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The sum of k distanced tribonacci numbers

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Abstract

We investigate a recurrence formula of a tribonacci number by any $k > 0$ step distanced tribonacci numbers. We also study the sum of the first t tribonacci numbers which are k step apart for any $t > 0$.

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1. Introduction

A tribonacci number T_n satisfies the recurrence $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ starting from $T_i = 1, 1, 2$ $(1 \leq i \leq 3)$. The numbers are extended to non positive index n that $T_0 = 0$, $T_{-1} = 0$, $T_{-2} = 1$ etc. The tribonacci numbers have many applications in various areas such as environment, biology, chemistry, arts and music, just like the fibonacci numbers [6]. Some articles including [3], [4], [5] and [7] were devoted to the research of tribonacci numbers. In particular it was proved in [1] that T_n satisfies a k distanced tribonacci rule that

$$T_{n+3k} = a_k T_{n+2k} + b_k T_{n+k} + T_n \quad (k > 0) \tag{1-1}$$

where a_k, b_k satisfies $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ and $b_{k+3} = -b_{k+2} - b_{k+1} + b_k$ with three initials $a_i = 1, 3, 7$ and $b_i = 1, 1, -5$ $(1 \leq i \leq 3)$ respectively. For instance, when $1 \leq k \leq 10$, the coefficients (a_k, b_k) are as follows

Table 1

k	(a_k, b_k)	k	(a_k, b_k)	k	(a_k, b_k)	k	(a_k, b_k)	k	(a_k, b_k)
1	(1, 1)	2	(3, 1)	3	(7, −5)	4	(11, 5)	5	(21, 1)
6	(39, −11)	7	(71, 15)	8	(131, −3)	9	(241, −23)	10	(443, 41)

Moreover a_k, b_k can be expressed by tribonacci numbers that $a_k = 3T_k - T_{k-6}$ and $b_k = -3T_{-k} + T_{-k-6} = -a_{-k}$.

For any $k, t > 0$, the t degree k distanced sum means a sum of the $t+1$ tribonacci numbers that are k step apart. Write

$$s_t^{(k,r)} = T_r + T_{k+r} + \cdots + T_{kt+r} = \sum_{i=0}^{t} T_{ki+r}$$

for some $r \geq 1$. When $1 \leq r \leq k \leq 10$, [2] proved

$$s_t^{(k,r)} = \frac{1}{a_k + b_k}(T_{k(t+1)+r} + (b_k + 1)T_{kt+r} + T_{k(t-1)+r} - \lambda^{(k,r)}) \tag{1-2}$$

where $\lambda^{(k,r)}$ called the tail of $s_t^{(k,r)}$ was found by experimental search:

Table 2

k	$\{\lambda^{(k,r)}\}_{r=1}^{k}$	k	$\{\lambda^{(k,r)}\}_{r=1}^{k}$
1	$\{1\}$	6	$\{-12, 6, 4, -2, 8, 10\}$
2	$\{0, 2\}$	7	$\{-29, 13, 9, -7, 15, 17, 25\}$
3	$\{-1, 1, 1\}$	8	$\{-48, 16, 16, -16, 16, 16, 16, 48\}$
4	$\{-4, 4, 4, 4\}$	9	$\{-87, 35, 21, -31, 25, 15, 9, 49, 73\}$
5	$\{-7, 3, 5, 1, 9\}$	10	$\{-176, 66, 44, -66, 44, 22, 0, 66, 88, 154\}$

Another recurrence of $s_t^{(k,r)}$ by $s_j^{(k,r)}$ is

$$s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + s_{t-3}^{(k,r)} + \lambda^{(k,r)}. \tag{1-3}$$

In particular if $k = r = 4$, then since

$$s_{t-1}^{(4,4)} = \sum_{i=0}^{t} T_{4i}$$

and $(a_4, b_4) = (11, 5)$, (1-2) shows

$$\sum_{i=0}^{t} T_{4i} = \frac{1}{T_4^2}(T_{4t+4} + 6T_{4t} + T_{4t-4} - T_4),$$

proved in [5].

A goal of the work is to study t degree k distanced sum $s_t^{(k,r)}$ for any integers $k, t > 0$. We shall investigate the coefficients a_k, b_k in (1-1) as well as the tail $\lambda^{(k,r)}$ in (1-2), since theses integers play important roles for $s_t^{(k,r)}$. And by letting $R_t^{(k)}$ as the sum of t degree k distanced sum $s_t^{(k,r)}$ for all $1 \leq r \leq k$, we investigate sequential relationships of $R_t^{(k)}$ for each $t \geq 0$.

2. Recurrence formula for t degree k distanced sum $s_t^{(k,r)}$

By a k-tribo table, we mean a rectangle table with k columns consisting of all tribonacci numbers in order (left below). So if $1 \le r \le k$, the t degree k distanced sum $s_t^{(k,r)} = \sum_{i=0}^{t} T_{ki+r}$ can be regarded as a sum of $t+1$ entries in rth column. We call the table composed of all $s_t^{(k,r)}$ a k distanced sum table (right below).

k-tribo table				
T_1	$\cdots$	T_r	$\cdots$	T_k
T_{k+1}	$\cdots$	T_{k+r}	$\cdots$	T_{2k}
T_{2k+1}	$\ddots$	T_{2k+r}	$\cdots$	

t	k distanced sum table				
0	$s_0^{(k,1)}$	$\cdots$	$s_0^{(k,r)}$	$\cdots$	$s_0^{(k,k)}$
1	$s_1^{(k,1)}$	$\cdots$	$s_1^{(k,r)}$	$\cdots$	$s_1^{(k,k)}$
2	$s_2^{(k,1)}$	$\ddots$	$s_2^{(k,r)}$	$\cdots$	$s_2^{(k,k)}$

We begin to investigate relations between a_k and b_k in (1-1).

