Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4057–4064

http://www.ripublication.com/gjpam.htm

A note on degenerate Changhee-Genocchi polynomials and numbers

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Abstract

In this paper, we consider the degenerate Changhee-Genocchi polynomials and numbers, and give some identities for these numbers and polynomials.

AMS subject classification: 11B68, 11S40, 11S80.
Keywords: The generalized q-Daehee numbers attached to χ, the generalized q-Bernoulli numbers attached to χ, the p-adic q-integral on $\mathbb{Z}_p$, λ-q-Daehee polynomials

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1. Introduction

As is well known, the Euler polynomials are defined by he generating function to be

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!},\ \text{(see [2, 4])}. \tag{1.1}$$

When $x=0$, $E_n=E_n(0)$ are called *Euler numbers*.

The *Genocchi polynomials* are defined by the generating function to be

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}. \tag{1.2}$$

When $x=0$, $G_n=G_n(0)$ are called the *Genocchi numbers* (see [2, 6, 7, 12]).

From (1.2), we note that

$$G_0=0,\ \frac{G_{n+1}(x)}{n+1}=E_n(x),\ (n\geq 0).$$

In [3], L. Carlitz considered degenerate Euler polynomials which are defined by the generating function to be

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.3}$$

When $x=0$, $\mathcal{E}_{n,\lambda}=\mathcal{E}_{n,\lambda}(0)$ are called the *degenerate Euler numbers*.

Note that

$$\lim_{\lambda\to 0}\mathcal{E}_{n,\lambda}(x)=E_n(x),\ (n\geq 0).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (1.3), we get

$$\begin{aligned}\frac{2}{e^t+1}e^{xt}&=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\frac{1}{n!}\frac{1}{\lambda^n}(e^{\lambda t}-1)^n\\&=\sum_{n=0}^{\infty}\mathcal{E}_{n,\lambda}(x)\sum_{m=n}^{\infty}S_2(m,n)\lambda^{m-n}\frac{t^m}{m!}\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m}\mathcal{E}_{n,\lambda}(x)S_2(m,n)\lambda^{m-n}\right)\frac{t^m}{m!},\end{aligned} \tag{1.4}$$

where $S_2(m,n)$ is the Stirling numbers of the second kind.

By (1.1) and (1.4), we get

$$E_m(x)=\sum_{n=0}^{m}\mathcal{E}_{n,\lambda}(x)S_2(m,n)\lambda^{m-n},\ (m\geq 0). \tag{1.5}$$

Recently, Changhee polynomials are defined by the generating function to be

$$\frac{2}{t+2}(1+t)^x = \sum_{n=0}^{\infty} Ch_n(x)\frac{t^n}{n!}, \text{ (see [5, 8, 9, 11, 13]).} \tag{1.6}$$

From (1.6), we have

$$\begin{aligned} \frac{2}{e^t+1}e^{xt} &= \sum_{n=0}^{\infty} Ch_n(x)\frac{1}{n!}(e^t-1)^n \\ &= \sum_{n=0}^{\infty} Ch_n(x)\sum_{m=n}^{\infty} S_2(m,n)\frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} Ch_n(x)S_2(m,n)\right)\frac{t^m}{m!}. \end{aligned} \tag{1.7}$$

Thus, by (1.1) and (1.7), we get

$$E_m(x) = \sum_{n=0}^{m} Ch_n(x)S_2(m,n), \ (m \geq 0). \tag{1.8}$$

The degenerate Changhee polynomials (called λ-Changhee polynomials) are given by the generating function to be

$$\frac{2\lambda}{2\lambda+\log(1+\lambda t)}\left(\lambda^{-1}\log(1+\lambda t)+1\right)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x)\frac{t^n}{n!} \text{ (see [10]).} \tag{1.9}$$

When $x = 0$, $Ch_{n,\lambda} = Ch_{n,\lambda}(0)$ are called *λ-Changhee numbers*.

In view of (1.2), we construct degenerate Changhee-Genocchi polynomials and numbers and investigate some properties of these numbers and polynomials in this paper.

2. Degenerate Changhee-Genocchi numbers and polynomials

Now, we consider the *degenerate Changhee-Genocchi polynomials* which are given by the generating function to be

$$\frac{2\lambda\log\left(1+\lambda^{-1}\log(1+\lambda t)\right)}{2\lambda+\log(1+\lambda t)}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^x = \sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.10}$$

When $x = 0$, $CG_{n,\lambda} = CG_{n,\lambda}(0)$ are called *degenerate Changhee-Geoncchi numbers*.

