A note on degenerate Changhee-Genocchi polynomials and numbers

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Abstract

In this paper, we consider the degenerate Changhee-Genocchi polynomials and numbers, and give some identities for these numbers and polynomials.

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1. Introduction

As is well known, the Euler polynomials are defined by the generating function to be

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad (\text{see } [2, 4]).
\]  

(1.1)

When \(x = 0\), \(E_n = E_n(0)\) are called Euler numbers.

The Genocchi polynomials are defined by the generating function to be

\[
\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}.
\]  

(1.2)

When \(x = 0\), \(G_n = G_n(0)\) are called the Genocchi numbers (see [2, 6, 7, 12]).

From (1.2), we note that

\[
G_0 = 0, \quad \frac{G_{n+1}(x)}{n+1} = E_n(x), \quad (n \geq 0).
\]

In [3], L. Carlitz considered degenerate Euler polynomials which are defined by the generating function to be

\[
\frac{2}{(1 + \lambda t)^{1/2} + 1} \left(1 + \lambda t\right)^{1/2} e^{xt} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{t^n}{n!}.
\]  

(1.3)

When \(x = 0\), \(E_{n,\lambda} = E_{n,\lambda}(0)\) are called the degenerate Euler numbers.

Note that

\[
\lim_{\lambda \to 0} E_{n,\lambda}(x) = E_n(x), \quad (n \geq 0).
\]

By replacing \(t\) by \(\frac{1}{\lambda} (e^{\lambda t} - 1)\) in (1.3), we get

\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,\lambda}(x) \frac{1}{n! \lambda^n} (e^{\lambda t} - 1)^n
\]

\[
= \sum_{n=0}^{\infty} E_{n,\lambda}(x) \sum_{m=n}^{\infty} S_2(m, n) \lambda^{m-n} \frac{t^m}{m!}
\]

\[
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} E_{n,\lambda}(x) S_2(m, n) \lambda^{m-n} \right) \frac{t^m}{m!},
\]  

(1.4)

where \(S_2(m, n)\) is the Stirling numbers of the second kind.

By (1.1) and (1.4), we get

\[
E_m(x) = \sum_{n=0}^{m} E_{n,\lambda}(x) S_2(m, n) \lambda^{m-n}, \quad (m \geq 0).
\]  

(1.5)
Degenerate Changhee-Genocchi polynomials and numbers

Recently, Changhee polynomials are defined by the generating function to be
\[ \frac{2}{t + 2} (1 + t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}, \quad \text{see} \ [5, 8, 9, 11, 13]. \] (1.6)

From (1.6), we have
\[ \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} Ch_n(x) \frac{1}{n!} (e^t - 1)^n \\
= \sum_{n=0}^{\infty} Ch_n(x) \sum_{m=n}^{\infty} S_2(m, n) \frac{t^m}{m!} \\
= \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} Ch_n(x) S_2(m, n) \right) \frac{t^m}{m!}. \] (1.7)

Thus, by (1.1) and (1.7), we get
\[ E_m(x) = \sum_{n=0}^{m} Ch_n(x) S_2(m, n), \quad (m \geq 0). \] (1.8)

The degenerate Changhee polynomials (called \( \lambda \)-Changhee polynomials) are given by the generating function to be
\[ \frac{2\lambda}{2\lambda + \log(1 + \lambda t)} \left( \lambda^{-1} \log(1 + \lambda t) + 1 \right)^x = \sum_{n=0}^{\infty} Ch_{n,\lambda}(x) \frac{t^n}{n!}, \quad \text{see} \ [10]. \] (1.9)

When \( x = 0 \), \( Ch_{n,\lambda} = Ch_{n,\lambda}(0) \) are called \( \lambda \)-Changhee numbers.

In view of (1.2), we construct degenerate Changhee-Genocchi polynomials and numbers and investigate some properties of these numbers and polynomials in this paper.

2. Degenerate Changhee-Genocchi numbers and polynomials

Now, we consider the degenerate Changhee-Genocchi polynomials which are given by the generating function to be
\[ \frac{2\lambda \log(1 + \lambda^{-1} \log(1 + \lambda t))}{2\lambda + \log(1 + \lambda t)} \left( 1 + \lambda^{-1} \log(1 + \lambda t) \right)^x = \sum_{n=0}^{\infty} CG_{n,\lambda}(x) \frac{t^n}{n!}. \] (2.10)

When \( x = 0 \), \( CG_{n,\lambda} = CG_{n,\lambda}(0) \) are called degenerate Changhee-Geoncchi numbers.

