Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 4021–4039

http://www.ripublication.com/gjpam.htm

Riesz Almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ Lacunary $\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$ sequence spaces defined by a Orlicz function

P. Thirunavukkarasu
P.G. and Research Department of Mathematics,
Periyar E.V.R. College (Autonomous),
Tiruchirappalli–620 023, India.

N. Subramanian
Department of Mathematics,
SASTRA University,
Thanjavur 613 401, India.

N. Rajagopal
Department of Mathematics,
SASTRA University,
Thanjavur 613 401, India.

Abstract

In this paper we introduce a new concept for generalized almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$-Riesz spaces strong P-convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of generalized almost $\left(\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}\right)$ convergence in $\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}$-Riesz space and also some inclusion theorems are discussed.

AMS subject classification: 40A05, 40C05, 40D05.
Keywords: Analytic sequence, Orlicz function, double sequences, chi sequence, Riesz space, Riesz convergence, Pringsheim convergence.

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^3 for the set of all complex triple sequences (x_{mnk}), where $m, n, k \in \mathbb{N}$, the set of positive integers. Then, w^3 is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in *Apostol [1]* and double sequence spaces is found in *Hardy [7]*, *Subramanian et al. [8–14]*, and many others. Later on investigated by some initial work on triple sequence spaces is found in *Sahiner et al. [15]* , *Esi et al. [2–6]*, *Subramanian et al. [16–25]* and many others.

Let (x_{mnk}) be a triple sequence of real or complex numbers. Then the series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series $\sum_{m,n,k=1}^{\infty} x_{mnk}$ give one space is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where

$$S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq} (m, n, k = 1, 2, 3, \ldots).$$

A sequence $x = (x_{mnk})$is said to be triple analytic if

$$sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.$$

The vector space of all triple analytic sequences are usually denoted by Λ^3. A sequence $x = (x_{mnk})$ is called triple entire sequence if

$$|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.$$

The vector space of all triple entire sequences are usually denoted by Γ^3. Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric

$$d(x, y) = sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k : 1, 2, 3, \ldots \right\}, \tag{1.1}$$

forall $x = \{x_{mnk}\}$ and $y = \{y_{mnk}\}$ in Γ^3. Let $\phi = \{\text{finite sequences}\}$.

Consider a triple sequence $x = (x_{mnk})$. The $(m, n, k)^{th}$ section $x^{[m,n,k]}$ of the sequence is defined by

$$x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq} \delta_{ijq}$$

for all $m, n, k \in \mathbb{N}$,

$$\delta_{mnk} = \begin{bmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{bmatrix}$$

with 1 in the $(m, n, k)^{th}$ position and zero otherwise.

A sequence $x = (x_{mnk})$ is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \to 0$ as $m, n, k \to \infty$. The triple entire sequences will be denoted by Γ^3.

2. Definitions and Preliminaries

A triple sequence $x = (x_{mnk})$ has limit 0 (denoted by $P - limx = 0$)
(i.e) $|x_{mnk}|^{1/m+n+k} \to 0$ as $m, n, k \to \infty$. We shall write more briefly as P-convergent to 0.

Definition 2.1. A modulus function was introduced by Nakano [26]. We recall that a modulus f is a function from $[0, \infty) \to [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0,\ y \geq 0$,

(3) f is increasing,

(4) f is continuous from the right at 0. Since

$$|f(x) - f(y)| \leq f(|x - y|),$$

it follows from here that f is continuous on $[0, \infty)$.

Definition 2.2. Let

$$(q_{rst}), (\overline{q_{rst}}), (\overline{\overline{q_{rst}}})$$

be sequences of positive numbers and

$$Q_r = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1s} & 0... \\ q_{21} & q_{22} & \cdots & q_{2s} & 0... \\ . \\ . \\ . \\ q_{r1} & q_{r2} & \cdots & q_{rs} & 0... \\ 0 & 0 & ...0 & 0 & 0... \end{bmatrix} = q_{11} + q_{12} + \cdots + q_{rs} \neq 0,$$

$$\overline{Q}_s = \begin{bmatrix} \overline{q}_{11} & \overline{q}_{12} & \cdots & \overline{q}_{1s} & 0\ldots \\ \overline{q}_{21} & \overline{q}_{22} & \cdots & \overline{q}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{q}_{r1} & \overline{q}_{r2} & \cdots & \overline{q}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0,$$

