Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4075-4087

http://www.ripublication.com/gjpam.htm

Interaction of Adjacent Isentropic Gas Flows in Prandtl-Meyer's Wave and in The Field of Quasi-One-Dimensional Flow

Pavel Viktorovich Bulat[1*], Mikhail Vladimirovich Chernyshev[2]

[1]University ITMO, Kronverksky pr., 49, Saint Petersburg, 197101, Russia.

[2]Federal State Educational Institution of Higher Professional Education "Saint-Petersburg State Polytechnical University" St. Petersburg Polytechnic str., 29, 195251, Russia.

Abstract

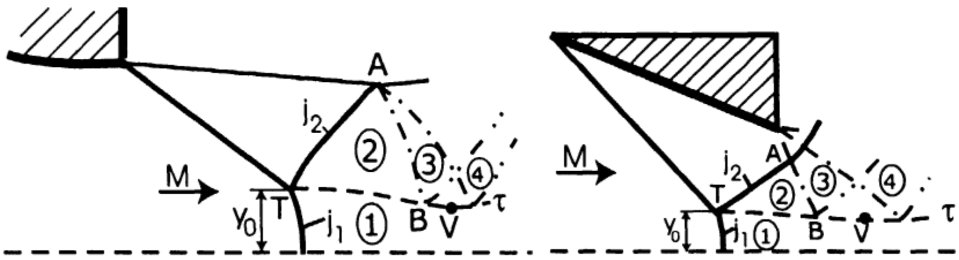
Objective is to develop a method for calculating the geometry of tangential discontinuity after triple point of shock waves, that interacts with fan of refraction waves. The interaction of co-existing isentropic flows in Prandtl-Meyer's wave and in the region of quasi-one-dimensional flow limited by tangential discontinuity is being researched. Based on conditions of compatibility on a tangential discontinuity, that separates flows behind the main and reflected shock in triple point, the equation describing the form of tangential discontinuity has been acquired and parametric analysis of its solution was carried out. The acquired results can be used in development of approximate analytical model and theoretical analysis of gas flows inside of jets, nozzles and canals. For instance calculation of Mach stem in flat overexpanded jet or supersonic flow onto a wedge of known sizes.

Keywords: Prandtl-Mayer's wave, supersonic jet, Mach reflection, tangential discontinuity.

INTRODUCTION

Aim of the research – development of a method for calculating geometry of tangential discontinuity behind triple point of shock waves that interact with fan of refraction waves.

The need to state and solve the problem of combination of stationary supersonic current and quasi-one-dimensional flow arises during analysis of gas flow with irregular (Mach) shock waves reflection. For instance, in flat, overexpanded enough jet (Fig. 1,a) the flow behind major shock j_1 (region 1) is usually considered quasi-one-dimensional [1]-[4]. The analysis of flow in a flat narrowing canal between two symmetrical wedges (Fig. 1,b) is similar [5]-[7].



a) b)

Figure 1. To research flows with Mach reflection, the described model can be applied: a) flat overexpanded jet; b) narrowing canal in between symmetrical wedges.

The analysis of aforementioned flows, usually indicates that due to slight curvature of reflective shock j_2, the flow behind it (region 2) is almost isentropic.

Comparison of the calculations results based on experimental, numerical, approximate analytical data of other authors [8]-[11], shows that gas flow behind reflected shock is described with high accuracy by the ratio on Prandtl-Meyer's wave with straight characteristics of first family. Additionally, the Prandtl-Meyer's flow model (but now - with straight characteristics of second family) is also applicable to describe the flow in fan of characteristics 3.

The interaction of fan with tangential discontinuity τ is modeled like a reflection of Prandtl-Meyer's wave from the edge of quasi-one-dimensional region. If reflected disturbances are absent then the interaction results in conjugation of different than in region 2 direction with quasi-one-dimensional canal (the "virtual nozzle", Fig.2).