Note that
\[ \lim_{\lambda \to 0} \frac{2\lambda \log(1 + \lambda^{-1} \log(1 + \lambda t))}{2\lambda + \log(1 + \lambda t)} \left( 1 + \lambda^{-1} \log(1 + \lambda t) \right)^x = \frac{2 \log(1 + t)}{1 + t} (1 + t)^x. \]
Now, we define Changhee-Genocchi polynomials as follow:

\[ \frac{2 \log(1 + t)}{2 + t} (1 + t)^x = \sum_{n=0}^{\infty} CG_n(x) \frac{t^n}{n!}. \]  \hspace{1cm} (2.11)

Thus, we note that \( \lim_{\lambda \to 0} CG_{n,\lambda}(x) = CG_n(x) \). By replacing \( t \) by \( \frac{1}{\lambda}(e^{\lambda t} - 1) \) in (2.10), we get

\[ \frac{2 \log(1 + t)}{2 + t} (1 + t)^x = \sum_{l=0}^{\infty} \frac{CG_l(x)}{l!} \left( \frac{1}{\lambda} \log(1 + \lambda t) \right)^l \]  \hspace{1cm} (2.12)

By (2.11) and (2.12), we get

\[ CG_n(x) = \sum_{l=0}^{n} \frac{S_2(n, l) \lambda^{n-l}}{l!} CG_{l,\lambda}(x), \quad (n \geq 0). \]  \hspace{1cm} (2.13)

From (1.2), we note that

\[ \frac{2 \log (1 + \log(1 + \lambda t)^{1/2})}{2 + \log(1 + \lambda t)^{1/2}} \left( 1 + \log(1 + \lambda t)^{1/2} \right)^x \]  
\[ = \sum_{n=0}^{\infty} CG_n(x) \frac{1}{n!} \left( \frac{1}{\lambda} \log(1 + \lambda t) \right)^n \]  
\[ = \sum_{n=0}^{\infty} CG_n(x) \sum_{m=n}^{\infty} \frac{S_1(m, n) \lambda^{m-n} t^m}{m!} \]  
\[ = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} CG_n(x) S_1(m, n) \lambda^{m-n} \right) \frac{t^m}{m!}, \]  \hspace{1cm} (2.14)

where \( S_1(m, n) \) is the Stirling numbers of the first kind.

Therefore, by (2.10) and (2.14), we get

\[ CG_{n,\lambda}(x) = \sum_{n=0}^{m} CG_n(x) S_1(m, n) \lambda^{m-n}, \quad (m \geq 0). \]  \hspace{1cm} (2.15)
Degenerate Changhee-Genocchi polynomials and numbers

It is not difficult to show that
\[
\sum_{n=0}^{\infty} CG_{n,\lambda}(x) \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} CG_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{x}{m} \left( \log(1 + \lambda t) \right)^m \right).
\] (2.16)

Now, we observe that
\[
\sum_{m=0}^{\infty} \binom{x}{m} \left( \log(1 + \lambda t) \right)^m = \sum_{m=0}^{\infty} (x)_m \frac{1}{m!} \lambda^{-m} (\log(1 + \lambda t))^m
\]
\[
= \sum_{m=0}^{\infty} (x)_m \lambda^{-m} \sum_{k=m}^{\infty} S_1(k, m) \frac{\lambda^k t^k}{k!}
\]
\[
= \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} (x)_m \lambda^{k-m} S_1(k, m) \right) \frac{t^k}{k!}.
\] (2.17)

By (2.16) and (2.17), we get
\[
\sum_{n=0}^{\infty} CG_{n,\lambda}(x) \frac{t^n}{n!} = \left( \sum_{l=0}^{\infty} CG_{l,\lambda} \frac{t^l}{l!} \right) \left( \sum_{m=0}^{\infty} \binom{x}{m} \left( \log(1 + \lambda t) \right)^m \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} \sum_{m=0}^{k} (x)_m \lambda^{k-m} S_1(k, m) \binom{n}{k} CG_{n-k} \right) \frac{t^n}{n!}.
\] (2.18)

By comparing the coefficients on the both sides of (2.18), we get
\[
CG_{n,\lambda}(x) = \sum_{k=0}^{n} \sum_{m=0}^{k} (x)_k \lambda^{k-m} S_1(k, m) CG_{n-k} \binom{n}{k},
\] (2.19)

where \( n \geq 0 \).

