{M}_c$ is the natural constraint set (Nehari manifold) defined by

$$\mathcal{M}_{m,c} = \{u \in \mathcal{V}^{m+1} : N_{m,c}(u) = 0\},$$

with the functional $N_{m,c}(u) = (J'_{m,c}(u), u) = 2I_{m,c}(u) - (n+2)K_c(u)$. Moreover, the set of ground state solutions $\mathcal{G}_c$ can be characterized as

$$\begin{aligned}
\mathcal{G}_{m,c} &= \{u \in \mathcal{M}_{m,c} : d(m, c) = J_{m,c}(u)\}\\
&= \left\{u \in \mathcal{X} \setminus \{0\} : d(m, c) = \left(\frac{n}{n+2}\right) I_{m,c}(u) = \left(\frac{n}{2}\right) K_c(u)\right\} \subset \mathcal{M}_{m,c}.
\end{aligned}$$

On the other hand, a direct computation shows for $0 < |c| < c_m$ that d has the following variational characterization

$$d(m, c) = \left(\frac{n2^{\frac{2}{n}}}{(n+2)^{\frac{n+2}{n}}}\right) \mathcal{I}_{m,c}^{\frac{n+2}{n}} > 0.$$

The first result we present gives another characterization for $d(m, c)$, which will be clever to prove some basic properties of d, which are the analogous ones of those proved by J. Quintero in [5] or [6] (see also J. Shatah ([7]).

Lemma 4.2. Let $m \geq 1$ and $0 < |c| \leq c_m$.

1. $d(m, c)$ exists and is positive.

2. $\inf \left\{ \left(\frac{n}{n+2} \right) I_{m,c}(u) : N_{m,c}(u) \le 0, \ u \ne 0 \right\} = \inf\{J_{m,c}(u) : \ u \in \mathcal{M}_{m,c}\}$.

3. There exists $v_c \in \mathcal{V}^{m+1} \setminus \{0\}$ such that $d(m, c) = J_{m,c}(v_c)$. Moreover, v_c is a weak solution of (1).

4. Let $\{u_k\} \subset \mathcal{V}^{m+1}$ be such that

$$\left(\frac{n}{n+2} \right) I_{m,c}(u_k) \to d(m, c) \text{ and } J_{m,c}(u_k) \to d_1 \le d(m, c).$$

Then there exist a subsequence of $\{u_k\}$ which we denote the same, a sequence $\{(x_k, y_k)\}$ and $v_c \in \mathcal{M}_c$ such that $u_k(\cdot + x_k, \cdot + y_k)$ converges strongly in $\mathcal{V}^{m+1}$ to v_c and $d_1 = d(m, c) = \left(\frac{n}{n+2} \right) I_{m,c}(v_c)$.

Proof.

1. For $u \in \mathcal{M}_{m,c}$ we have that $J_{m,c}(u) = \left(\frac{n}{n+2} \right) I_{m,c}(u) \ge 0$. This implies that $d(m, c)$ exists and $d(m, c) \ge 0$. Now, using the homogeneity of F and H, and Young's inequality appropriately, we have that

$$|K_c(u)| \le cC(n)\|u\|^{n+2}_{\mathcal{V}^{m+1}}.$$

This fact implies that

$$\begin{aligned} J_{m,c}(u) = \left(\frac{n}{n+2} \right) I_{m,c}(u) &= \left(\frac{n}{2} \right) |K_c(u)| \\ &\le \left(\frac{cnC(n)}{2} \right) \|u\|^{n+2}_{\mathcal{V}^{m+1}} \le \left(\frac{cnC(n)}{2C_1^{\frac{n+2}{2}}} \right) \left(I_{m,c}(u) \right)^{\frac{n+2}{2}} \\ &= cC_3(m, n) \left(I_{m,c}(u) \right)^{\frac{n+2}{2}} . \end{aligned}$$

This inequality implies that

$$\left(\frac{n}{n+2} \right) I_{m,c}(u) \ge \frac{C_4(n, m)}{c^{\frac{2}{n}}} > 0,$$

and so

$$d(m, c) \ge \frac{C_4(n, m)}{c^{\frac{2}{n}}} > 0. \tag{47}$$

2. For $u \in \mathcal{V}$ such that $N_{m,c}(u) \leq 0$ we have that $K_c(u) > 0$. If we define $\alpha \in [0, 1)$ by
$$\alpha = \left(\frac{2}{n+2}\right)\frac{I_{m,c}(u)}{K_c(u)},$$
then, we have that $N_{m,c}(\alpha u) = 0$. In other words, $\alpha u \in \mathcal{M}_{m,c}$. So,
$$\inf\{J_{m,c}(u) : \ u \in \mathcal{M}_{m,c}\} \leq J_{m,c}(\alpha u) = \left(\frac{n\alpha^2}{n+2}\right) I_{m,c}(u) \leq \left(\frac{n}{n+2}\right) I_{m,c}(u).$$
Thus we obtain the first inequality,
$$\inf\{J_{m,c}(u) : \ u \in \mathcal{M}_{m,c}\} \leq \inf\left\{\left(\frac{n}{n+2}\right) I_{m,c}(u) : \ N_{m,c}(u) \leq 0\right\}.$$
Now let $u \in \mathcal{M}_{m,c}$. Then $J_{m,c}(u) = \left(\frac{n}{n+2}\right) I_{m,c}(u)$ because $N_{m,c}(u) = 0$. So,
$$\inf\{J_{m,c}(u) : \ u \in \mathcal{M}_{m,c}\} \geq \inf\left\{\left(\frac{n}{n+2}\right) I_{m,c}(u) : \ N_{m,c}(u) \leq 0, \ u \neq 0\right\},$$
so, we get the desired conclusion.

3. Proof of this fact follows by the same arguments used by J. Quintero and O. Montoya in [2] to show the existence of travelling waves with finite energy (concentration-compactness arguments).