$$\overline{\overline{Q}}_t = \begin{bmatrix} \overline{\overline{q}}_{11} & \overline{\overline{q}}_{12} & \cdots & \overline{\overline{q}}_{1s} & 0\ldots \\ \overline{\overline{q}}_{21} & \overline{\overline{q}}_{22} & \cdots & \overline{\overline{q}}_{2s} & 0\ldots \\ . & & & & \\ . & & & & \\ . & & & & \\ \overline{\overline{q}}_{r1} & \overline{\overline{q}}_{r2} & \cdots & \overline{\overline{q}}_{rs} & 0\ldots \\ 0 & 0 & \ldots 0 & 0 & 0\ldots \end{bmatrix} = \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \neq 0.$$

Then the transformation is given by

$$T_{rst} = \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_r \overline{Q}_s \overline{\overline{Q}}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left| x_{mnk} \right|^{1/m+n+k}$$

is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - lim_{rst} T_{rst}(x) = 0, 0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - lim x = 0$.

Definition 2.3. Let $\lambda = (\lambda_{m_i})$, $\mu = (\mu_{n_\ell})$ and $\gamma = (\gamma_{k_j})$ be three non-decreasing sequences of positive real numbers such that each tending to ∞ and

$$\lambda_{m_i+1} \leq \lambda_{m_i} + 1, \lambda_1 = 1, \quad \mu_{n_\ell+1} \leq \mu_{n_\ell} + 1, \mu_1 = 1 \quad \gamma_{k_j+1} \leq \gamma_{k_j} + 1, \gamma_1 = 1.$$

Let

$$I_{m_i} = \left[m_i - \lambda_{m_i} + 1, m_i \right], I_{n_\ell} = \left[n_\ell - \mu_{n_\ell} + 1, n_\ell \right]$$

and $I_{k_j} = \left[k_j - \gamma_{k_j} + 1, k_j \right]$. For any set $K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, the number

$$\delta_{\lambda,\mu,\gamma}(K) = lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}} \left| \left\{ (i, j) : i \in I_{m_i}, j \in I_{n_\ell}, k \in I_{k_j}, (i, \ell, j,) \in K \right\} \right|,$$

is called the (λ, μ, γ)-density of the set K provided the limit exists.

Definition 2.4. A triple sequence $x = (x_{mnk})$ of numbers is said to be (λ, μ, γ)-statistical convergent to a number ξ provided that for each $\epsilon > 0$,

$$lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \left| \left\{ (i, \ell, j) \in I_{m_i n_\ell k_j} : q_m \overline{q}_n \overline{\overline{q}}_k \left| x_{mnk} - \xi \right| \geq \epsilon \right\} \right| = 0,$$

(i.e) the set

$$K(\epsilon) = \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{Q_i \overline{Q_\ell} \overline{\overline{Q}}_j} \left|\left\{(i, \ell, j) \in I_{m_i n_\ell k_j} : q_m \overline{q}_n \overline{\overline{q}}_k \left|x_{mnk} - \xi\right| \geq \epsilon\right\}\right|$$

has (λ, μ, γ)-density zero. In this case the number ξ is called the (λ, μ, γ)-statistical limit of the sequence $x = (x_{mnk})$ and we write $St_{(\lambda,\mu,\gamma)} lim_{m,n,k\to\infty} = \xi$.

Definition 2.5. The triple sequence $\theta_{i,\ell,j} = \left\{\left(m_i, n_\ell, k_j\right)\right\}$ is called triple lacunary if there exist three increasing sequences of integers such that

$$\begin{aligned} m_0 = 0, h_i = m_i - m_{r-1} \to \infty \text{ as } i \to \infty \text{ and} \\ n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty. \\ k_0 = 0, \overline{h_j} = k_j - k_{j-1} \to \infty \text{ as } j \to \infty. \end{aligned}$$

Let

$$m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i \overline{h_\ell h_j},$$

and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \left\{(m, n, k) : m_{i-1} < m < m_i \,\text{and}\, n_{\ell-1} < n \leq n_\ell \,\text{and}\, k_{j-1} < k \leq k_j\right\},$$

$$q_k = \frac{m_k}{m_{k-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

Using the notations of lacunary sequence and Riesz mean for triple sequences. $\theta_{i,\ell,j} = \left\{\left(m_i, n_\ell, k_j\right)\right\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive real numbers such that

$$Q_{m_i} = \sum_{m\in(0,m_i]} p_{m_i}, Q_{n_\ell} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, Q_{n_j} = \sum_{k\in(0,k_j]} p_{k_j}$$

and

$$H_i = \sum_{m\in(0,m_i]} p_{m_i}, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell}, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j}.$$