Lemma 2.1. Let $a_k + b_k = \Gamma_k$ and $a_k - b_k = \Delta_k$. Then

$$\Gamma_{k+3} = \Gamma_{k+2} + \Gamma_{k+1} + \Gamma_k - 2(b_{k+2} + b_{k+1})$$

and

$$\Delta_{k+3} = \Delta_{k+2} + \Delta_{k+1} + \Delta_k + 2(b_{k+2} + b_{k+1}).$$

Proof. Let $\mu_k = 2(b_{k+2} + b_{k+1})$ for convenience. Since $a_{k+3} = a_{k+2} + a_{k+1} + a_k$ and $b_{k+3} = -b_{k+2} - b_{k+1} + b_k$, we have

$$\begin{aligned}\Gamma_{k+3} &= a_{k+3} + b_{k+3} = a_{k+2} + a_{k+1} + a_k - b_{k+2} - b_{k+1} + b_k \\ &= (a_{k+2} - b_{k+2}) + (a_{k+1} - b_{k+1}) + a_k + b_k = \Delta_{k+2} + \Delta_{k+1} + \Gamma_k \\ &= (a_{k+2} + b_{k+2}) + (a_{k+1} + b_{k+1}) + (a_k + b_k) - 2b_{k+2} - 2b_{k+1} \\ &= \Gamma_{k+2} + \Gamma_{k+1} + \Gamma_k - \mu_k\end{aligned}$$

and similarly

$$\begin{aligned}\Delta_{k+3} &= (a_{k+2} + b_{k+2}) + (a_{k+1} + b_{k+1}) + a_k - b_k \\ &= \Gamma_{k+2} + \Gamma_{k+1} + \Delta_k \\ &= (a_{k+2} - b_{k+2}) + (a_{k+1} - b_{k+1}) + (a_k - b_k) + 2b_{k+2} + 2b_{k+1} \\ &= \Delta_{k+2} + \Delta_{k+1} + \Delta_k - \mu_k.\end{aligned}$$

∎

So it follows that

$$\Gamma_{k+3} - \Gamma_k = \Delta_{k+2} + \Delta_{k+1}$$

and

$$\Delta_{k+3} - \Delta_k = \Gamma_{k+2} + \Gamma_{k+1}.$$

For example, from $a_i = 1, 3, 7$ and $b_i = 1, 1, -5$ ($i = 1, 2, 3$), $\Gamma_i = 2, 4, 2$ and $\Delta_i = 0, 2, 12$, so Lemma 2.1 shows

$$\{\Gamma_i\} = \{2, 4, 2, 16, 22, 28, 86, \ldots\}$$

and

$$\{\Delta_i\} = \{0, 2, 12, 6, 20, 50, 56, \ldots\}.$$

A recurrence formula of $s_t^{(k,r)}$ by $s_j^{(k,r)}$ is as follows.

Theorem 2.2.

$$\begin{aligned} s_t^{(k,r)} &= (a_k + 1)s_{t-1}^{(k,r)} - \Delta_k s_{t-2}^{(k,r)} \\ &\quad - (b_k - 1)s_{t-3}^{(k,r)} - s_{t-4}^{(k,r)} \end{aligned}$$

for any $t > 0$. In particular $s_t^{(1,1)} = 2s_{t-1}^{(1,1)} - s_{t-4}^{(1,1)}$.

Proof. When $k = 3$ we have

$$\begin{aligned} 8s_3^{(3,1)} - 12s_2^{(3,1)} + 6s_1^{(3,1)} - s_0^{(3,1)} &= 8(178) - 12(29) + 6(5) - 1 = s_4^{(3,1)}, \\ 8s_3^{(3,2)} - 12s_2^{(3,2)} + 6s_1^{(3,2)} - s_0^{(3,2)} &= 8(326) - 12(52) + 6(8) - 1 = s_4^{(3,2)}, \\ 8s_3^{(3,3)} - 12s_2^{(3,3)} + 6s_1^{(3,3)} - s_0^{(3,3)} &= 8(600) - 12(96) + 6(15) - 2 = s_4^{(3,3)}. \end{aligned}$$

By assuming

$$s_t^{(3,r)} = 8s_{t-1}^{(3,r)} - 12s_{t-2}^{(3,r)} + 6s_{t-3}^{(3,r)} - s_{t-4}^{(3,r)}$$

for some t, the identity (1-1) and $(a_3, b_3) = (7, -5)$ imply

$$\begin{aligned} s_{t+1}^{(3,r)} &= s_t^{(3,r)} + T_{3(t+1)+r} \\ &= (8s_{t-1}^{(3,r)} - 12s_{t-2}^{(3,r)} + 6s_{t-3}^{(3,r)} - s_{t-4}^{(3,r)}) \\ &\quad + (7T_{3t+r} - 5T_{3(t-1)+r} + T_{3(t-2)+r}) \\ &= (8s_{t-1}^{(3,r)} + 8T_{3t+r}) - (12s_{t-2}^{(3,r)} + 12T_{3(t-1)+r}) \\ &\quad + (6s_{t-3}^{(3,r)} + 6T_{3(t-2)+r}) \\ &\quad - s_{t-4}^{(3,r)} - (T_{3t+r} - 7T_{3(t-1)+r} + 5T_{3(t-2)+r}) \\ &= 8s_t^{(3,r)} - 12s_{t-1}^{(3,r)} + 6s_{t-2}^{(3,r)} - s_{t-4}^{(3,r)} - T_{3(t-3)+r} \\ &= 8s_t^{(3,r)} - 12s_{t-1}^{(3,r)} + 6s_{t-2}^{(3,r)} - s_{t-3}^{(3,r)}. \end{aligned}$$