4. 4. Note that
$$J_{m,c}(u_k) = \left(\frac{n}{n+2}\right) I_{m,c}(u_k) + \frac{1}{n+2} N_{m,c}(u_k) \to d_1 \leq d(c).$$
Then for k large enough, $N_{m,c}(u_k) \leq 0$, implying by the previous result that the sequence $\{u_k\}$ is a minimizing sequence for $\theta(c)$. Then , there exist a subsequence of $\{u_k\}$ which we denote the same, a sequence $\{x_k\}_k$ and $v_c \in \mathcal{M}_{m,c}$ such that $u_k(\cdot + x_k)$ converges strongly in $\mathcal{V}^{m+1}$ to v_c. In particular, $N_{m,c}(v_c) = 0$. Then we conclude that $d_1 = d(m, c) = \left(\frac{n}{n+2}\right) I_{m,c}(v_c)$. ■

Proposition 4.3. Let $0 < |c| < 1$ and $a_{2,j} > a_{1,j}$ for $1 \leq j \leq m$. If we set Σ_m on the space $\mathcal{V}^{m+1}$ as
$$\Sigma_m(u) = \sum_{i=0}^{m} \int_{\mathbb{R}} a_{1,j} (\partial_x^{j+1} u)^2.$$
Then, we have that

1. If $c_1 < c_2$ with $(c_1, c_2) \subset (0, 1)$. Then $d(c)$ and $\Sigma_m(u^c)$ are uniformly bounded for $c \in [c_1, c_2]$ and $u^c \in \mathcal{G}_{m,c}$.
2. If $c_1 < c_2$ with $c_2 - c_1$ close to zero and $u^{c_i} \in \mathcal{G}_{c_i}$,

$$K_{c_1}^{\frac{2}{n}}(u^{c_2})d(m, c_1) \le K_{c_2}^{\frac{2}{n}}(u^{c_2})d(m, c_2) + \left(\frac{K_{c_2}^{\frac{2}{n}}(u^{c_2})(c_2^2 - c_1^2)}{2}\right)\Sigma_m(u^{c_2}) + o(c_2 - c_1),$$

and

$$K_{c_2}^{\frac{2}{n}}(u^{c_1})d(m, c_2) \le K_{c_1}^{\frac{2}{n}}(u^{c_1})d(m, c_1) - \left(\frac{K_{c_1}^{\frac{2}{n}}(u^{c_1})(c_2^2 - c_1^2)}{2}\right)\Sigma_m(u^{c_1}) + o(c_2 - c_1).$$

Proof.

1. Let $c_1 < c_2$ be such that $(c_1, c_2) \subset (0, 1)$. Let $u \in \mathcal{V}^{m+1}$ be such that $\mathcal{G}_{m,c}(u) \neq 0$. Note that

$$N_c(t_c u) = 0 \Leftrightarrow t^n = \left(\frac{2}{n+2}\right)\frac{I_{m,c}(u)}{K_c(u)}$$

Then by the characterization of $d(m, c)$, we have that

$$d(m, c) \le J_{m,c}(tu) = \left(\frac{n}{n+2}\right)\left(\frac{2}{n+2}\right)^{\frac{2}{n}}\frac{(I_{m,c}(u))^{\frac{n+2}{n}}}{K_c^{\frac{2}{n}}(u)} \le M(n)\frac{(I_0(u))^{\frac{n+2}{n}}}{c_1^{\frac{2}{n}}K_0^{\frac{2}{n}}(u)},$$

where $K_0(u)$ only depends on c_1 and c_2. On the other hand, from (47), we have that

$$d(m, c) \ge \frac{C_4(n, m)}{c^{\frac{2}{n}}} \ge \frac{C_4(n, m)}{c_2^{\frac{2}{n}}} > 0.$$

In other words, $d(m, \cdot)$ is uniformly bounded for $c \in [c_1, c_2] \subset [0, 1]$. Now, let $u^c \in \mathcal{G}_{m,c}$. Then

$$\min_{1\le j\le m}\{a_{2,j} - c^2 a_{1,j}\}\Sigma_m(u^c) \le \left(\frac{n+2}{n}\right)d(m, c) = I_{m,c}(u^c) \le \max_{1\le j\le m}\{a_{2,j} - c^2 a_{1,j}\}\Sigma_m(u^c),$$

meaning that $\Sigma_m(u^c)$ is uniformly bounded for $c \in [c_1, c_2] \subset [0, 1]$.

2. We want t such that $N_{c_1}(tu^{c_2}) = 0$ to assert that

$$d(m, c_1) \le \left(\frac{n}{n+2}\right)I_{m,c_1}(tu^{c_2}).$$

Note that

$$\begin{aligned}
N_{c_1}(tu^{c_2}) &= 2t^2 I_{m,c_1}(u^{c_2}) - (n+2)t^{n+2}K_{c_1}(u^{c_2})\\
&= t^2\left[N_{c_2}(u^{c_2}) + (c_2^2-c_1^2)\Sigma_m(u^{c_2}) + (n+2)(K_{c_2}(u^{c_2}) - t^n K_{c_1}(u^{c_2}))\right]\\
&= t^2\left[(c_2^2-c_1^2)\Sigma_m(u^{c_2}) + (n+2)(K_{c_2}(u^{c_2}) - t^n K_{c_1}(u^{c_2}))\right].
\end{aligned}$$

So we need t to be,

$$\begin{aligned}
t^n &= \left(\frac{K_{c_2}(u^{c_2})}{K_{c_1}(u^{c_2})}\right)\left(1 + \frac{(c_2^2-c_1^2)}{(n+2)K_{c_2}(u^{c_2})}\Sigma_m(u^{c_2})\right)\\
&= \left(\frac{K_{c_2}(u^{c_2})}{K_{c_1}(u^{c_2})}\right)\left(1 + \frac{n(c_2^2-c_1^2)}{2(n+2)d(m,c_2)}\Sigma_m(u^{c_2})\right).
\end{aligned}$$

Then we have that

$$\begin{aligned}
d(m,c_1) &\le \left(\frac{n}{n+2}\right) I_{m,c_1}(tu^{c_2})\\
&= t^2\left(\left(\frac{n}{n+2}\right) I_{m,c_2}(u^{c_2}) + \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma_m(u^{c_2})\right)\\
&\le t^2\left(d(m,c_2) + \frac{n(c_2^2-c_1^2)}{2(n+2)}\Sigma_m(u^{c_2})\right)\\
&\le d(m,c_2)\left(\frac{K_{c_2}(u^{c_2})}{K_{c_1}(u^{c_2})}\right)^{\frac{2}{n}}\left(1 + \frac{n(c_2^2-c_1^2)}{2(n+2)d(m,c_2)}\Sigma_m(u^{c_2})\right)^{\frac{n+2}{n}}.
\end{aligned}$$

Moreover, since

$$(1+x)^{\frac{n+2}{n}} = 1 + \left(\frac{n+2}{n}\right)x + O(x^2),$$

for x small, we conclude for $c_1 - c_2$ close to zero that

$$\begin{aligned}
K_{c_1}^{\frac{2}{n}}(u^{c_2})d(m,c_1) \le{}& K_{c_2}^{\frac{2}{n}}(u^{c_2})d(m,c_2)\\
&+ \left(\frac{K_{c_2}^{\frac{2}{n}}(u^{c_2})(c_2^2-c_1^2)}{2}\right)\Sigma_m(u^{c_2}) + o(c_2-c_1).
\end{aligned}$$