Clearly,

$$H_i = Q_{m_i} - Q_{m_{i-1}}, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}}, \overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}}.$$

If the Riesz transformation of triple sequences is RH-regular, and $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as

$$i \to \infty, \overline{H} = \sum_{n\in(0,n_\ell]} p_{n_\ell} \to \infty$$

as

$$\ell \to \infty, \overline{\overline{H}} = \sum_{k\in(0,k_j]} p_{k_j} \to \infty$$

as $j \to \infty$, then

$$\theta^{'}_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} = \{(Q_{m_i} Q_{n_j} Q_{k_k})\}$$

is a triple lacunary sequence. If the assumptions $Q_r \to \infty$ as $r \to \infty$, $\overline{Q}_s \to \infty$ as $s \to \infty$ and $\overline{\overline{Q}}_t \to \infty$ as $t \to \infty$ may be not enough to obtain the conditions $H_i \to \infty$ as $i \to \infty$, $\overline{H}_\ell \to \infty$ as $\ell \to \infty$ and $\overline{\overline{H}}_j \to \infty$ as $j \to \infty$ respectively. For any lacunary sequences (m_i), (n_ℓ) and (k_j) are integers.

Throughout the paper, we assume that

$$\begin{aligned} Q_r &= q_{11} + q_{12} + \cdots + q_{rs} \to \infty \, (r \to \infty) \, , \overline{Q}_s \\ &= \overline{q}_{11} + \overline{q}_{12} + \cdots + \overline{q}_{rs} \to \infty \, (s \to \infty) \, , \overline{\overline{Q}}_t \\ &= \overline{\overline{q}}_{11} + \overline{\overline{q}}_{12} + \cdots + \overline{\overline{q}}_{rs} \to \infty \, (t \to \infty) \, , \end{aligned}$$

such that $H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty$ as

$$i \to \infty, \overline{H}_\ell = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty$$

as $\ell \to \infty$ and

$$\overline{\overline{H}}_j = Q_{k_j} - Q_{k_{j-1}} \to \infty$$

as $j \to \infty$.

Let

$$\begin{aligned} Q_{m_i, n_\ell, k_j} &= Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}, \, H_{i\ell j} = H_i \overline{H}_\ell \overline{\overline{H}}_j, \\ I^{'}_{i\ell j} &= \left\{ (m, n, k) : Q_{m_{i-1}} < m < Q_{m_i}, \overline{Q}_{n_{\ell-1}} < n < Q_{n_\ell} \text{ and } \overline{Q}_{k_{j-1}} < k < \overline{Q}_{k_j} \right\}, \\ V_i &= \frac{Q_{m_i}}{Q_{m_{i-1}}}, \overline{V}_\ell = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}} \end{aligned}$$

and

$$\overline{\overline{V}}_j = \frac{Q_{k_j}}{Q_{k_{j-1}}}$$

and

$$V_{i\ell j} = V_i \overline{V}_\ell \overline{\overline{V}}_j.$$

If we take

$$q_m = 1, \overline{q}_n = 1 \text{ and } \overline{\overline{q}}_k = 1$$

for all m, n and k then $H_{i\ell j}$, $Q_{i\ell j}$, $V_{i\ell j}$ and $I^{'}_{i\ell j}$ reduce to $h_{i\ell j}$, $q_{i\ell j}$, $v_{i\ell j}$ and $I_{i\ell j}$.

Let f be an Orlicz function and $p = (p_{mnk})$ be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

$$\begin{aligned} &\left[\Gamma^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, \theta_{i\ell j}, q, f, p \right] \\ &= \left\{ P - lim_{i,\ell,j \to \infty} \frac{1}{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i \in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j \in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i, n+\ell, k+j} \right| \right)^{p_{mnk}} \right] \right\} = 0, \end{aligned}$$

uniformly in i, ℓ and j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \left\{ x = (x_{mnk}) : P - sup_{i,\ell,j} \frac{1}{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}} \frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \sum_{\ell \in I_{i\ell j}} \sum_{j\in I_{i\ell j}} \times q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left| x_{m+i,n+\ell,k+j} \right|^{p_{mnk}} \right] < \infty \right\},$$

uniformly in i, ℓ and j.