Now let us consider any integer k and $t \geq 1$. Then it follows that

$$\begin{aligned}
&(a_k+1)s_{t-1}^{(k,r)} - \Delta_k s_{t-2}^{(k,r)} - (b_k-1)s_{t-3}^{(k,r)} - s_{t-4}^{(k,r)} \\
&= a_k(s_{t-1}^{(k,r)} - s_{t-2}^{(k,r)}) + b_k(s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}) \\
&\quad + s_{t-1}^{(k,r)} + (s_{t-3}^{(k,r)} - s_{t-4}^{(k,r)}) \\
&= a_k T_{k(t-1)+r} + b_k T_{k(t-2)+r} + s_{t-1}^{(k,r)} + T_{k(t-3)+r} \\
&= T_{kt+r} + s_{t-1}^{(k,r)} = s_t^{(k,r)}
\end{aligned}$$

due to $s_j^{(k,r)} - s_{j-1}^{(k,r)} = T_{kj+r}$ for all j and (1-1). In particular we have

$$\begin{aligned}
s_t^{(1,1)} &= T_1 + \cdots + T_t + T_{t+1} = (T_1 + \cdots + T_t) + (T_t + T_{t-1} + T_{t-2}) \\
&= s_{t-1}^{(1,1)} + (T_t + T_{t-1} + T_{t-2}) + [T_{t-3} + \cdots + T_1] - [T_{t-3} + \cdots + T_1] \\
&= s_{t-1}^{(1,1)} + s_{t-1}^{(1,1)} - s_{t-4}^{(1,1)} \\
&= 2s_{t-1}^{(1,1)} - s_{t-4}^{(1,1)} = (a_1+1)s_{t-1}^{(1,1)} - s_{t-4}^{(1,1)}
\end{aligned}$$

since $(a_1, b_1) = (1, 1)$. This completes the proof. ■

For example, the 7 degree 5 distanced sum

$$s_7^{(5,3)} = T_3 + T_{5+3} + \cdots + T_{7\cdot 5+3}$$

is

$$\begin{aligned}
s_7^{(5,3)} &= (a_5+1)s_6^{(5,3)} - (a_5 - b_5)s_5^{(5,3)} + (b_5-1)s_4^{(5,3)} - s_3^{(5,3)} \\
&= 22 \cdot 191074895 - 20(9077294) - 20486 = 4022081324.
\end{aligned}$$

This can be compared to (1-2) and (1-3) involving tail $\lambda^{(5,3)}$ that

$$\begin{aligned}
s_7^{(5,3)} &= \frac{1}{22}(80641778674 + 2 \cdot 3831006429 + 181997601 - 5) = 4022081324 \text{ or} \\
s_7^{(5,3)} &= 21 \cdot 191074895 + 9077294 + 431230 + 5 \\
&= 4022081324.
\end{aligned}$$

3. Sum $R_t^{(k)}$ of t degree k distanced sums

Let $R_t^{(k)} = \sum_{r=1}^{k} s_t^{(k,r)}$ be the sum of t degree k distanced sums $s_t^{(k,r)}$ for $1 \leq r \leq k$. From the k distanced sum table, $R_t^{(k)}$ can be regarded as the sum of tth row.

Lemma 3.1. $R_t^{(k)} = \sum_{i=1}^{k(t+1)} T_i$ and $R_t^{(k)} - R_{t-1}^{(k)} = \sum_{j=1}^{k} T_{kt+j}$ for any $k, t > 0$.

Proof. We clearly have

$$R_t^{(k)} = \sum_{r=1}^{k} s_t^{(k,r)} = \sum_{r=1}^{k}\sum_{i=0}^{t} T_{ki+r} = \sum_{j=1}^{kt+k} T_j,$$

and

$$R_t^{(k)} - R_{t-1}^{(k)} = \sum_{j=1}^{k(t+1)} T_j - \sum_{j=1}^{kt} T_j = T_{kt+1} + T_{kt+2} + \cdots + T_{kt+k}.$$

■

A recurrence formula of $R_t^{(k)}$ by $R_j^{(k)}$ $(t-3 \le j \le t-1)$ is as follows.

Theorem 3.2. $R_{t+3}^{(k)} = a_k R_{t+2}^{(k)} + b_k R_{t+1}^{(k)} + R_t^{(k)} + \frac{1}{2}\Gamma_k$ for any $k, t > 0$.