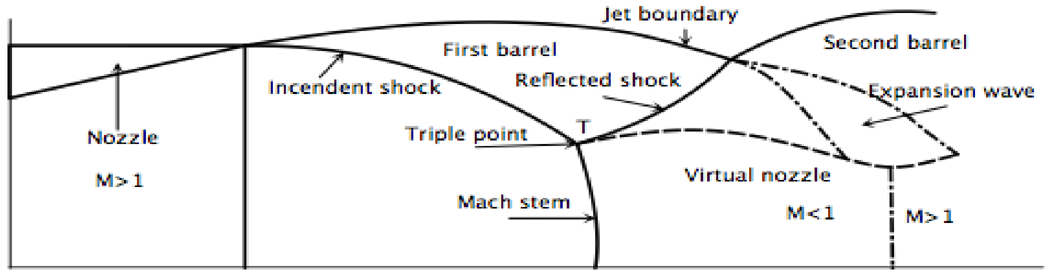


Figure 2. Virtual nozzle behind Mach stem in flat jet – the region of quasi-one-dimensional flow.

When solving problems of analytical analysis of gas jets with irregular reflection of shock waves [4],[12],[13], it is required to calculate the "virtual" nozzle geometry – region limited by Prandtl-Meyer's wave arriving on the tangential discontinuity and plane of symmetry. With known ratios of Prandtl-Meyer and conditions of dynamic compatibility, describing derivative parameters of the flow [14], solving problem of conjugation Prandtl-Meyer's wave with quasi-one-dimensional region, allows to analytically describe variation of flow parameters of studied wave and acquire data for solving problem of Prandtl-Meyer's wave interaction with catching-up shocks [15], counter shocks [16] and other disturbances.

The position of triple-point and value of Mach stem is chosen based on comparison between consumption function behind Mach stem and cross-section of a throat of a virtual nozzle(they has to be equal).

The calculation method of Mach disk, in case of axial symmetry, is described by different laws [17], based on so-called stationary Mach configuration of shock wave.

The task of engineering triple configuration of shock waves with optimal parameters, with specific set value of Mach stem, arises during development of detonation engines [18], especially with stationary or rearward detonation [19].

This work sets and solves the problem of coexistence of two isentropic currents: quasi-one-dimensional flow and Prandtl-Meyer's wave, that are separated by tangential discontinuity. Parameter of conjugated flows are tied by required equality of pressures on sides of discontinuity. The analysis and classification of results from conjugation based on parameters of the problem (Mach variables on sides of discontinuity before interaction, initial incline angle of discontinuity τ, value of gas adiabatic γ). Since quasi-one-dimensional flow can only be conjugated with a wave that has specific form of flow lines.

The acquired results can be useful for analysis of a number supersonics flows and verification of calculated data.

MATHEMATICAL MODEL

At the start point O of a studied interaction ($x=0$, Fig. 3,a), there is a tangential discontinuity τ, directed at a negative (Fig. 3,a) or positive (Fig. 3,б) angle θ_0 to horizontal ($y=0$). Tangential discontinuity can appear due to interaction between shock waves, that create triple configuration. Discontinuity τ separates flow regions 2 and 3.