As before, we want t such that $N_{c_2}(tu^{c_1}) = 0$. In this case

$$t^n = \left(\frac{K_{c_2}(u^{c_2})}{K_{c_1}(u^{c_2})}\right)\left(1 - \frac{n(c_2^2-c_1^2)}{2(n+2)d(m,c_2)}\Sigma_m(u^{c_2})\right).$$

Then we have that

$$d(m,c_2) \le d(m,c_1)\left(\frac{K_{c_1}(u^{c_1})}{K_{c_2}(u^{c_1})}\right)^{\frac{2}{n}}\left(1-\frac{n(c_2^2-c_1^2)}{2(n+2)d(c_1)}\Sigma_m(u^{c_1})\right)^{\frac{n+2}{n}}.$$

But $(1-x)^{\frac{n+2}{n}} = 1-\left(\frac{n+2}{n}\right)x + O(x^2)$, for x small. Then, for $c_1 - c_2$ close to zero we have that

$$K_{c_2}^{\frac{2}{n}}(u^{c_1})d(m,c_2) \le K_{c_1}^{\frac{2}{n}}(u^{c_1})d(m,c_1) - \left(\frac{K_{c_1}^{\frac{2}{n}}(u^{c_1})(c_2^2-c_1^2)}{2}\right)\Sigma_m(u^{c_1}) + o(c_2-c_1).$$

■

4.3. Convexity of d

In this section we will prove that $d(m,\cdot)$ is strictly convex for $a_{2,j} > a_{1,j}$ for $1 \le j \le m$, and $0 < |c| < 1$, near zero and near 1. We start analyzing the behavior of $d(m,\cdot)$ and $d'(m,\cdot)$ near zero and 1, where $'$ denotes derivation with respect to c. We only consider $G_{i,m}$ in the class **[C2]** since computation are similar for $G_{i,m}$ in the class **[C1]**.

Theorem 4.4. Let $0 < |c| < 1$, $a_{2,j} > a_{1,j}$ for $1 \le j \le m$ and $u \in \mathcal{G}_{m,c}$. Then we have that

1.
$$d'(m,c) = -\left(c\Sigma_m(u) + K_c'(u)\right) \tag{48}$$

2.
$$\lim_{c\to 0^+} d(m,c) = \infty, \quad \text{and} \quad \lim_{c\to 0^+} d'(m,c) = -\infty.$$

Proof.

1. The first part follows by taking appropriate limits in Proposition (4.3) part 2, using that $2d(m,c) = nK_c(u)$ for $u \in \mathcal{G}_{m,c}$, and that

$$\frac{d}{dc}\left(K_c^{\frac{2}{n}}(u)\right) = \frac{2}{n}\left(K_c(u)\right)^{\frac{2}{n}-1}K_c'(u)$$

2. First note that inequality (47) in the previous section implies that

$$\lim_{c\to 0^+} d(m,c) = +\infty.$$

On the other hand, from previous fact, the homogeneity of F and H, we conclude that

$$\begin{aligned}
\lim_{c\to 0^+} K_c(u) &= \lim_{c\to 0^+} \int_{\mathbb{R}} (-F(-cu_x, u_x, u_{xx}) + cH(u_x, u_{xx}))\, dx = 0,\\
\lim_{c\to 0^+} \left|K_c'(u)\right| &= \lim_{c\to 0^+} \left|\int_{\mathbb{R}} \left(u_x F_p(-cu_x, u_x, u_{xx}) + H(u_x, u_{xx})\right)\, dx\right|\\
&= \lim_{c\to 0^+} \left|\int_{\mathbb{R}} H(u_x, u_{xx})\, dx\right|\\
&= \lim_{c\to 0^+} \left|\frac{1}{c}K_c(u) + \frac{1}{c}\int_{\mathbb{R}} F(-cu_x, u_x, u_{xx})\, dx\right|\\
&= \lim_{c\to 0^+} \left|\frac{2}{nc}d(m,c) + \frac{1}{c}\int_{\mathbb{R}} F(-cu_x, u_x, u_{xx})\, dx\right| = \infty,
\end{aligned}$$

where we are using the convergence dominated Theorem, the homogeneity of F, H, F_p, and that $u_x \in H^m(\mathbb{R})$ for $m \geq 2$ ($m \geq 1$ for $G_{i,m}$ in the class **[C1]**). Using these facts we are able to conclude that

$$\lim_{c\to 0^+} d'(m,c) = -\infty.$$

■

Finally, we study the behavior of $d(m,\cdot)$ and $d'(m,\cdot)$ near 1.

Proposition 4.5. Let $n + 4 > 2n_1$ and $0 < |c| < 1$. Then we have that

$$\lim_{c\to 1^-} d(m,c) = 0. \tag{49}$$

Moreover, if $n + 2n_1 < 4$, then

$$\lim_{c\to 1^-} d'(m,c) = 0. \tag{50}$$

Proof. 1. Set $\rho^2 = a_{2,0} - c^2 a_{1,0}$ and $\alpha = (1-c^2)^{-\frac{n+n_1+1}{2(n+2)}}$. Then for given $u \in \mathcal{V}^{m+1}$, we define $v \in \mathcal{V}^{m+1}$ for $u(x) = \alpha v(z)$, where $z = \rho x$. A direct computation shows that

$$\begin{aligned}
I_{m,c}(u) &= (1-c^2)^{\frac{n-2n_1+4}{2(n+2)}} \tilde{I}_{m,c}(v)\\
K_c(u) &= K_c(v)
\end{aligned}$$

where $\tilde{I}_{m,c}$ is defined by

$$\tilde{I}_{m,c}(v) = \int_{\mathbb{R}} \left(v_z^2 + (a_{2,1} - c^2 a_{1,1}) v_{zz}^2 + \sum_{j=2}^{m} (1-c^2)^{j-1} (a_{2,j} - c^2 a_{1,j}) \left(\partial_z^{j+1} v\right)^2 \right) dz$$

Moreover, it follows that

$$\mathcal{I}_{m,c} = (1-c^2)^{\frac{n-2n_1+4}{2(n+2)}} \tilde{\mathcal{I}}_{m,c}, \quad \text{where } \tilde{\mathcal{I}}_{m,c} = \inf\{\tilde{I}_{m,c}(v) : K_c(v) = 1\}.$$