Proof. Note that $\Gamma_k = a_k + b_k$ is even for all $k \ge 1$, for

$$\Gamma_{k+3} = \Gamma_{k+2} + \Gamma_{k+1} + \Gamma_k - 2(b_{k+2} + b_{k+1})$$

in Lemma 2.1 with $\Gamma_1 = 2$, $\Gamma_2 = 4$ and $\Gamma_3 = 2$. We begin to consider the k distanced sum table for $k = 3, 4$. Since $a_3 = 7$, $b_3 = -5$ and $\frac{\Gamma_3}{2} = 1$, it is easy to see $1104 = 7 \cdot 177 - 5 \cdot 28 + 4 + 1$, i.e.,

$$R_3^{(3)} = 7R_{t+2}^{(3)} - 5R_{t+1}^{(3)} + R_t^{(3)} + 1.$$

Similarly the 4 distanced sum table shows that

$$R_3^{(4)} = 11R_2^{(4)} + 5R_1^{(4)} + R_0^{(4)} + 8 = a_4 R_2^{(4)} + b_4 R_1^{(4)} + R_0^{(4)} + \frac{1}{2}\Gamma_4.$$

t	$s_t^{(3,1)}$	$s_t^{(3,2)}$	$s_t^{(3,3)}$	$R_t^{(3)}$
0	1	1	2	4
1	5	8	15	28
2	29	52	96	177
3	178	326	600	1104

t	$s_t^{(4,1)}$	$s_t^{(4,2)}$	$s_t^{(4,3)}$	$s_t^{(4,4)}$	$R_t^{(4)}$
0	1	1	2	4	8
1	8	14	26	48	96
2	89	163	300	552	1104
3	1016	1868	3436	6320	12640

Now let $k > 0$ be any integer and assume

$$R_t^{(k)} = a_k R_{t-1}^{(k)} + b_k R_{t-2}^{(k)} + R_{t-3}^{(k)} + \frac{1}{2}\Gamma_k$$

is true for some $t \geq 3$. Then due to Lemma 3.1, we have

$$\begin{aligned} R_{t+1}^{(k)} &= \sum_{i=1}^{k(t+2)} T_i \\ &= \sum_{i=1}^{k(t+1)} T_i + \sum_{j=1}^{k} T_{k(t+1)+j} \\ &= R_t^{(k)} + \sum_{j=1}^{k} T_{k(t+1)+j}. \end{aligned}$$

By applying the identity (1-1), it follows that

$$\begin{aligned} R_{t+1}^{(k)} &= R_t^{(k)} + \sum_{j=1}^{k}(a_k T_{kt+j} + b_k T_{k(t-1)+j} + T_{k(t-2)+j}) \\ &= R_t^{(k)} + a_k \sum_{j=1}^{k} T_{kt+j} + b_k \sum_{j=1}^{k} T_{k(t-1)+j} \\ &\quad + \sum_{j=1}^{k} T_{k(t-2)+j} \\ &= R_t^{(k)} + a_k(R_t^{(k)} - R_{t-1}^{(k)}) + b_k(R_{t-1}^{(k)} - R_{t-2}^{(k)}) \\ &\quad + (R_{t-2}^{(k)} - R_{t-3}^{(k)}) \end{aligned}$$

because of Lemma 3.1 again. So the mathematical induction shows

$$\begin{aligned} R_{t+1}^{(k)} &= a_k R_t^{(k)} + b_k R_{t-1}^{(k)} + R_{t-2}^{(k)} + [R_t^{(k)} - a_k R_{t-1}^{(k)} - b_k R_{t-2}^{(k)} - R_{t-3}^{(k)}] \\ &= a_k R_t^{(k)} + b_k R_{t-1}^{(k)} + R_{t-2}^{(k)} + \frac{\Gamma_k}{2}. \end{aligned}$$

■

Besides the recurrence in Theorem 3.2, we have another form of $R_t^{(k)}$ by $s_j^{(k,r)}$.

Theorem 3.3.

$$R_t^{(k)} = s_{t+1}^{(k,1)} + \sum_{j=1}^{k-3} s_t^{(k,j)} - 1$$

for all $k \geq 3$.

Proof. By employing the identities

$$s_t^{(k,r)} + s_t^{(k,r+1)} + s_t^{(k,r+2)} = s_t^{(k,r+3)} (1 \leq r \leq k-3)$$

and

$$s_t^{(k,k-2)} + s_t^{(k,k-1)} + s_t^{(k,k)} = s_{t+1}^{(k,1)} - 1 \text{ in [2]},$$

we have

$$\begin{aligned} R_t^{(3)} &= s_t^{(3,1)} + s_t^{(3,2)} + s_t^{(3,3)} = s_{t+1}^{(3,1)} - 1, \\ R_t^{(4)} &= s_t^{(4,1)} + s_t^{(4,2)} + s_t^{(4,3)} + s_t^{(4,4)} \\ &= s_{t+1}^{(4,1)} + s_t^{(4,1)} - 1, \end{aligned}$$

and

$$R_t^{(5)} = s_{t+1}^{(5,1)} + s_t^{(5,1)} + s_t^{(5,2)} - 1.$$

Hence for any $k > 5$, it follows immediately that

$$\begin{aligned} R_t^{(k)} &= s_t^{(k,1)} + \cdots + s_t^{(k,k-3)} + s_{t+1}^{(k,1)} - 1 \\ &= s_{t+1}^{(k,1)} + \sum_{j=1}^{k-3} s_t^{(k,j)} - 1. \end{aligned}$$

■

Corollary 3.4. $s_t^{(4,1)} + s_t^{(4,3)} + s_t^{(4,4)} = \frac{1}{2}(s_{t+1}^{(4,2)} + s_{t-1}^{(4,2)})$ for all $t \geq 0$.