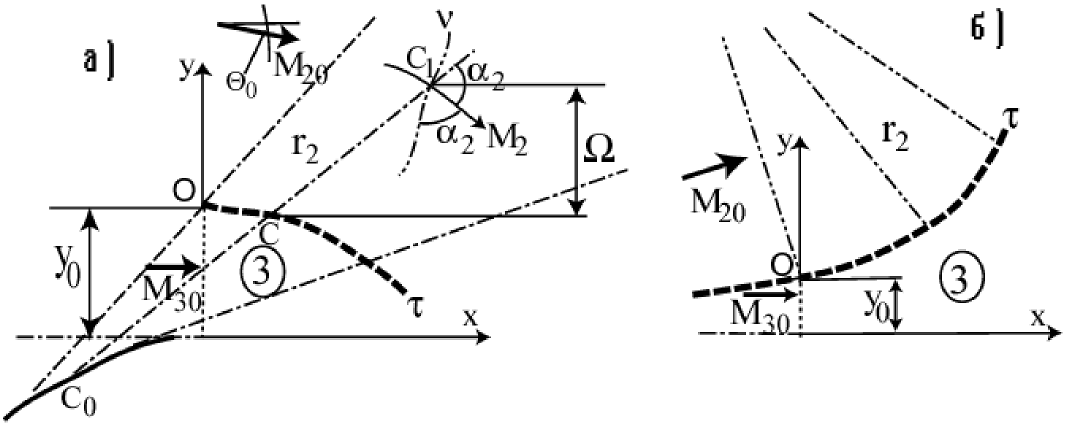


Figure 3. Schematic of wave conjugation with straight-line characteristic of first (a) and second (b) family and quasi-one-dimensional region.

It's assumed that in the top one (with initial Mach number $M_{20}>1$ while $x=0$ at discontinuity τ), the Parndtl-Meyer's wave r_2 occurs with straight-line characteristics of first ($\chi=1$, Fig. 3,a) or second ($\chi=-1$, Fig. 3,b) family. It's also assumed that for calculating of parameters in region 3 formulas for quasi-one-dimensional isentropic flow are applicable. Gas flow in area 3 can be subsonic ($M_{30}<1$, Fig. 3,a) or supersonic ($M_{30}>1$, Fig. 3,b).

Requirement for pressure equality on sides of discontinuity τ ties Mach numbers:

$\pi(M_2)/\pi(M_{20})=\pi(M_3)/\pi(M_{30})$, where $\pi(M)=\left[1+0.5(\gamma-1)M^2\right]^{-\gamma/(\gamma-1)}$ – isentropic function of pressure, γ –parameter of gas adiabatic, therefore

$$\frac{1+\varepsilon\left(M_2^2-1\right)}{1+\varepsilon\left(M_3^2-1\right)}=\frac{1+\varepsilon\left(M_{20}^2-1\right)}{1+\varepsilon\left(M_{30}^2-1\right)}=K\,, \tag{1}$$

where $\varepsilon=(\gamma-1)/(\gamma+1)$. Also, the Mach number M_2 is dependent on angle θ of discontinuity τ:

$$\theta=\theta_0-\chi\left(\omega(M_2)-\omega(M_{20})\right), \tag{2}$$

and Mach number M_3 is bound to it coordinate $y(x)$, that defines the width of region 3:

$$y/y_0=q(M_{30})/q(M_3)\,. \tag{3}$$

here $\omega(M)=1/\sqrt{\varepsilon}\arctan\sqrt{\varepsilon\left(M^2-1\right)}$ – Prandtl-Meyer's function, $q(M)=M\left[\mu(M)\right]^{-1/2\varepsilon}$ – consumption function, $\mu(M)=1+\varepsilon\left(M^2-1\right)$.

From ratios(1-3) the differential equations follow, that define the form of tangential discontinuity and variability of Mach numbers $M_2(x)$ and $M_3(x)$ on its sides:

$$\frac{dy}{dx}=\tan\,\theta\,, \tag{4}$$

$$\frac{d\theta}{dx}=-\frac{\chi M_3^2\sqrt{M_2^2-1}\,\tan\,\theta}{M_2^2\left(M_3^2-1\right)y}\,, \tag{5}$$

$$\frac{dM_2}{dx}=\frac{\left(1+\varepsilon\left(M_2^2-1\right)\right)\,\tan\,\theta}{(1-\varepsilon)M_2\left(M_3^2-1\right)y}\,, \tag{6}$$

$$\frac{dM_3}{dx}=\frac{M_3^2\left(1+\varepsilon\left(M_3^2-1\right)\right)\,\tan\,\theta}{(1-\varepsilon)\left(M_3^2-1\right)y}\,. \tag{7}$$

At fixed values of adiabatic γ the solution of equations (4-7) are defined by parameters $\chi=\pm1$, $M_{20}>1$, $M_{30}>0$ и $\theta_0\in(-\pi/2;\pi/2)$. The solution of the problem is sufficiently defined by form of tangential discontinuity shown on Fig. 3. a-n, for each diagram the values of M_{20}, M_{30}, χ are fixe and different values of angle θ_0.