We claim that

$$\lim_{c\to 1^-} \tilde{\mathcal{I}}_{m,c} = \tilde{\mathcal{I}}_{m,1}, \quad \text{with } \tilde{\mathcal{I}}_{m,1} = \inf\{\tilde{I}_{m,1}(v) : K_1(v) = 1\}.$$

In fact, note that for any $v \in \mathcal{V}^{m+1}$ with $K_1(v) = (-1)^{n+1}$, we have that

$$K_c\left(\frac{v}{\left((-1)^{n+1}K_c(v)\right)^{\frac{1}{n+2}}}\right) = (-1)^{n+1},$$

and so we get that

$$\frac{1}{\left((-1)^{n+1}K_c(v)\right)^{\frac{2}{n+2}}}\tilde{I}_{m,c}(v) = \tilde{I}_{m,c}\left(\frac{v}{\left((-1)^{n+1}K_c(v)\right)^{\frac{1}{n+2}}}\right) \geq \tilde{\mathcal{I}}_{m,c}$$

On the other hand,

$$K_c(v) = \int_{\mathbb{R}} (-F(-cv_z, v_z, v_{zz}) + cH(v_z, v_{zz}))\, dz \to K_1(v) = (-1)^{n+1}, \quad c \to 1^-.$$

Moreover, for any $v \in \mathcal{V}^{m+1}$ with $K_1(v) = (-1)^{n+1}$ we have that

$$\tilde{I}_{m,1}(v) = \limsup_{c\to 1^-} \frac{1}{\left((-1)^{n+1}K_c(v)\right)^{\frac{2}{n+2}}}\tilde{I}_{m,c}(v) \geq \limsup_{c\to 1^-} \tilde{\mathcal{I}}_{m,c},$$

which implies that

$$\tilde{\mathcal{I}}_{m,1} \geq \limsup_{c\to 1^-} \tilde{\mathcal{I}}_{m,c}, \tag{51}$$

Now, let u_c and v_c be such that $K_c(u_c) = K_c(v_c) = (-1)^{n+1}$ and that

$$I_{m,c}(u_c) = \mathcal{I}_{m,c} = (1-c^2)^{\frac{n+4-2n_1}{2(n+2)}} \tilde{\mathcal{I}}_{m,c} = (1-c^2)^{\frac{n+4-2n_1}{2(n+2)}} \tilde{I}_{m,c}(v_c).$$

Then, using that $\tilde{I}_{m,c}(w) \geq \tilde{I}_{m,1}(w)$ and that

$$K_1\left(\frac{v}{\left((-1)^{n+1}K_1(v)\right)^{\frac{1}{n+2}}}\right) = (-1)^{n+1},$$

we have that

$$\frac{\tilde{I}_{m,c}\,(v_c)}{\left((-1)^{n+1}K_1(v_c)\right)^{\frac{2}{n+2}}} = \tilde{I}_{m,c}\left(\frac{v_c}{\left((-1)^{n+1}K_1(v_c)\right)^{\frac{1}{n+2}}}\right)$$
$$\geq \tilde{I}_{m,1}\left(\frac{v_c}{\left((-1)^{n+1}K_1(v_c)\right)^{\frac{1}{n+2}}}\right) \geq \tilde{\mathcal{I}}_{m,1}.$$

From this and estimate (51), after taking lim sup as $c \to 1^{-1}$, we conclude

$$\limsup_{c\to 1^-} \tilde{\mathcal{I}}_{m,c} = \limsup_{c\to 1^-} c^{\frac{2}{n+2}}\tilde{I}_{m,c}(v_c) \geq \tilde{\mathcal{I}}_{m,1} \geq \limsup_{c\to 1^-} \tilde{\mathcal{I}}_{m,c},$$

and so

$$\lim_{c\to 1^-} \tilde{\mathcal{I}}_{m,c} = \tilde{\mathcal{I}}_{m,1},$$

as claimed. So, if we assume that $n+4 > 2n_1$, then we conclude that

$$\lim_{c\to 1^-} \mathcal{I}_{m,c} = \lim_{c\to 1^-} (1-c^2)^{\frac{n+4-2n_1}{2(n+2)}}\tilde{\mathcal{I}}_{m,c} = 0.$$

Moreover, we conclude that

$$\lim_{c\to 1^-} d(m,c) = \lim_{c\to 1^-}\left(\frac{n2^{\frac{2}{n}}}{(n+2)^{\frac{n+2}{n}}}\right)\mathcal{I}_{m,c}^{\frac{n+2}{n}} = 0.$$

2. By (48) in the previous Theorem, to get (50) we only need to show that for any $u \in \mathcal{G}_{m,c_j}$,

$$\lim_{c_j\to 1^-} \Sigma_m(u) = 0.$$

Let $c_j \to 1^-$, as $j \to \infty$ and assume that $u \in \mathcal{G}_{m,c_j}$. Then

$$d(m,c_j) = \left(\frac{n}{n+2}\right) I_{m,c_j}(u) = \left(\frac{n}{2}\right) K_{c_j}(u).$$

Then a simple computation shows that

$$I_{m,c_j}\left(\left(\frac{n}{2d(m,c_j)}\right)^{\frac{1}{n+2}} u\right) = \mathcal{I}_{m,c_j} \quad\text{and}\quad K_{c_j}\left(\left(\frac{n}{2d(m,c_j)}\right)^{\frac{1}{n+2}} u\right) = (-1)^{n+1}.$$

In other words, $w = \left(\frac{n}{2d(m,c_j)}\right)^{\frac{1}{n+2}} u$ is a minimizer for I_{m,c_j}. Now, we define $v \in \mathcal{V}^{m+1}$ by the formula $w(x,y) = \alpha v(z)$, where $z = (1-c_j^2)^{\frac{1}{2}}x$ and $\alpha = (1-c_j^2)^{-\frac{n+1+n_1}{2(n+2)}}$. Then we have that v is a minimizer of $\tilde{\mathcal{I}}_{m,c_j}$. We must recall that

$$\lim_{c\to 1^-} \tilde{\mathcal{I}}_{m,c} = \tilde{\mathcal{I}}_{m,1}.$$

Thus in particular, the following functions are bounded in $L^2(\mathbb{R}^2)$ as $j \to \infty$,

$$v_z, \;\; v_{zz}, \;\; (1-c_j^2)^{\frac{j-1}{2}}\partial_z^{j+1}v, \;\; 2 \leq j \leq m.$$