Let f be an Orlicz function, $p = p_{mnk}$ be any factorable triple sequence of strictly positive real numbers and and $q_m, \overline{q}_n$ and $\overline{\overline{q}}_k$ be sequences of positive numbers and $Q_r = q_{11} + \cdots q_{rs}$, $\overline{Q}_s = \overline{q}_{11} \cdots \overline{q}_{rs}$ and $\overline{\overline{Q}}_t = \overline{\overline{q}}_{11} \cdots \overline{\overline{q}}_{rs}$,

If we choose $q_m = 1, \overline{q}_n = 1$ and $\overline{\overline{q}}_k = 1$ for all m, n and k, then we obtain the following sequence spaces.

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \left\{ P - lim_{i,\ell,j\to\infty} \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \sum_{m=1}^{i} \sum_{n=1}^{\ell} \times \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{p_{mnk}} \right] = 0 \right\},$$

uniformly in i, ℓ and j.

$$\left[\Lambda^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right] = \left\{ P - sup_{i,\ell,j} \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{1}{Q_i \overline{Q}_\ell \overline{\overline{Q}}_j} \times \sum_{m=1}^{i} \sum_{n=1}^{\ell} \sum_{k=1}^{j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{p_{mnk}} \right] < \infty \right\},$$

uniformly in i, ℓ and j.

3. Main Results

Theorem 3.1. If f be any Orlicz function and a bounded factorable positive triple number sequence p_{mnk} then $\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$ is linear space.

Proof. The proof is easy. Theorefore omit the proof. ∎

Theorem 3.2. For any Orlicz function f, we have

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subset \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof. Let $x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right]$ so that for each i, ℓ and j

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] = \left\{ P - lim_{i,\ell,j\to\infty} \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i,\ell j}} \sum_{i\in I_{i\ell j}} \times \sum_{\ell\in I_{i\ell j}} \sum_{j\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k \left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{p_{mnk}}\right] = 0 \right\},$$

uniformly in i, ℓ and j.

Since f is continuous at zero, for $\varepsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for every t with $0 \le t \le \delta$. We obtain the following,

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}}\left(h_{i\ell j}\epsilon\right) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} \sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}} \text{ and } \times \left|x_{m+i,n+\ell,k+j} - 0\right| > \delta\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right] \times \frac{1}{h_{i\ell j}}\left(h_{i\ell j}\epsilon\right) + \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{h_{i\ell j}} K\delta^{-1} f(2)\, h_{i\ell j} \times \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Hence i, ℓ and j goes to infinity, we are granted $x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right]$. ∎

Theorem 3.3. Let $\theta_{i,\ell,j} = \left\{m_i, n_\ell, k_j\right\}$ be a triple lacunary sequence and $q_i, \overline{q}_\ell\overline{\overline{q}}_j$ with

$$liminf_i V_i > 1, \;\; liminf_\ell \overline{V_\ell} > 1 \text{ and } liminf_j V_j > 1$$

then for any Orlicz function

$$f, \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, , f, q, p\right] \subseteq \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

Proof. Suppose $liminf_i V_i > 1,\ \ liminf_\ell \overline{V_\ell} > 1$ and $liminf_j \overline{\overline{V}}_j > 1$ then there exists $\delta > 0$ such that

$$V_i > 1+\delta,\ \ \overline{V_\ell} > 1+\delta$$

and $\overline{\overline{V}}_j > 1+\delta$. This implies

$$\frac{H_i}{Q_{m_i}} \geq \frac{\delta}{1+\delta},\ \ \frac{\overline{H}_\ell}{\overline{Q}_{n_\ell}} \geq \frac{\delta}{1+\delta}$$

and

$$\frac{\overline{\overline{H}}_j}{\overline{\overline{Q}}_{k_j}} \geq \frac{\delta}{1+\delta}.$$

Then for

$$x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, f, q, p\right],$$

we can write for each i, ℓ and j.