Proof. Since $s_t^{(k,k-1)} + s_t^{(k,k)} + s_{t+1}^{(k,1)} = s_{t+1}^{(k,2)}$ for any k, we have

$$\begin{aligned} s_t^{(4,2)} + s_{t-1}^{(4,2)} &= \left[s_{t-1}^{(4,3)} + s_{t-1}^{(4,4)} + s_t^{(4,1)}\right] + \left[s_t^{(4,1)} - 1 - s_{t-1}^{(4,3)} - s_{t-1}^{(4,4)}\right] \\ &= 2s_t^{(4,1)} - 1. \end{aligned}$$

Therefore it follows that

$$\begin{aligned} s_{t+1}^{(4,2)} + s_{t-1}^{(4,2)} &= (s_t^{(4,3)} + s_t^{(4,4)} + s_{t+1}^{(4,1)}) + s_{t-1}^{(4,2)} \\ &= s_t^{(4,3)} + s_t^{(4,4)} + (s_t^{(4,2)} + s_t^{(4,3)} + s_t^{(4,4)} + 1) + s_{t-1}^{(4,2)} \\ &= 2(s_t^{(4,3)} + s_t^{(4,4)}) + (s_t^{(4,2)} + s_{t-1}^{(4,2)}) + 1 \\ &= 2(s_t^{(4,3)} + s_t^{(4,4)}) + 2s_t^{(4,1)} \\ &= 2(s_t^{(4,1)} + s_t^{(4,3)} + s_t^{(4,4)}). \end{aligned}$$

■

4. Recurrence formula for tail set

We now focus on the integer $\lambda^{(k,r)}$ satisfying

$$s_t^{(k,r)} = a_k s_{t-1}^{(k,r)} + b_k s_{t-2}^{(k,r)} + s_{t-3}^{(k,r)} + \lambda^{(k,r)}.$$

The $\lambda^{(k,r)}$ is called a tail of $s_t^{(k,r)}$, and Table 2 is the list of $\lambda^{(k,r)}$ for $1 \le k \le 10$. We keep the notations $\Gamma_k = a_k + b_k$ and $\Delta_k = a_k - b_k$.

Lemma 4.1. [2] For any $1 \le r \le k$, we have the followings.

(1) $\lambda^{(k,r)} + \lambda^{(k,r+1)} + \lambda^{(k,r+2)} = \lambda^{(k,r+3)}$ for $1 \le r \le k-3$ (tribonacci rule).

(2) $\lambda^{(k,k-2)} + \lambda^{(k,k-1)} + \lambda^{(k,k)} = \lambda^{(k,1)} + \Gamma_k$.

(3) $\lambda^{(k,k-1)} + \lambda^{(k,k)} + \lambda^{(k,1)} = \lambda^{(k,2)}$, $\lambda^{(k,k)} + \lambda^{(k,1)} + \lambda^{(k,2)} = \lambda^{(k,3)}$ (cyclic rule).

Owing to the tribonacci rule of $\lambda^{(k,r)}$, the tail set $\{\lambda^{(k,r)}\}_{r=1}^{k}$ $(1 \le k \le 10)$ is determined by the following three initials.

k		k		k		k	
3	$(-1, 1, 1)$	4	$(-4, 4, 4)$	5	$(-7, 3, 5)$	6	$(-12, 6, 4)$
7	$(-29, 13, 9)$	8	$(-48, 16, 16)$	9	$(-87, 35, 21)$	10	$(-176, 66, 44)$

Theorem 4.2. For all $1 \le r \le k$, we have the followings.

(1) $\lambda^{(k,r)} = T_{2k+r} + (1 - a_k)T_{k+r} + (1 - \Gamma_k)T_r$.

(2) $s_t^{(k,r)} = \dfrac{1}{\Gamma_k}(T_{k(t+1)+r} + (b_k + 1)T_{kt+r} + T_{k(t-1)+r} - \lambda^{(k,r)})$.

Proof. The identity

$$\lambda^{(k,r)} = s_t^{(k,r)} - a_k s_{t-1}^{(k,r)} - b_k s_{t-2}^{(k,r)} - s_{t-3}^{(k,r)}$$

shows that $\lambda^{(k,r)}$ does not depend on t. Thus if we assume $t = 3$ then

$$\lambda^{(k,r)} = s_3^{(k,r)} - a_k s_2^{(k,r)} - b_k s_1^{(k,r)} - s_0^{(k,r)}.$$

But since

$$\begin{aligned} s_0^{(k,r)} &= T_r, \\ s_1^{(k,r)} &= T_r + T_{k+r}, \\ s_2^{(k,r)} &= T_r + T_{k+r} + T_{2k+r} \end{aligned}$$

and

$$\begin{aligned} s_3^{(k,r)} &= T_r + T_{k+r} + T_{2k+r} + T_{3k+r} \\ &= 2T_r + (1 + b_k)T_{k+r} + (1 + a_k)T_{2k+r}, \end{aligned}$$

we have

$$\begin{aligned} \lambda^{(k,r)} &= (2T_r + (1 + b_k)T_{k+r} + (1 + a_k)T_{2k+r}) \\ &\quad - a_k(T_r + T_{k+r} + T_{2k+r}) - b_k(T_r + T_{k+r}) - T_r \\ &= T_{2k+r} + (1 - a_k)T_{k+r} + (1 - \Gamma_k)T_r. \end{aligned}$$