DISCUSSION

For start, let's take a look at the case of simple wave, "induced" by region of quasi-one-dimensional flow($\chi = 1$) at subsonic ($M_{30} < 1$) and supersonic ($M_{30} > 1$**)** flow in this region.

If $\chi = 1$, $M_{30} < 1$ and $\theta_0 > 0$ (Fig. 4,a-b) then from equations (4-7) follows, that in regions 2 and 3 the flow is decelerated, r_2 – compression wave, and convexity of tangential discontinuity τ is facing downwards. Flows in regions 2 and 3 are decelerate to a value $M_2 = 1$ (curves 1-3 Fig. 4.a) or to a limit $M_3 \to 0$ (curves 1-3 Fig. 4.b), or till discontinuity τ rotates at an angle $\theta = \pi/2$ (curves 4-5 Fig. 4.a-b).

The implementation of quasi-one-dimensional flow in the last case is open to a debate. The result of conjugation of flows is based on value of K. If $K < K_*$ ($K_* = 1/(1-\varepsilon)$ corresponds to $M_2 = 1$ и $M_3 = 0$) then is possible for a flow to be decelerated in region 2 down to Mach number $M_2 = 1$, if $K > K_*$ then flow can decelerated in region 3 to a limit $M_3 \to 0$, if $K = K_*$ conditions $M_2 \to 1$ and $M_3 \to 0$ are applied simultaneously. Deceleration of the flow to a sonic speed in the upper region of discontinuity τ happens at finitely distant points A_i (Fig. 4,a) If the interaction leads to infinite expansion of region 3($M_3 \to 0$), the discontinuity τ continues infinitely and has a slanted asymptote. Angle of slope θ_{fin} for discontinuity and the end of interaction (at points A_i in the first case, and on asymptote – in the second) can be calculated using general formula:

$$\theta_{fin} = \theta_0 - \chi\left(\omega\left(M_2^{fin}\right) - \omega\left(M_{20}\right)\right), \quad (8)$$

where $\chi = 1$, $M_2^{fin} = 1$ if $K < K_*$, if $K > K_*$ value $M_2^{fin} > 1$ is defined by formula(3) if $M_3 = M_3^{fin} = 0$.

If the value calculated by formula (8) is $\theta_{fin} > \pi/2$, then interaction of flows ends not because of their complete s deceleration, but because of discontinuity's τ rotation to an angle of $\pi/2$ (curves 4-5 Fig. 4,a-b). Mach number on this rotated discontinuity are defined by equations:

$$\theta_0 - \left(\omega(M_2) - \omega(M_{20})\right) = \pi/2$$

on the upper region and (1) for lower region. Further rotation of discontinuity at an angle of $\theta > \pi/2$ doesn't have physical sense.

If $\chi = 1$, $M_{30} < 1$ and $\theta_0 < 0$ (Fig.4, c-d) the both flows accelerate, and wave r_2 is a wave of refraction, flow lines of which(including discontinuity τ) have a convexity directed upwards. Equations (4-7) allow for two outcomes for interaction. The first one is acceleration of subsonic flow to a critical speed ($M_3 = 1$ at points B_i Fig. 4,c) while the region 3 is contracts to a critical width y_*. The curvature of discontinuity τ at point B_i is negative and infinite large, the angle of its slope θ_{fin} is defined by formula (8), and it's Mach number M_2^{fin} on its upper region by formula (1) with $M_3 = 1$.