$$A_{i,\ell,j} = \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}}\sum_{m\in I_{i,\ell,j}}\sum_{n\in I_{i,\ell,j}}\sum_{k\in I_{i,\ell,j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$= \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{i-1}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}}\sum_{m=m_{i-1}+1}^{m_i}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$
$$- \frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{H_{i\ell j}}\sum_{k=k_j+1}^{k_j}\sum_{n=n_{\ell-1}+1}^{n_\ell}\sum_{m=1}^{m_{k-1}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}$$

$$= \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{Q}_{k_j}}{H h_{i\ell j}}$$

$$\times \left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$

$$\times \left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{kj-1}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} f \left[\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{k_{j-1}}}{H_{i\ell j}} \left(\frac{1}{Q_{k_{j-1}}} \sum_{m=m_{i-1}+1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{Q}_{n_{\ell-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{Q}_{n_{\ell-1}}} \sum_{m=m_{k-1}+1}^{m_k} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} f \left[\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{\overline{\overline{Q}}_{m_{k-1}}}{H_{i\ell j}} \left(\frac{1}{\overline{\overline{Q}}_{m_{k-1}}} \sum_{k=1}^{k_j} \sum_{n=n_{\ell-1}+1}^{n_\ell} \sum_{m=1}^{m_{k-1}} f \left[\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right).$$

Since

$$x \in \left[\Gamma^3_{R_{\lambda_{m_i} \mu_{n_\ell} \gamma_{k_j}}}, f, q, p \right],$$

the last three terms tend to zero uniformly in m, n, k in the sense, thus, for each i, ℓ and j

$$A_{i,\ell,j} = \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}}{H_{i\ell j}}$$

$$\times \left(\frac{1}{Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$- \frac{1}{\lambda_i \mu_\ell \gamma_j} \frac{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}}$$

$$\times \left(\frac{1}{Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_{i-1}} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_{j-1}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f \left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right) + O(1).$$

Since

$$\frac{1}{\lambda_i \mu_\ell \gamma_j} H_{i\ell j} = \frac{1}{\lambda_i \mu_\ell \gamma_j} Q_{m_i} \overline{Q}_{n_\ell} \overline{\overline{Q}}_{k_j} - \frac{1}{\lambda_i \mu_\ell \gamma_j} Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}$$

we are granted for each i, ℓ and j the following

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}{H_{i\ell j}} \leq \frac{1+\delta}{\delta}$$

and

$$\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}{H_{i\ell j}} \leq \frac{1}{\delta}.$$

The terms

$$\left(\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{Q_{m_i}\overline{Q}_{n_\ell}\overline{\overline{Q}}_{k_j}}\sum_{m=1}^{m_i}\sum_{n=1}^{n_\ell}\sum_{k=1}^{k_j} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+r,n+s,k+u}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

and

$$\left(\frac{1}{\lambda_i\mu_\ell\gamma_j}\frac{1}{Q_{m_{i-1}}\overline{Q}_{n_{\ell-1}}\overline{\overline{Q}}_{k_{j-1}}}\sum_{m=1}^{m_{i-1}}\sum_{n=1}^{n_{\ell-1}}\sum_{k=1}^{k_{j-1}} q_m\overline{q}_n\overline{\overline{q}}_k\left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}}\right)$$

are both gai sequences for all r, s and u. Thus $A_{i\ell j}$ is a entire sequence for each i, ℓ and j. Hence

$$x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, p\right].$$

■

Theorem 3.4. Let $\theta_{i,\ell,j} = \left\{m_i, n_\ell, k_j\right\}$ be a triple lacunary sequence and $q_m\overline{q}_n\overline{\overline{q}}_k$ with $limsup_i V_i < \infty$, $limsup_\ell \overline{V}_\ell < \infty$ and $limsup_j \overline{\overline{V}}_j < \infty$ then for any Orlicz function

$$f, \quad \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p\right] \subseteq \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p\right].$$

Proof. Since

$$limsup_i V_i < \infty, \; limsup_\ell \overline{V}_\ell < \infty$$

and $limsup_j \overline{\overline{V}}_j < \infty$ there exists $H > 0$ such that $V_i < H$, $\overline{V_\ell} < H$ and $\overline{\overline{V}}_j < H$ for all i, ℓ and j. Let

$$x \in \left[\Gamma^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}, q, f, p\right]$$

and $\epsilon > 0$. Then there exist $i_0 > 0$, $\ell_0 > 0$ and $j_0 > 0$ such that for every $a \geq i_0$, $b \geq \ell_0$ and $c \geq j_0$ and for all i, ℓ and j.