On the other hand when $1 \le k \le 10$ the identity (2) is observed from Table 1 and Table 2. Now let us consider any positive integer k. Since

$$\begin{aligned} \lambda^{(k,r)} &= \sum_{i=0}^{t+1} T_{ki+r} - a_k \sum_{i=0}^{t} T_{ki+r} - b_k \sum_{i=0}^{t-1} T_{ki+r} - \sum_{i=0}^{t-2} T_{ki+r} \\ &= (T_{k(t-1)+r} + T_{kt+r} + T_{k(t+1)+r}) - a_k \sum_{i=0}^{t} T_{ki+r} - b_k \sum_{i=0}^{t-1} T_{ki+r}, \end{aligned}$$

we have

$$\begin{aligned} &T_{k(t+1)+r} + (b_k + 1)T_{kt+r} + T_{k(t-1)+r} - \lambda^{(k,r)} \\ &\quad = T_{k(t+1)+r} + (b_k + 1)T_{kt+r} + T_{k(t-1)+r} \\ &\quad - (T_{k(t-1)+r} + T_{kt+r} + T_{k(t+1)+r}) + a_k \sum_{i=0}^{t} T_{ki+r} + b_k \sum_{i=0}^{t-1} T_{ki+r} \\ &= b_k T_{kt+r} + a_k \sum_{i=0}^{t} T_{ki+r} + b_k \sum_{i=0}^{t-1} T_{ki+r} \\ &= (a_k + b_k) \sum_{i=0}^{t} T_{ki+r} \\ &= \Gamma_k s_t^{(k,r)}. \end{aligned}$$

This completes the proof. ■

For example when $i = 8, 9, 10$, $(a_i, b_i) = (131, -3), (241, -23)$ and $(443, 41)$ so that $a_{11} = 815$ and $b_{11} = -41 + 23 - 3 = -21$, thus $\Gamma_{11} = 795$. Hence

$$\begin{aligned}\lambda^{(11,1)} &= (1 - \Gamma_{11})T_1 + (1 - a_{11})T_{12} + T_{23} \\ &= -793 - 814(504) + 410744 = -305, \\ \lambda^{(11,2)} &= (1 - \Gamma_{11})T_2 + (1 - a_{11})T_{13} + T_{24} \\ &= -793 - 814(927) + 755476 = 105\end{aligned}$$

and

$$\lambda^{(11,3)} = (1 - \Gamma_{11})T_3 + (1 - a_{11})T_{14} + T_{25} = 81.$$

Then the cyclic rule of $\lambda^{(k,r)}$ shows that $\lambda^{(11,4)} = -305 + 105 + 81 = -119$, $\lambda^{(11,5)} = 105 + 81 - 119 = 67$, etc. So for instance we have

$$\begin{aligned}\{\lambda^{(11,r)}\} &= \{-305, 105, 81, -119, 67, 29, -23, 73, 79, 129, 281\}, \\ \{\lambda^{(12,r)}\} &= \{-564, 212, 132, -220, 124, 36, -60, 100, 76, 116, 292, 484\},\end{aligned}$$

and so on.

Theorem 4.3. For every $k \geq 1$, $\displaystyle\sum_{r=1}^{k} \lambda^{(k,r)} = \frac{1}{2}\Gamma_k$.

Proof. By means of Theorem 3.2 and $R_t^{(k)} = \displaystyle\sum_{r=1}^{k} s_t^{(k,r)}$, we have

$$\begin{aligned}0 &= \left[a_k R_{t+2}^{(k)} + b_k R_{t+1}^{(k)} + R_t^{(k)} + \frac{\Gamma_k}{2}\right] - R_{t+3}^{(k)} \\ &= \Bigg[a_k \sum_{r=1}^{k} s_{t+2}^{(k,r)} + b_k \sum_{r=1}^{k} s_{t+1}^{(k,r)} \\ &\quad + \sum_{r=1}^{k} s_t^{(k,r)} + \frac{\Gamma_k}{2}\Bigg] - R_{t+3}^{(k)} \\ &= \sum_{r=1}^{k} (a_k s_{t+2}^{(k,r)} + b_k s_{t+1}^{(k,r)} + s_t^{(k,r)}) + \frac{1}{2}\Gamma_k - R_{t+3}^{(k)} \\ &= \sum_{r=1}^{k} (s_{t+3}^{(k,r)} - \lambda^{(k,r)}) + \frac{\Gamma_k}{2} - R_{t+3}^{(k)} \\ &= \left(R_{t+3}^{(k)} - \sum_{r=1}^{k} \lambda^{(k,r)}\right) + \frac{\Gamma_k}{2} - R_{t+3}^{(k)},\end{aligned}$$

so $\sum_{r=1}^{k} \lambda^{(k,r)} = \frac{\Gamma_k}{2}$. This completes the proof. ■

Indeed when $k = 8, 9, 10$ Table 1 and 2 show that $\sum_{r=1}^{k} \lambda^{(k,r)} = 64, 109, 242$ and $a_k + b_k = 128, 218, 484$. Lemma 4.1 implies that the tail $\lambda^{(k,r)}$ satisfies tribonacci rule if $1 \leq r \leq k - 3$, and cyclic rule if $k - 2 \leq r \leq k$. We construct a new tribonacci type sequence $\{\lambda_*^{(k,u)} \mid u \geq 1\}$ having initials $\lambda_*^{(k,u)} = \lambda^{(k,u)}$ for $u = 1, 2, 3$. ■

Lemma 4.4. Let $\{U_n\}$ be a tribonacci type sequence. Then for any k, U_k is a linear combination of U_1, U_2, U_3 with coefficients $(T_{k-3}, (T_{k-4} + T_{k-3}), T_{k-2})$.