Note that

$$\lim_{\lambda\to 0}\frac{2\lambda\log\left(1+\lambda^{-1}\log(1+\lambda t)\right)}{2\lambda+\log(1+\lambda t)}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^x = \frac{2\log(1+t)}{1+t}(1+t)^x.$$

Now, we define *Changhee-Genocchi polynomials* as follow:

$$\frac{2\log(1+t)}{2+t}(1+t)^x = \sum_{n=0}^{\infty} CG_n(x)\frac{t^n}{n!}. \tag{2.11}$$

Thus, we note that $\lim_{\lambda\to 0} CG_{n,\lambda}(x) = CG_n(x)$. By replacing t by $\frac{1}{\lambda}(e^{\lambda t}-1)$ in (2.10), we get

$$\begin{aligned}\frac{2\log(1+t)}{2+t}(1+t)^x &= \sum_{l=0}^{\infty} CG_{l,\lambda}(x)\frac{1}{l!}(e^{\lambda t}-1)^l\lambda^{-l}\\ &= \sum_{l=0}^{\infty} CG_{l,\lambda}(x)\lambda^{-l}\sum_{n=l}^{\infty} S_2(n,l)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n} S_2(n,l)\lambda^{n-l}CG_{l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we get

$$CG_n(x) = \sum_{l=0}^{n} S_2(n,l)\lambda^{n-l}CG_{l,\lambda}(x),\ (n \geq 0). \tag{2.13}$$

From (1.2), we note that

$$\begin{aligned}&\frac{2\log\left(1+\log(1+\lambda t)\right)^{\frac{1}{\lambda}}}{2+\log(1+\lambda t)^{\frac{1}{\lambda}}}\left(1+\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^x\\ &= \sum_{n=0}^{\infty} CG_n(x)\frac{1}{n!}\left(\frac{1}{\lambda}\log(1+\lambda t)\right)^n\\ &= \sum_{n=0}^{\infty} CG_n(x)\sum_{m=n}^{\infty} S_1(m,n)\lambda^{m-n}\frac{t^m}{m!}\\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} CG_n(x)S_1(m,n)\lambda^{m-n}\right)\frac{t^m}{m!},\end{aligned} \tag{2.14}$$

where $S_1(m,n)$ is the Stirling numbers of the first kind.

Therefore, by (2.10) and (2.14), we get

$$CG_{n,\lambda}(x) = \sum_{n=0}^{m} CG_n(x)S_1(m,n)\lambda^{m-n},\ (m \geq 0). \tag{2.15}$$

It is not difficult to show that

$$\sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!} = \left(\sum_{l=0}^{\infty} CG_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^m\right). \tag{2.16}$$

Now, we observe that

$$\begin{aligned}\sum_{m=0}^{\infty}\binom{x}{m}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^m &= \sum_{m=0}^{\infty}(x)_m\frac{1}{m!}\lambda^{-m}\left(\log(1+\lambda t)\right)^m\\ &= \sum_{m=0}^{\infty}(x)_m\lambda^{-m}\sum_{k=m}^{\infty}S_1(k,m)\lambda^k\frac{t^k}{k!}\\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}(x)_m\lambda^{k-m}S_1(k,m)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.17}$$

By (2.16) and (2.17), we get

$$\begin{aligned}\sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!} &= \left(\sum_{l=0}^{\infty} CG_{l,\lambda}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty}\binom{x}{m}\left(\log(1+\lambda t)^{\frac{1}{\lambda}}\right)^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(x)_m\lambda^{k-m}S_1(k,m)\binom{n}{k}CG_{n-k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.18}$$

By comparing the coefficients on the both sides of (2.18), we get

$$CG_{n,\lambda}(x) = \sum_{k=0}^{n}\sum_{m=0}^{k}(x)_k\lambda^{k-m}S_1(k,m)CG_{n-k}\binom{n}{k}, \tag{2.19}$$

where $n \geq 0$.