On the other hand, we also have that

$$\int_{\mathbb{R}^2} u_x^2\,dx = \left(\frac{2d(m,c_j)}{n}\right)^{\frac{2}{n+2}} (1-c_j^2)^{-\frac{n+n_1+1}{n+2}}(1-c_j^2)^{\frac{1}{2}} \int_{\mathbb{R}^2} v_x^2\,dx$$

$$= \left(\frac{2d(m,c_j)}{n}\right)^{\frac{2}{n+2}} (1-c_j^2)^{-\frac{n+2n_1}{2(n+2)}} \int_{\mathbb{R}^2} v_x^2\,dx$$

$$= \left(\frac{2}{n+2}\right)^{\frac{2}{n}} (1-c_j^2)^{-\frac{n+2n_1}{2(n+2)}} \left(\mathcal{I}_{m,c_j}\right)^{\frac{2}{n}} \int_{\mathbb{R}^2} v_x^2\,dx$$

$$= \left(\frac{2}{n+2}\right)^{\frac{2}{n}} (1-c_j^2)^{-\frac{n+2n_1}{2(n+2)}}(1-c_j^2)^{\frac{n-2n_1+4}{n(n+2)}} \left(\tilde{\mathcal{I}}_{m,c_j}\right)^{\frac{2}{n}} \int_{\mathbb{R}^2} v_x^2\,dx$$

$$= \left(\frac{2}{n+2}\right)^{\frac{2}{n}} (1-c_j^2)^{-\frac{n^2-(2-2n_1)n+4n_1-8}{2n(n+2)}} \left(\tilde{\mathcal{I}}_{m,c_j}\right)^{\frac{2}{n}} \int_{\mathbb{R}^2} v_x^2\,dx$$

$$= \left(\frac{2}{n+2}\right)^{\frac{2}{n}} (1-c_j^2)^{\frac{4-n-2n_1}{2n}} \left(\tilde{\mathcal{I}}_{m,c_j}\right)^{\frac{2}{n}} \int_{\mathbb{R}^2} v_x^2\,dx.$$

Similar computations give us for $j \geq 0$ that,

$$\int_{\mathbb{R}} (\partial_x^{j+1}u)^2\,dx = \left(\frac{2}{n+2}\right)^{\frac{2}{n}} (1-c_j^2)^{\frac{4-n-2n_1}{2n}}(1-c_j^2)^j \left(\tilde{\mathcal{I}}_{m,c_j}\right)^{\frac{2}{n}} \int_{\mathbb{R}} (\partial_x^{j+1}v)^2\,dx$$

As a consequence of these bounds, we have for any $u_j \in \mathcal{G}_{m,c_j}$ and $n + 2n_1 < 4$ that

$$\lim_{c_j \to 1^-} \Sigma_m(u_j) = 0.$$

So, the conclusion of the Proposition follows from (48) in Theorem (4.4) and (49). ■

Moreover, from the previous result we have the convexity of d.

Proposition 4.6. For $n + 2n_1 < 4$ and $n + 4 > 2n_1$, we have that $d(m,\cdot)$ is strictly convex for $0 < c < 1$, near zero and near 1. Moreover, if $n_1 = 0$, we have that $d(m,\cdot)$ is strictly convex for $0 < c < 1$, near zero and near 1, when $1 \leq n < 4$.

Now we are in position to establish the basic criteria to prove the stability result, which is a direct implication of Proposition 4.3, Proposition 4.6 and the following Shatah's result.

Lemma 4.7. **(Shatah's Lemma [7])** Suppose that h is a strictly convex function in a neighborhood of c_0. Then given $\epsilon > 0$, there exists $N(\epsilon) > 0$ such that for $|c_\epsilon - c_0| = \epsilon$,

1. If $c_\epsilon < c_0 < c$ and $|c - c_0| < \frac{\epsilon}{2}$,
$$\frac{h(c_\epsilon) - h(c)}{c_\epsilon - c} \leq \frac{h(c_0) - h(c)}{c_0 - c} - \frac{1}{N(\epsilon)}$$
2. If $c < c_0 < c_\epsilon$ and $|c - c_0| < \frac{\epsilon}{2}$,
$$\frac{h(c_\epsilon) - h(c)}{c_\epsilon - c} \geq \frac{h(c_0) - h(c)}{c_0 - c} + \frac{1}{N(\epsilon)}$$

Proposition 4.8. Let $n + 2n_1 < 4$, $n + 4 > 2n_1$ and $0 < c_0 < 1$ near zero 0 or 1. Then for c close to c_0, there exists $\eta(c) > 0$ with $\eta(c_0) = 0$ such that

$$d(m, c) - d(m, c_0) \geq (c_0 - c)\left[c_0\Sigma_m(u^{c_0}) + \left(\frac{d}{dc}K_c(u^{c_0})\right)_{c=c_0}\right] + \eta(c).$$

Proof. Let $c < c_0$, c close to c_0. Then by Shatah's Lemma, for $c < c_0 < c_1$

$$\frac{d(m, c) - d(m, c_1)}{c - c_1} \leq \frac{d(m, c_0) - d(m, c_1)}{c_0 - c_1} - \frac{1}{N(\epsilon)} \tag{52}$$

By Proposition 4.3, using $c_0 < c_1$

$$K_{c_1}^{\frac{2}{n}}(u^{c_0})d(m, c_1) \leq K_{c_0}^{\frac{2}{n}}(u^{c_0})d(m, c_0) - \left(\frac{K_{c_0}^{\frac{2}{n}}(u^{c_0})(c_1^2 - c_0^2)}{2}\right)\Sigma_m(u^{c_0}) + o(c_2 - c_1).$$

Moreover, we also have that

$$\left(\frac{K_{c_1}^{\frac{2}{n}}(u^{c_0}) - K_{c_0}^{\frac{2}{n}}(u^{c_0})}{K_{c_0}^{\frac{2}{n}}(u^{c_0})}\right)d(m, c_1) \leq d(m, c_0) - d(m, c_1) - \left(\frac{c_1^2 - c_0^2}{2}\right)\Sigma(u^{c_0}) + o(c_1 - c_0).$$

or equivalently,

$$\frac{d(m, c_0) - d(m, c_1)}{c_0 - c_1} \leq \left(\frac{K_{c_1}^{\frac{2}{n}}(u^{c_0}) - K_{c_0}^{\frac{2}{n}}(u^{c_0})}{K_{c_0}^{\frac{2}{n}}(u^{c_0})(c_0 - c_1)}\right)d(m, c_1) - \left(\frac{c_1 + c_0}{2}\right)\Sigma(u^{c_0}) + \frac{o(c_1 - c_0)}{c_1 - c_0}.$$