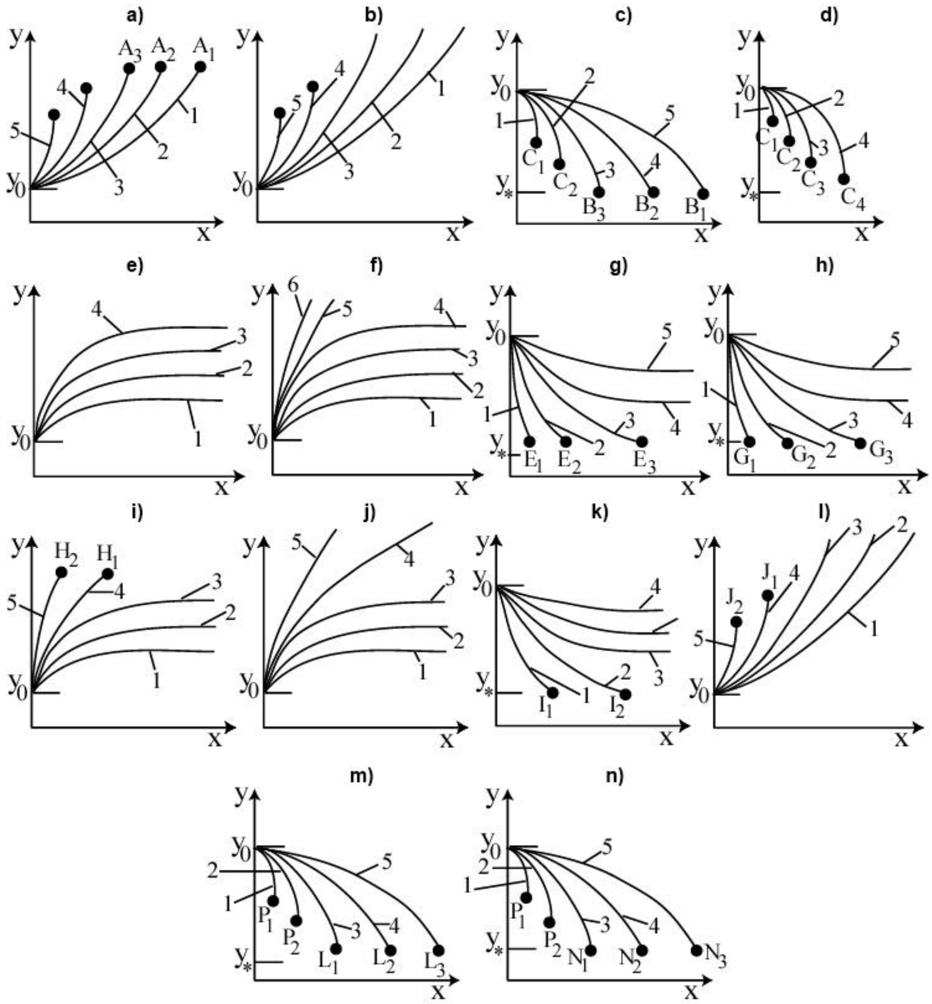


Figure 4. Diagram of tangential discontinuity, that separates flows with different values of its parameters.

If calculated angle of slope is $\theta_{fin} < -\pi/2$ then discontinuity τ starts to flow vertically before the flow in region 3 and reaches critical speed (point C_i Fig. 3c-d).

The final Mach number M_2^{fin} is defined by equation:

$\theta_0 + \omega(M_{20}) - \omega(M_2) = -\pi/2$ and Mach number $M_3^{fin} < 1$ by equation (1).

If $\chi = 1$, $M_{30} > 1$ и $\theta_0 > 0$ (Fig. 4, e-f) in the region 3 the supersonic flow expands, and in area 2 refraction wave forms. The discontinuity τ has a convexity directed upwards and it's asymptote is horizontal (curves 1-4). Mach number M_2^{fin} and M_3^{fin} on asymptotic discontinuity are defined by (8) and (1) with $\theta_{fin} = 0$.