Now, we observe that
\[
\log \left( 1 + \frac{1}{\lambda^{-1} \log(1 + \lambda t)} \right)
\]
\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \lambda^{-m} (\log(1 + \lambda t))^m
\]
\[
= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \lambda^{-m} m! \sum_{k=m}^{\infty} S_1(k, m) \lambda^k t^k
\]
\[
= \sum_{k=1}^{\infty} \left( \sum_{m=1}^{k} \frac{(-1)^k}{m!} \lambda^{k-m} S_1(k, m) \right) \frac{t^k}{k!}
\]
\[
= \sum_{m=1}^{\infty} \left( \sum_{m=1}^{k} (m-1)!(\log t)^{m-1} \lambda^{k-m} S_1(k, m) \right) \frac{t^k}{k!}.
\] (2.20)
From (1.8), (2.10) and (2.20), we have

\[
\begin{align*}
\sum_{n=0}^{\infty} CG_{n,\lambda}(x) \frac{t^n}{n!} &= \frac{2\lambda \log \left( 1 + \lambda^{-1} \log(1 + \lambda t) \right)}{2\lambda + \log(1 + \lambda t)} \left( 1 + \lambda^{-1} \log(1 + \lambda t) \right)^x \\
&= \left( \sum_{l=0}^{\infty} Ch_{l,\lambda}(x) \frac{t^l}{l!} \right) \left( \sum_{k=1}^{\infty} \left( \sum_{m=1}^{k} (m-1)!(-1)^{m-1} \lambda^{k-m} S_1(k,m) \right) \frac{t^k}{k!} \right) \\
&= \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \sum_{m=1}^{k} (m-1)!(-1)^{m-1} \lambda^{k-m} S_1(k,m) Ch_{n-k,\lambda}(x) \binom{n}{k} \right) \frac{t^n}{n!}.
\end{align*}
\]  

(2.21)

By (2.10), we easily get \( CG_{0,\lambda}(x) = 0 \). Comparing the coefficients on the both sides of (2.21), we have

\[
CG_{n,\lambda}(x) = \sum_{k=1}^{n} \sum_{m=1}^{k} (m-1)!(-1)^{m-1} \lambda^{k-m} S_1(k,m) Ch_{n-k,\lambda}(x) \binom{n}{k},
\]  

(2.22)

where \( n \in \mathbb{N} \).

Note that

\[
\lim_{\lambda \to 0} CG_{n,\lambda}(x) = \sum_{k=1}^{n} (k-1)!(-1)^{k-1} Ch_{n-k,\lambda}(x) \binom{n}{k} = CG_n(x), \quad (n \in \mathbb{N}).
\]

From (2.12), we have

\[
\frac{2t}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} S_2(m,l) \lambda^{m-l} CG_{l,\lambda}(x) \right) \frac{1}{m!} (e^t - 1)^m \\
&= \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} S_2(m,l) \lambda^{m-l} CG_{l,\lambda}(x) \sum_{n=0}^{\infty} S_2(n,m) \frac{t^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left( \sum_{m=0}^{n} \sum_{l=0}^{m} S_2(m,l) \lambda^{m-l} S_2(n,m) CG_{l,\lambda}(x) \right) \frac{t^n}{n!}.
\]

(2.23)

From (1.2) and (2.23), we have

\[
G_n(x) = \sum_{m=0}^{n} \sum_{l=0}^{m} S_2(m,l) S_2(n,m) \lambda^{m-l} CG_{l,\lambda}(x),
\]

(2.24)

where \( n \geq 0 \).
Degenerate Changhee-Genocchi polynomials and numbers

Note that

\[ G_n(x) = \lim_{\lambda \to 0} \sum_{m=0}^{n} \sum_{l=0}^{m} S_2(m, l) S_2(n, m) \lambda^{m-l} C_{G_1, \lambda}(x) \]

\[ = \sum_{m=0}^{n} S_2(n, m) C G_n(x). \]  

(2.25)

By (2.14), we get

\[ \frac{2t}{e^t + 1} e^{xt} = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} C G_n(x) S_1(m, n) \lambda^{m-n} \right) \frac{1}{m!} \lambda^{-m} (e^{\lambda t} - 1)^m \]

\[ = \sum_{m=0}^{\infty} \left( \sum_{n=0}^{m} C G_n(x) S_1(m, n) \lambda^{m-n} \right) \sum_{k=m}^{\infty} S_2(k, m) \lambda^{k-m} \frac{t^n}{n!} \]

\[ = \sum_{k=0}^{\infty} \left( \sum_{m=0}^{k} \sum_{n=0}^{m} C G_n(x) S_1(n, m) S_2(k, m) \lambda^{k-n} \right) \frac{t^k}{k!}. \]  

(2.26)

Thus, by (2.26), we get

\[ G_k(x) = \sum_{m=0}^{k} \sum_{n=0}^{m} C G_n(x) S_1(n, m) S_2(k, m) \lambda^{k-n}, \ (k \geq 0). \]

References