Using inequality (52) and the continuity of d as $c_1 \to c_0$, we have that

$$\begin{aligned}\frac{d(m,c)-d(m,c_0)}{c-c_0} &\leq -\frac{d(m,c_0)}{K_{c_0}^{\frac{2}{n}}(u^{c_0})}\left(\frac{d}{dc}K_c^{\frac{2}{n}}(u^{c_0})\right)_{c=c_0} - c_0\Sigma(u^{c_0}) - \frac{1}{N(\epsilon)} \\ &\leq -\frac{2d(m,c_0)}{nK_{c_0}(u^{c_0})}\left(\frac{d}{dc}K_c(u^{c_0})\right)_{c=c_0} - c_0\Sigma(u^{c_0}) - \frac{1}{N(\epsilon)},\end{aligned}$$

and so, using that $2d(m,c_0) = nK_{c_0}(u^{c_0})$, we conclude that

$$d(m,c) - d(m,c_0) \geq (c_0 - c)\left(\frac{d}{dc}\left(K_c(u^{c_0})\right)_{c=c_0} + c_0\Sigma(u^{c_0})\right) + \frac{c-c_0}{N(\epsilon)}.$$

Now, let $c_0 < c$ be c close to c_0 with $c_1 < c_0 < c$. Shatah's Lemma implies that,

$$\frac{d(m,c)-d(m,c_1)}{c-c_1} \geq \frac{d(m,c_0)-d(m,c_1)}{c_0-c_1} + \frac{1}{N(\epsilon)}$$

Using Proposition 4.3, we have that

$$K_{c_1}^{\frac{2}{n}}(u^{c_0})d(m,c_1) \leq K_{c_0}^{\frac{2}{n}}(u^{c_0})d(m,c_0) + \left(\frac{K_{c_0}^{\frac{2}{n}}(u^{c_0})(c_0^2 - c_1^2)}{2}\right)\Sigma_m(u^{c_0}) + o(c_0 - c_1).$$

From the same type of arguments, we conclude that

$$d(m,c) - d(m,c_0) \geq (c_0 - c)\left(\frac{d}{dc}\left(K_c(u^{c_0})\right)_{c=c_0} + c_0\Sigma(u^{c_0})\right) + \frac{c-c_0}{N(\epsilon)}.$$

■

5. Orbital Stability of the Solitary Waves

As we discussed in section 4, we have the existence and uniqueness of solutions for the Cauchy problem associated with the m-Benney-Luke equation (1) with initial condition $(u_0, u_1) \in \mathcal{V}^{m+1} \times H^m(\mathbb{R})$. Now, we look for the modulated equation associated with the m-Benney-Luke equation (3). In other words, if we have solution of the form $u(x,t) = v(x - ct, t)$, then v satisfies the modulated equation

$$\begin{aligned}&M_{1,m}(v_{tt}) - 2cM_{1,m}(v_{xt}) + c^2M_{1,m}(v_{xx}) - M_{2,m}(v_{xx}) \\ &\quad - \partial_x(G_{1,m}(v_t, v_x, v_{xx}, v_{xxx})) \\ &\quad + c\partial_x(G_{1,m}(v_x, v_x, v_{xx}, v_{xxx})) - \partial_t(G_{2,m}((v_x, v_{xx}, v_{xxx})) \\ &\quad + c\partial_x(G_{2,m}((v_x, v_{xx}, v_{xxx})) = 0, \qquad (53)\end{aligned}$$

where we are using that the condition (24) implies that

$$G_{1,m}(v_t - cv_x, v_x, v_{xx}, v_{xxx}) = G_{1,m}(v_t, v_x, v_{xx}, v_{xxx}) - cG_{1,m}(v_x, v_x, v_{xx}, v_{xxx}), \tag{54}$$

$$G_{1,m}(-cv_x, v_x, v_{xx}, v_{xxx}) = -cG_{1,m}(v_x, v_x, v_{xx}, v_{xxx}) \tag{55}$$

Moreover, using the condition (24) we have for $\tilde{q} = (v_x, v_{xx}, v_{xxx})$ that

$$\begin{aligned}\int_{\mathbb{R}} \partial_x (G_{1,m}(v_t, \tilde{q}))v_t \, dx &= -\int_{\mathbb{R}} G_{1,m}(v_t, \tilde{q})v_{tx} \, dx \\ &= -\int (v_t \partial_1 G_{2,m}(\widetilde{q}) - \partial_x \left(v_t \partial_2 G_{2,m}(\widetilde{q})\right) + \partial_{xx} \left(v_t \partial_3 G_{2,m}(\widetilde{q})\right))v_{tx} \, dx \\ &= -\int (v_{tx} \partial_1 G_{2,m}(\widetilde{q}) + \partial_x \left(v_{txx} \partial_2 G_{2,m}(\widetilde{q})\right) + \partial_{xx} \left(v_{xxxt} \partial_3 G_{2,m}(\widetilde{q})\right))v_t \, dx \\ &= -\int_{\mathbb{R}} \partial_t (G_{2,m}(\tilde{q}))v_t \, dx.\end{aligned}$$

Now, we see directly that

$$\begin{aligned}K_1'(u)(v) &= -\int_{\mathbb{R}} \left(-c \left(\partial_p F\right)(-cu_x, u_x, u_{xx}) v_x + \left(\partial_q F\right)(-cu_x, u_x, u_{xx}) v_x \right. \\ &\qquad \left. + (\partial_r F)(-cu_x, u_x, u_{xx}) v_{xx}\right) dx \\ &= -\int_{\mathbb{R}} \left(-c \left(\partial_p F\right)(-cu_x, u_x, u_{xx}) + \left(\partial_q F\right)(-cu_x, u_x, u_{xx}) \right. \\ &\qquad \left. - (\partial_r F(-cu_x, u_x, u_{xx}))_x\right) v_x \, dx \\ &= -\int_{\mathbb{R}} G_{1,m}(-cu_x, u_x, u_{xx}, u_{xxx}) v_x \, dx, \end{aligned} \tag{56}$$

$$K_2'(u)(v) = \int_{\mathbb{R}} G_{2,m}(u_x, u_{xx}, u_{xxx}) v_x \, dx, \tag{57}$$