Proof. By adapting the usual inner product notation $\circ$, it is clear to have

$$(T_1, T_0 + T_1, T_2) \circ (U_1, U_2, U_3) = U_1 + U_2 + U_3 = U_4$$

and

$$(T_2, T_1 + T_2, T_3) \circ (U_1, U_2, U_3) = U_1 + 2U_2 + 2U_3 = U_2 + U_3 + U_4 = U_5.$$

So if we assume $(T_{k-3}, (T_{k-4} + T_{k-3}), T_{k-2}) \circ (U_1, U_2, U_3) = U_k$ for all $k \leq n$ then

$$\begin{aligned} U_{n+1} &= [(T_{n-3}, (T_{n-4} + T_{n-3}), T_{n-2}) + (T_{n-4}, (T_{n-5} + T_{n-4}), T_{n-3}) \\ &\quad + (T_{n-5}, (T_{n-6} + T_{n-5}), T_{n-4})] \circ (U_1, U_2, U_3) \\ &= (T_{n-2}, T_{n-3} + T_{n-2}, T_{n-1}) \circ (U_1, U_2, U_3). \end{aligned}$$

■

Theorem 4.5. Let $\{\lambda_*^{(k,u)} | u \geq 1\}$ be a tribonacci type sequence begins with three initials $\lambda_*^{(k,u)} = \lambda^{(k,u)}$. If $u = qk + r$ $(q \geq 0, 1 \leq r \leq k)$ then

$$\lambda_*^{(k,u)} = \lambda^{(k,r)} + \Gamma_k \sum_{i=0}^{q-1} T_{ki+r},$$

so $\lambda_*^{(k,u)} \equiv \lambda^{(k,r)} \pmod{\Gamma_k}$. Moreover we have

$$\lambda_*^{(k,u)} = T_{kq+r} + (b_k + 1)T_{k(q-1)+r} + T_{k(q-2)+r}$$

where $\Gamma_k = a_k + b_k$.

Proof. If $q = 0$ then $\lambda_*^{(k,u)} = \lambda^{(k,u)}$ satisfies the tribonacci rule, so Lemma 4.4 shows

$$\lambda_*^{(k,u)} = (T_{u-3}, T_{u-4} + T_{u-3}, T_{u-2}) \circ (\lambda^{(k,1)}, \lambda^{(k,2)}, \lambda^{(k,3)}) = \lambda^{(k,u)}.$$

Let $u = k + 1$ or $u = k + 2$. Then Lemma 4.1 shows that

$$\begin{aligned}\lambda_*^{(k,k+1)} &= \lambda_*^{(k,k)} + \lambda_*^{(k,k-1)} + \lambda_*^{(k,k-2)}\\ &= \lambda^{(k,k)} + \lambda^{(k,k-1)} + \lambda^{(k,k-2)}\\ &= \Gamma_k + \lambda^{(k,1)} = T_1\Gamma_k + \lambda^{(k,1)},\\ \lambda_*^{(k,k+2)} &= \lambda_*^{(k,k+1)} + \lambda_*^{(k,k)} + \lambda_*^{(k,k-1)}\\ &= \Gamma_k + \lambda^{(k,1)} + \lambda^{(k,k)} + \lambda^{(k,k-1)}\\ &= \Gamma_k + \lambda^{(k,2)} = T_2\Gamma_k + \lambda^{(k,2)}.\end{aligned}$$

And if $u = k + 3$ then the cyclic rule of $\lambda^{(k,r)}$ in Lemma 4.1 show that

$$\begin{aligned}\lambda_*^{(k,k+3)} &= \lambda_*^{(k,k+2)} + \lambda_*^{(k,k+1)} + \lambda_*^{(k,k)}\\ &= \Gamma_k + \lambda^{(k,2)} + \Gamma_k + \lambda^{(k,1)} + \lambda^{(k,k)}\\ &= 2\Gamma_k + \lambda^{(k,3)}\\ &= T_3\Gamma_k + \lambda^{(k,3)}.\end{aligned}$$

So continuing, we have

$$\lambda_*^{(k,k+k)} = T_k\Gamma_k + \lambda^{(k,k)}.$$

It is also easy to see that

$$\begin{aligned}\lambda_*^{(k,2k+1)} &= \lambda_*^{(k,2k)} + \lambda_*^{(k,2k-1)} + \lambda_*^{(k,2k-2)}\\ &= (T_k\Gamma_k + \lambda^{(k,k)}) + (T_{k-1}\Gamma_k + \lambda^{(k,k-1)}) + (T_{k-2}\Gamma_k + \lambda^{(k,k-2)})\\ &= (T_k + T_{k-1} + T_{k-2})\Gamma_k + [\lambda^{(k,k)} + \lambda^{(k,k-1)} + \lambda^{(k,k-2)}]\\ &= T_{k+1}\Gamma_k + (\Gamma_k + \lambda^{(k,1)})\\ &= (T_{k+1} + T_1)\Gamma_k + \lambda^{(k,1)},\\ \lambda_*^{(k,2k+2)} &= \lambda_*^{(k,2k+1)} + \lambda_*^{(k,2k)} + \lambda_*^{(k,2k-1)}\\ &= ((T_{k+1} + T_1)\Gamma_k + \lambda^{(k,1)}) + (T_k\Gamma_k + \lambda^{(k,k)}) + (T_{k-1}\Gamma_k + \lambda^{(k,k-1)})\\ &= (T_{k+1} + T_1 + T_k + T_{k-1})\Gamma_k + (\lambda^{(k,1)} + \lambda^{(k,k)}\lambda^{(k,k-1)})\\ &= (T_{k+2} + T_2)\Gamma_k + \lambda^{(k,2)},\\ \lambda_*^{(k,2k+3)} &= \lambda_*^{(k,2k+2)} + \lambda_*^{(k,2k+1)} + \lambda_*^{(k,2k)}\\ &= ((T_{k+2} + T_2)\Gamma_k + \lambda^{(k,2)}) + ((T_{k+1} + T_1)\Gamma_k + \lambda^{(k,1)}) + (T_k\Gamma_k + \lambda^{(k,k)})\\ &= [(T_{k+2} + T_{k+1} + T_k) + (T_2 + T_1)]\Gamma_k + [\lambda^{(k,2)} + \lambda^{(k,1)} + \lambda^{(k,k)}]\\ &= (T_{k+3} + T_3)\Gamma_k + \lambda^{(k,3)}\end{aligned}$$