Now, we observe that

$$\begin{aligned}&\log\left(1+\lambda^{-1}\log(1+\lambda t)\right)\\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^{-m}\left(\log(1+\lambda t)\right)^m\\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\lambda^{-m}m!\sum_{k=m}^{\infty}S_1(k,m)\lambda^k\frac{t^k}{k!}\\ &= \sum_{k=1}^{\infty}\left(\sum_{m=1}^{k}\frac{(-1)^k}{m}m!\lambda^{k-m}S_1(k,m)\right)\frac{t^k}{k!}\\ &= \sum_{m=1}^{\infty}\left(\sum_{m=1}^{k}(m-1)!(-1)^{m-1}\lambda^{k-m}S_1(k,m)\right)\frac{t^k}{k!}.\end{aligned} \tag{2.20}$$

From (1.8), (2.10) and (2.20), we have

$$\begin{aligned}\sum_{n=0}^{\infty} CG_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2\lambda\log\left(1+\lambda^{-1}\log(1+\lambda t)\right)}{2\lambda+\log(1+\lambda t)}\left(1+\lambda^{-1}\log(1+\lambda t)\right)^x \\ &= \left(\sum_{l=0}^{\infty} Ch_{l,\lambda}(x)\frac{t^l}{l!}\right)\left(\sum_{k=1}^{\infty}\left(\sum_{m=1}^{k}(m-1)!(-1)^{m-1}\lambda^{k-m}S_1(k,m)\right)\frac{t^k}{k!}\right) \\ &= \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\sum_{m=1}^{k}(m-1)!(-1)^{m-1}\lambda^{k-m}S_1(k,m)Ch_{n-k,\lambda}(x)\binom{n}{k}\right)\frac{t^n}{n!}.\end{aligned} \tag{2.21}$$

By (2.10), we easily get $CG_{0,\lambda}(x) = 0$. Comparing the coefficients on the both sides of (2.21), we have

$$CG_{n,\lambda}(x) = \sum_{k=1}^{n}\sum_{m=1}^{k}(m-1)!(-1)^{m-1}\lambda^{k-m}S_1(k,m)Ch_{n-k}(x)\binom{n}{k}, \tag{2.22}$$

where $n \in \mathbb{N}$.

Note that

$$\begin{aligned}\lim_{\lambda\to 0} CG_{n,\lambda}(x) &= \sum_{k=1}^{n}(k-1)!(-1)^{k-1}Ch_{n-k,\lambda}(x)\binom{n}{k} \\ &= CG_n(x),\ \ (n \in \mathbb{N}).\end{aligned}$$

From (2.12), we have

$$\begin{aligned}\frac{2t}{e^t+1}e^{xt} &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}S_2(m,l)\lambda^{m-l}CG_{l,\lambda}(x)\right)\frac{1}{m!}(e^t-1)^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}S_2(m,l)\lambda^{m-l}CG_{l,\lambda}(x)\sum_{n=m}^{\infty}S_2(n,m)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\sum_{l=0}^{m}S_2(m,l)\lambda^{m-l}S_2(n,m)CG_{l,\lambda}(x)\right)\frac{t^n}{n!}.\end{aligned} \tag{2.23}$$

From (1.2) and (2.23), we have

$$G_n(x) = \sum_{m=0}^{n}\sum_{l=0}^{m}S_2(m,l)S_2(n,m)\lambda^{m-l}CG_{l,\lambda}(x), \tag{2.24}$$

where $n \geq 0$.

Note that

$$\begin{aligned} G_n(x) &= \lim_{\lambda\to 0}\sum_{m=0}^{n}\sum_{l=0}^{m} S_2(m,l)S_2(n,m)\lambda^{m-l}CG_{l,\lambda}(x) \\ &= \sum_{m=0}^{n} S_2(n,m)CG_n(x). \end{aligned} \tag{2.25}$$

By (2.14), we get

$$\begin{aligned} &\frac{2t}{e^t+1}e^{xt} \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} CG_n(x)S_1(m,n)\lambda^{m-n}\right)\frac{1}{m!}\lambda^{-m}(e^{\lambda t}-1)^m \\ &= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} CG_n(x)S_1(n,m)\lambda^{m-n}\right)\sum_{k=m}^{\infty} S_2(k,m)\lambda^{k-m}\frac{t^n}{n!} \\ &= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k}\sum_{n=0}^{m} CG_n(x)S_1(n,m)S_2(k,m)\lambda^{k-n}\right)\frac{t^k}{k!}. \end{aligned} \tag{2.26}$$

Thus, by (2.26), we get

$$G_k(x) = \sum_{m=0}^{k}\sum_{n=0}^{m} CG_n(x)S_1(n,m)S_2(k,m)\lambda^{k-n}, \ (k \geq 0).$$

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