So, using previous equalities and formula (55), we conclude that

$$\begin{aligned}\frac{d}{dt}(K_1(v)) &= -\int_{\mathbb{R}} G_{1,m}(-cv_x, v_t, v_{xx}, v_{xxx}) v_{xt} \, dx \\ &= c\int_{\mathbb{R}} \partial_x G_{1,m}(v_x, v_x, v_{xx}, v_{xxx}) v_x \, dx, \\ \frac{d}{dt}(K_2(v)) &= -\int_{\mathbb{R}} G_{2,m}(v_x, v_{xx}, v_{xxx}) v_{xt} \, dx \\ &= \int_{\mathbb{R}} \partial_x G_{2,m}(v_x, v_{xx}, v_{xxx}) v_t \, dx.\end{aligned}$$

Putting together previous facts after multiplying by v_t the modulated equation (53), we conclude that classical and also mild solutions v of (53) conserve the modulated energy functional on $\mathcal{V}^{m+2} \times H^{m+1}(\mathbb{R})$ given by

$$\mathcal{E}_{m,c}(v, v_t) = \frac{1}{2}\widetilde{\Sigma}_m(v_{xt}) + J_{m,c}(v) = \frac{1}{2}\int_{\mathbb{R}} M_{1,m}(v_t)v_t \, dx + J_{m,c}(v),$$

where $\widetilde{\Sigma}_m(v_x) = \Sigma_m(v)$. It is clear that $\mathcal{E}_0(u, u_t)$ corresponds to the energy for a solutions u of the m-Benney-Luke (1)

$$\mathcal{E}_0(u, u_t) = \int_{\mathbb{R}} \frac{1}{2} \left(M_{1,m}(u_t)u_t + M_{2,m}(u_x)u_x \right) dx + J_0(u).$$

The first result is associated with the existence of invariant set under the flow of the modulated equation (53).

Lemma 5.1. R_c^1, R_c^2 are invariant regions under the flow for the modulated equation (53)

$$R_c^i = \left\{ (u, v) \in \mathcal{V}^{m+1} \times H^m(\mathbb{R}) : \mathcal{E}_{m,c}(u, v) < d(c), \ \left(\frac{n}{n+2} \right) I_{m,c}(u) < d(c) \right\}$$
$$R_c^i = \left\{ (u, v) \in \mathcal{V}^{m+1} \times H^m(\mathbb{R}) : \mathcal{E}_{m,c}(u, v) < d(c), \ \left(\frac{n}{n+2} \right) I_{m,c}(u) > d(c) \right\}.$$

For the proof see for example references [4], [5], [6], [7]).

Now, we establish the main result to prove the orbital stability with respect to the ground state solutions in the case of strong surface tension $a - b = \sigma - \frac{1}{3} > 0$.

Lemma 5.2. Let $n + 2n_1 < 4$, $n + 4 > 2n_1$, $a_{2,j} > a_{1,j}$ for $1 \le j \le m$ and suppose that c_0 is near 0^+ or 1^- and $(u_0, u_1) \in \mathcal{V}^{m+1} \times H^m(\mathbb{R})$. If u is a solution of the Cauchy problem associated with the m-Benney-Luke equation (1), with initial data $u(0)(\cdot) = u_0$ and $u_t(0)(\cdot) = u_1$. Then for every N, there is $\delta(K)$ such that, if

$$\|u_0 - u^{c_0}\|_{\mathcal{V}^{m+2}} + \sqrt{\tilde{\Sigma}_m(u_1 + c_0(u^{c_0})_x)} < \delta(N),$$

then we have

$$d\left(m, c_0 + \frac{1}{N}\right) \le \left(\frac{n}{n+2}\right) I_{m,c_0}(u(t)) \le d\left(m, c_0 - \frac{1}{N}\right), \text{ for all } t \in \mathbb{R}.$$

Proof. Let N be fixed and let v^i for $i = 1, 2$ be defined as

$$u(t)(x, y) := v^i(t, x - c_i t, y), \quad c_i = c_0 + \frac{(-1)^i}{N},$$

Then v^i satisfies the modulated equation with $c = c_i$ and initial conditions

$$v(0, \cdot) = u_0(\cdot) \text{ and } v_t(0, \cdot) = u_1(\cdot) + c_i(u_0)_x(\cdot),$$

and energy

$$\mathcal{E}_{m,c_i}(v^i, \partial_t v^i) = \mathcal{E}_{m,c_i}(u_0, u_1 + c_i(u_0)_x) = \mathcal{E}_{m,c_i}(u, \partial_t v^i),$$

If we set $\delta(N) < 1$ such that

$$\sqrt{\tilde{\Sigma}(u_1 + c_0(u^{c_0})_x)} + \sqrt{\Sigma(u^{c_0} - u_0)} = O(\delta),$$

then using that $\sqrt{\tilde{\Sigma}_m}$ is a norm and $\tilde{\Sigma}_m(u_x) = \Sigma_m(u)$, we conclude from the triangular inequality that for some constant $C_1(c_i, c_0)$

$$\begin{aligned}\sqrt{\tilde{\Sigma}_m(u_1 + c_i(u_0)_x)} &\leq \sqrt{\tilde{\Sigma}_m(u_1 + c_0(u^{c_0})_x)} + |c_0 - c_i|\sqrt{\Sigma_m(u^{c_0})} + c_i\sqrt{\Sigma_m(u^{c_0} - u_0)} \\ &\leq |c_0 - c_i|\sqrt{\Sigma_m(u^{c_0})} + C_1\left(\sqrt{\tilde{\Sigma}_m(u_1 + c_0(u^{c_0})_x)} + \sqrt{\Sigma_m(u^{c_0} - u_0))}\right) \\ &\leq |c_0 - c_i|\sqrt{\Sigma_m(u^{c_0})} + O(\delta),\end{aligned}$$

which implies that

$$\tilde{\Sigma}(u_1 + c_i(u_0)_x) \leq |c_0 - c_i|^2\Sigma(u^{c_0}) + O(\delta). \tag{58}$$

We note that a direct computation shows that for some C_2 (depending only on c_0)

$$|I_{m,c_i}(u^{c_0}) - I_{m,c_i}(u_0)| \leq C_2\|u^{c_0} - u_0\|_{\mathcal{V}^{m+1}}(\|u^{c_0}\|_{\mathcal{V}^{m+1}} + \|u_0\|_{\mathcal{V}^{m+1}}) \leq O(\delta),$$

which implies that

$$I_{m,c_i}(u^{c_0}) = I_{m,c_i}(u_0) + O(\delta) = I_{m,c_0}(u^{c_0}) + (c_0^2 - c_i^2)\Sigma(u^{c_0}) + O(\delta). \tag{59}$$