At high values of M_{20}, θ_0, and $\theta_{fin} = 0$ the equation (8) may not have a real solution. This means that required angle for rotation of discontinuity τ, is bigger that can be allowed inside of refraction wave r_2. In this case discontinuity τ has noty horizontal but a slanted asymptote (curves 5-6, Fig 4,f), direction of which I defined by formula (8) with $M_2^{fin} \to \infty$. Mach numbers on the sides of such discontinuity approach infinity. The solution with slanted asymptote is possible if $M_{20} > 2.558$, where Mach number $M_{20} = 2.558$ is calculated by formula (8) with $\theta_0 = \pi/2$, $\theta_{fin} = 0$ and $M_2^{fin} \to \infty$. At $M_{20} < 2.558$ the asymptote of discontinuity τ is always horizontal (Fig. 3, e)

In case of $\chi = 1$, $M_{30} > 1$ and $\theta_0 < 0$ the compression of conjugated flows exists, and discontinuity τ has a convexity directed downwards. It's also possible for one of the flow to decelerated to a critical speed or for discontinuity to be rotated in way that it has horizontal asymptote. If $M_{20} < M_{30}$, then the flow in region 2 is decelerated (at point E_i Fig 4, g). Angle of slope θ_{fin} of discontinuity at these points is calculated via (8) with $M_2^{fin} = 1$. If calculated angle $\theta_{fin} > 0$, then flow doesn't decelerate to a critical speed, and discontinuity τ has horizontal asymptote (curves 4-5 Fig 4, g) parameters on which are calculated using condition $\theta_{fin} = 0$

If $M_{20} > M_{30}$ then flow in region 3 can be decelerated to a critical speed (at point G_i Fig. 4, h). The value of M_2^{fin} above this point is calculated via (1), and angle θ_{fin} via (8). However if calculated value $\theta_{fin} > 0$ then it means horizontal asymptote.

The parameter $\chi = -1$ corresponds to a simple Prandtl-Meyer's wave, that arrives without reflection on the edge of quasi-one-dimensional subsonic or supersonic flow region.

If $\chi = -1$, $M_{30} < 1$ and $\theta_0 > 0$ (Fig. 4, i-j) gas flow in region 2 contracts, and expands in subsonic region 3. The direction of discontinuity's convexity τ is different. The interaction occurs before flow's deceleration in region 2 till $M_2^{fin} = 1$ (if $K < K_*$) or in region 3 till limit $M_3^{fin} \to 0$ (if $K > K_*$), or till rotation of discontinuity τ to it's horizontal asymptote(in both case if the calculated angle via (8) $\theta_{fin} < 0$).

First case corresponds to a stop of interaction at some point on the discontinuity (point H_i, Fig. 4,i), the second corresponds to a slanted(Fig. 4,j), and third to a horizontal(curves 1-3 Fig.4, i-j) asymptote.

If $\chi = -1$, $M_{30} < 1$ and $\theta_0 < 0$ (Fig. 4,k) flow in both regions accelerates, and discontinuity's τ convexity is directed downwards. It's a special case of a wave arriving onto the edge of quasi-one-dimensional flow region. Flow acceleration cause either critical speed in region 3($M_3^{fin} = 1$ point I_i Fig. 4, k) or rotation of discontinuity to horizontal asymptote (if calculated via (8) angle $\theta_{fin} > 0$). The latter case is show by curves 3-5.