since $T_2 + T_1 = 2 = T_3$. Hence we have

$$\begin{aligned}\lambda_*^{(k,2k+4)} &= \lambda_*^{(k,2k+3)} + \lambda_*^{(k,2k+2)} + \lambda_*^{(k,2k+1)} \\ &= [(T_{k+3} + T_{k+2} + T_{k+1}) + (T_3 + T_2 + T_1)]\Gamma_k + [\lambda^{(k,3)} + \lambda^{(k,2)} + \lambda^{(k,1)}] \\ &= (T_{k+4} + T_4)\Gamma_k + \lambda^{(k,4)}.\end{aligned}$$

Therefore it follows that

$$\lambda_*^{(k,2k+k)} = (T_{k+k} + T_k)\Gamma_k + \lambda^{(k,k)}.$$

Now similarly

$$\begin{aligned}\lambda_*^{(k,3k+1)} &= ((T_{2k} + T_k)\Gamma_k + \lambda^{(k,k)}) + ((T_{2k-1} + T_{k-1})\Gamma_k + \lambda^{(k,k-1)}) \\ &\quad + ((T_{2k-2} + T_{k-2})\Gamma_k + \lambda^{(k,k-2)}) \\ &= (T_{2k+1} + T_{k+1})\Gamma_k + (\Gamma_k + \lambda^{(k,1)}) \\ &= (T_{2k+1} + T_{k+1} + T_1)\Gamma_k + \lambda^{(k,1)}.\end{aligned}$$

Continuing this process, for any $1 \leq r \leq k$ we clearly have

$$\begin{aligned}\lambda_*^{(k,3k+r)} &= (T_{2k+r} + T_{k+r} + T_r)\Gamma_k + \lambda^{(k,r)} \\ &= \sum_{i=0}^{2} T_{ki+r}\Gamma_k + \lambda^{(k,r)}.\end{aligned}$$

So for any $q \geq 0$, $1 \leq r \leq k$, it follows

$$\lambda_*^{(k,qk+r)} = \sum_{i=0}^{q-1} T_{ki+r}\Gamma_k + \lambda^{(k,r)}.$$

Furthermore the identity (1-1) implies that

$$\begin{aligned}\sum_{i=0}^{q-1} T_{ki+r} &= s_{q-1}^{(k,r)} \\ &= \frac{1}{\Gamma_k}(T_{k(q+1)+r} + (b_k + 1)T_{k(q-1)+r} + T_{k(q-2)} - \lambda^{(k,r)}),\end{aligned}$$

so that

$$\begin{aligned}\lambda_*^{(k,qk+r)} &= \left(\frac{1}{\Gamma_k}(T_{kq+r} + (b_k + 1)T_{k(q-1)+r} + T_{k(q-2)} - \lambda^{(k,r)})\right)\Gamma_k + \lambda^{(k,r)} \\ &= (T_{kq+r} + (b_k + 1)T_{k(q-1)+r} + T_{k(q-2)} - \lambda^{(k,r)}) + \lambda^{(k,r)} \\ &= T_{kq+r} + (b_k + 1)T_{k(q-1)+r} + T_{k(q-2)}.\end{aligned}$$

This completes the proof. ■

It shows that $\{\lambda_*^{(k,u)}\}$ is periodic by mod Γ_k. For instance, let $k = 9$. Then $\{\lambda_*^{(9,u)}\}$ starts from $(-87, 35, 21)$ that are the initials of $\{\lambda^{(9,u)}\}$. So we have

$$\lambda_*^{(9,8)} = T_5(-87) + (T_4 + T_5)(35) + T_6(21) = 49 = \lambda^{(9,8)}$$

and

$$\lambda_*^{(9,9)} = T_6(-87) + (T_5 + T_6)(35) + T_7(21) = 73 = \lambda^{(9,9)}.$$

If $u > 9$, since $a_9 + b_9 = 218$ we have

$$\begin{aligned}\lambda_*^{(9,10)} &= T_7(-87) + (T_6 + T_7)(35) + T_8(21) = 131 \\ &= \lambda^{(9,1)} + T_1\Gamma_9, \\ \lambda_*^{(9,11)} &= T_8(-87) + (T_7 + T_8)(35) + T_9(21) = 253 \\ &= \lambda^{(9,2)} + T_2\Gamma_9,\end{aligned}$$

and so on. On the other hand, we have

$$\begin{aligned}\lambda_*^{(5,33)} &= \Gamma_5 \sum_{i=0}^{5} T_{5i+3} + \lambda^{(5,3)} \\ &= 22(T_3 + T_8 + \cdots + T_{28}) + 5 \\ &= 199700473\end{aligned}$$

that equals

$$T_{33} + (b_5 + 1)T_{28} + T_{23} = 181997601 + 2 \cdot 8646064 + 410744.$$

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