From this fact, we have that

$$d(m, c_0) = \frac{n}{n+2}I_{m,c_0}(u^{c_0}) = \left(\frac{n}{n+2}\right)I_{m,c_i}(u_0) - \left(\frac{n}{n+2}\right)(c_0^2 - c_i^2)\Sigma(u^{c_0}) + O(\delta),$$

Recall that we proved that d is strictly decreasing, meaning that

$$d(m, c_2) < d(m, c_0) < d(m, c_1).$$

Using the previous equality and previous inequality, it is possible to choose δ small enough such that

$$d(m, c_2) < \left(\frac{n}{n+2}\right)I_{m,c_i}(u_0) < d(m, c_1).$$

Now note that $K_{c_0}(u^{c_0}) = \frac{2}{n}d(m, c_0)$. Then we also get that

$$\begin{aligned}J_{m,c_i}(u_0) &= J_{m,c_i}(u^{c_0}) + O(\delta) \\ &= J_{m,c_0}(u^{c_0}) + \frac{(c_0^2 - c_i^2)}{2}\Sigma_m(u^{c_0}) + \frac{(c_i - c_0)}{2c_0}K_{c_0}(u^{c_0}) + O(\delta) \\ &= d(m, c_0) + \frac{(c_0^2 - c_i^2)}{2}\Sigma_m(u^{c_0}) + \frac{2(c_0 - c_i)}{nc_0}d(c_0) + O(\delta).\end{aligned}$$

From the identity $(c_0 - c_i)^2 + c_0^2 - c_i^2 = 2c_0(c_0 - c_i)$ and using (58), we have for δ small enough that,

$$\begin{aligned}
\mathcal{E}_{m,c_i}(u_0, u_1 + c_i(u_0)_x) &= \frac{1}{2}\widetilde{\Sigma}_m(u_1 + c_i(u_0)_x) + J_{m,c_i}(u_0) \\
&\le \frac{(c_0 - c_i)^2}{2}\Sigma_m(u^{c_0}) + J_{m,c_i}(u_0) + O(\delta) \\
&\le \left(\frac{(c_0 - c_i)^2 + (c_0^2 - c_i^2)}{2}\right)\Sigma_m(u^{c_0}) + (c_0 - c_i)\frac{d}{dc}(K_c(u^{c_0}))_{c=c_0} \\
&\quad + d(m, c_0) + O(\delta) \\
&\le (c_0 - c_i)\left(c_0\Sigma_m(u^{c_0}) + \frac{d}{dc}(K_c(u^{c_0}))_{c=c_0}\right) + d(m, c_0) + O(\delta).
\end{aligned}$$

Now, from Proposition 4.8, we have that

$$(c_0 - c_i)\left(c_0\Sigma_m(u^{c_0}) + \frac{d}{dc}(K_c(u^{c_0}))_{c=c_0}\right) + d(m, c_0) \le d(m, c_i) - \eta(c_i).$$

and so, we conclude that

$$\mathcal{E}_{m,c_i}(u_0, u_1 + c_i(u_0)_x) \le O(\delta) + d(m, c_i) - \eta(c_i).$$

So, as a consequence of the previous estimate, we can choose $\delta > 0$ small enough such that

$$2\delta(N) < \min\left\{\eta\left(c_0 - \frac{1}{N}\right), \eta\left(c_0 + \frac{1}{N}\right)\right\},$$

to conclude that

$$\mathcal{E}_{m,c_i}(u_0, u_1 + c_i(u_0)_x) < d(m, c_i). \tag{60}$$

Then using the invariance given by Lemma (5.1), we have for all $t \in \mathbb{R}$ that

$$\mathcal{E}_{m,c_i}(u(t), v_t(t)) < d(m, c_i), \quad d(c_2) \le \left(\frac{n}{n+2}\right) I_{m,c_i}(u(t)) \le d(m, c_1).$$

Now, recall that $c_1 < c_0 < c_2$ and that $\Sigma_m(u) \ge 0$, we have that

$$\begin{aligned}
d(m, c_1) > \left(\frac{n}{n+2}\right) I_{m,c_1}(u(t)) &= \left(\frac{n}{n+2}\right)(I_{m,c_0}(u(t)) + (c_0^2 - c_1^2)\Sigma_m(u(t))) \\
\ge \left(\frac{n}{n+2}\right) I_{m,c_0}(u(t)).
\end{aligned}$$

In a similar fashion, we have that

$$\begin{aligned}
d(m, c_2) < \left(\frac{n}{n+2}\right) I_{m,c_2}(u(t)) &= \left(\frac{n}{n+2}\right)(I_{m,c_0}(u(t)) + (c_0^2 - c_2^2)\Sigma_m(u(t))) \\
\le \left(\frac{n}{n+2}\right) I_{m,c_0}(u(t))
\end{aligned}$$

In other words, we have the desired inequality,

$$d\left(m, c_0 + \frac{1}{N}\right) \leq \left(\frac{n}{n+2}\right) I_{m,c_0}(u(t)) \leq d\left(c_0 - \frac{1}{N}\right),$$

since $c_1 = c_0 - \dfrac{1}{N}$ and $c_2 = c_0 + \dfrac{1}{N}$. ■

Finally, as proved by J. Quintero *et al.* in [5] for the 2D-Benney-Luke equation, in [6] for a 2D-Benney-Luke-Paumond equation, and in [4] for the Benney-Luke model ($m = 1$), we are able to establish orbital stability for solitons of the m-Benney-Luke equation (1).

Theorem 5.3. (Orbital Stability) Let $n + 2n_1 < 4$, $n + 4 > 2n_1$, $a_{2,j} > a_{1,j}$ for $1 \leq j \leq m$ and $0 < c_0 < 1$ is near 0 or 1, then the ground state solitary waves solutions of the m-Benney-Luke equation (1) are stable in the following sense: Given $\epsilon > 0$, there exists $\delta(\epsilon)$ such that if $(u_0, u_1) \in \mathcal{V}^{m+1} \times H^m(\mathbb{R}^2)$ satisfies

$$\|u_0 - u^{c_0}\|_{\mathcal{V}^{m+1}} + \sqrt{\widetilde{\Sigma}(u_1 + c_0(u^{c_0})_x)} < \delta(\epsilon),$$

then a unique solution u of (1) with initial condition (u_0, u_1) exits for all $t \in \mathbb{R}$ and

$$\inf_{v \in \mathcal{G}_{c_0}} \|u(t) - v\|_{\mathcal{V}^{m+1}} + \sqrt{\widetilde{\Sigma}_m(u_t(t) + c_0 v_x)} < \epsilon, \text{ for all } t \in \mathbb{R}.$$

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