In case of $\chi = -1$, $M_{30} > 1$ and $\theta_0 > 0$ there is an expansion of both supersonic flows(Fig. 4,m). In marginal case of the expansion (curves 1-3), discontinuity τ approaches slanted asymptote, the angle of slope, which is calculated via (8) with $M_2^{fin} \to \infty$ (in this case the $M_3^{fin} \to \infty$). The are exceptions from this case when the calculated angle $\theta_{fin} > \pi/2$. This means that discontinuity τ rotates to a vertical direction (point J_i Fig. 4, l). If $\beta_r^{\lim}(M_{20}) > \pi/2$ then the interaction in question always leads to a vertical direction.

If $\chi = -1$, $M_{30} > 1$ and $\theta_0 < 0$ then supersonic flows in regions 2 and 3 contract, and discontinuity's τ convexity directed upwards (Fig. 3, m-n). Flows can decelerate to critical speeds ($M_2^{fin} = 1$ if $M_2 < M_3$ and $M_3^{fin} = 1$ if $M_2 > M_3$, points L_i and N_i Fig. 4,н and Fig. 4,o correspondingly). Angle of slope θ^{fin} of discontinuity τ at the end of interaction is calculated via (1) and (8). If calculated value is $\theta^{fin} < -\pi/2$, the deceleration of one flow doesn't occur, but discontinuity τ rotates to a vertical direction (point P_i Fig 4, m-n) instead. Mach numbers M_2^{fin} and M_3^{fin} at point P_i corresponds to supersonic flow

In special case ($\theta = 0$) at any value of Mach numbers $M_{30} > 0$ and $M_{20} > 1$ there is an infinitely long coexistence of constant flow regions, separated by tangential discontinuity.

CONCLUSION

The selection of parameters (Mach numbers and, angle of slope of discontinuity, parameter for direction of wave and gas adiabatic) clearly define the outcome of conjugation of two flows and their properties. Results of conjugation are various: acceleration and deceleration of one or both flows to a maximum, critical, or infinitely small speed, rotation of tangential discontinuity into vertical position in either direction, it's ending at some point or approach to horizontal or slanted asymptote. Some solutions have a mathematical rather than practical interest. However there is no problem solutions, during which angle of slope changes it's operator on any line of wave flow or direction of convexity. Therefore flow sections, that are actually described by the given model, shouldn't change the angle's operator or direction of flow lines' convexity. Hence the important conclusion – for Mach's stem to exist in cases shown on Fig. 1, the "virtual nozzle" throat's cross-section has to be position in region of fan of characteristics of refraction wave (Fig. 1, 3).

ACKNOWLEDGMENTS

This study was financially supported by the Ministry of Education and Science of the Russian Federation (the Agreement No. 14.575.21.0057), a unique identifier for Applied Scientific Research (project) RFMEFI57514X0057.

REFERENCES

[1] Melnikov D.A., Reflection of shock waves from the axis of symmetry. *Journal of USSR Academy of Science. Mechanics and Mechanical Engineering*, № 3 (1962), 24–30.

[2] Averenkova G.I., Ashratov E.A., Volkonskaya T.G., *Supersonic jets of ideal gas. Part 2*. Publishing house of Moscow State University, (1971).

[3] Avduevsky V.S., Ashratov E.A., Ivanov A.V., Pirumov U.G., *Supersonic non-isobaric jets*, Mashinostroenie, Moscow (1985).

[4] Omelchenko A.V., Uskov V.N., Chernyshev M. V., An approximate analytical model of flow in the first barrel of an overexpanded jet. *Technical Physics Letters*, **29,** №3 (2003), 243–245. http://link.springer.com/article/10.1134/1.1565647. doi: 10.1134/1.1565647

[5] Azevedo D.J., Liu C.S., Engineering approach to the prediction of shock patterns in bounded high-speed flows. *AIAA Journal*, **31**, №1 (1993), 83-90. doi: 10.2514/3.11322

[6] Medvedev A.E., Fomin V.M., Approximate analytical calculation of the Mach configuration of steady shock waves in a plane constricting channel, *Journal of Applied Mechanics and Technical Physics*. **39** №3 (1998), 369-374. doi: 10.1007/BF02468117

[7] Chanetz B., Chernyshev M.V., Durand A., Ivanov M.S., Khotyanovsky D.V., Kudryavtsev A.N., Omelchenko A.V., Uskov V.N., Analytical, numerical, and experimental investigation of shock wave reflection transition induced by variation of distance between wedges. *Proceedings of West East High Speed Flow Fields 2002 Conference* (Marseille, France, April 22-26, 2002). Barcelona, Spain: CIMNE, (2003), 274-281.

[8] Henderson L.F., Lozzi A., Experiments on transition to Mach reflection. *Journal of Fluid Mechanics*, **68,** №1 (1975), 139-155. DOI: http://dx.doi.org/10.1017/S0022112075000730

[9] Henderson LF., Lozzi A., Further experiments on transition to Mach reflection. *Journal of Fluid Mechanics*, **94,** №3 (1979), 541-559. doi: http://dx.doi.org/10.1017/S0022112079001178

[10] Hadjadj A., Kudryavtsev A.N., Ivanov M.S., Numerical Investigation of Shock-Reflection Phenomena in Overexpanded Supersonic Jets. *AIAA Journal*, **43**, №3 (2004), 570-577. http://arc.aiaa.org/doi/abs/10.2514/1.989

[11] Ivanov M.S., Markelov G.N., Kudryavtsev A.N., Gimelshein S.F. Numerical analysis of shock wave reflection transition in steady flows. *AIAA Journal*, **36,** №11 (1998), 2079-2086. doi: 10.2514/2.309

[12] Uskov V.N., Chernyshov M.V., Differential characteristics of the flow field in a plane overexpanded jet in the vicinity of the nozzle lip. *Journal of Applied Mechanics and Technical Physics*, **47,** №3 (2006), 366-376. doi: 10.1007/s10808-006-0064-6

[13] Silnikov M.V., Chernyshov M.V., Uskov V. N., Two-dimensional over-expanded jet flow parameters in supersonic nozzle lip vicinity. *Acta Astronautica*, **97** (2014), 38–41. http://www.sciencedirect.com/science/article/pii/S0 094576513004633. doi: 10.1016/j.actaastro.2013.12.010

[14] Adrianov A.L., Starykh A.L., Uskov V. N., *Interference of Stationary Gasdynamic Discontinuities*, Nauka, Novosibirsk (1995).

[15] Silnikov M.V., Chernyshov M.V., Uskov V. N., Analytical solutions for Prandtl–Meyer wave–oblique shock overtaking interaction. *Acta Astronautica*, **99 (**2014), 175–183. http://www.sciencedirect.com/science/article/pii/S0 094576514000824

[16] Meshkov V.R., Omel'chenko A.V., Uskov V.N. , The interaction of shock wave with counter rarefaction wave. *Vestnik Sankt-Peterburgskogo Universiteta. Ser 1. Matematika. Mekhanika. Astronomiya*, **2** (2002), 101-109.

[17] Bulat P.V., Uskov V. N., Mach Reflection of a Shock Wave from the Symmetry Axis of the Supersonic Nonisobaric Jet. *Research Journal of Applied Sciences, Engineering and Technology*, **8,** №1 (2014), 135–142. http://www.maxwellsci.com/jp/abstract.php?jid=RJASET&no=449&abs=19

[18] Bulat P.V., Uskov V.N., Shock and detonation wave in terms of view of the theory of interaction gasdynamic discontinuities. *Life Science Journal*, **11,** №8s (2014), 307-310. http://www.lifesciencesite.com/lsj/life1108s/068_249 21life1108s14_307_310.pdf

[19] Bulat P.V., Uskov V.N. (2014). The problem of constructing the optimal configuration of shock and detonation waves. *Life Science Journal*, **11,** №8s (2014), 311-314. http://www.lifesciencesite.com/lsj/life1108s/069_24922life 1108s14_311_314.pdf