$$A'_{abc} = \frac{1}{\lambda_i\mu_\ell\gamma_j H_{abc}}\sum_{m\in I_{a,b,c}}\sum_{n\in I_{a,b,c}}\sum_{k\in I_{a,b,c}} q_m\overline{q}_n\overline{\overline{q}}_k \times \left[f\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}\right]^{p_{mnk}} \to 0 \text{ as } m,n,k \to \infty.$$

Let $G' = \max\{A'_{a,b,c} : 1 \le a \le i_0,\ \ 1 \le b \le \ell_0$ and $1 \le c \le j_0\}$ and p, r and t be such that $m_{i-1} < p \le m_i,\ \ n_{\ell-1} < r \le n_\ell$ and $k_{j-1} < t \le k_j$. Thus we obtain the following:

$$\frac{1}{\lambda_i \mu_\ell \gamma_j Q_p \overline{Q}_r \overline{\overline{Q}}_t} \sum_{m=1}^{p} \sum_{n=1}^{r} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$

$$\le \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_\ell} \sum_{k=1}^{k_j} \left[\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}}$$

$$\le \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{a=1}^{i} \sum_{b=1}^{\ell} \sum_{c=1}^{j}$$

$$\times \left(\sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left(\left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} \right)$$

$$= \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{Q}_{k_{j-1}}} \sum_{a=1}^{i_0} \sum_{b=1}^{\ell_0} \sum_{c=1}^{j_0} H_{a,b,c} A'_{a,b,c}$$

$$+ \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{k-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} H_{a,b,c} A'_{a,b,c}$$

$$\le \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{i_0}} \overline{\overline{Q}}_{k_{i_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}$$

$$\times \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} H_{a,b,c} A'_{a,b,c}$$

$$\le \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j m_{i-1} n_{\ell-1} k_{j-1}} + \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{j_{j-1}}}$$

$$\times \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} H_{a,b,c} A'_{a,b,c}$$

$$\le \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \left(sup_{a \ge i_0 \bigcup b \ge \ell_0 \bigcup c \ge j_0} A'_{a,b,c} \right)$$

$$\times \frac{1}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} \sum_{(i_0 < a \le i) \bigcup (\ell_0 < b \le \ell) \bigcup (j_0 < c \le j)} H_{a,b,c}$$

$$\leq \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \frac{\epsilon}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}}$$
$$\times \sum_{(i_0<a\leq i)\bigcup(\ell_0<b\leq\ell)\bigcup(j_0<c\leq j)} H_{a,b,c}$$
$$= \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + V_i \overline{V}_\ell \overline{\overline{V}}_j \epsilon \leq \frac{G' Q_{m_{i_0}} \overline{Q}_{n_{\ell_0}} \overline{\overline{Q}}_{k_{j_0}}}{\lambda_i \mu_\ell \gamma_j Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}}} + \epsilon H^3.$$

Since

$$Q_{m_{i-1}} \overline{Q}_{n_{\ell-1}} \overline{\overline{Q}}_{k_{j-1}} \to \infty$$

as $i, \ell, j \to \infty$ approaches infinity, it follows that

$$\frac{1}{\lambda_i \mu_\ell \gamma_j Q_p \overline{Q}_r \overline{\overline{Q}}_t} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} q_m \overline{q}_n \overline{\overline{q}}_k \left[f\left((m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/m+n+k} \right]^{p_{mnk}} = 0,$$

uniformly in i, ℓ and j. Hence

$$x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p \right].$$

■

Corollary 3.5. Let $\theta_{i,\ell,j} = \left\{ m_i, n_\ell, k_j \right\}$ be a triple lacunary sequence and $q_m \overline{q}_n \overline{\overline{q}}_k$ be sequences of positive numbers. If $1 < lim_{i\ell j} V_{i\ell j} \leq lim_{i\ell j} sup V_{i\ell j} < \infty$, then for any Orlicz function

$$f, \quad \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q, f, p \right] = \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, q, f, p \right].$$

Definition 3.6. Let $\theta_{i,\ell,j} = \left\{ m_i, n_\ell, k_j \right\}$ be a triple lacunary sequence. The triple number sequence x is said to be $S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j} \right]} - P$ convergent to 0 provided that for every $\epsilon > 0$,

$$P{-}lim_{i\ell J} \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} sup_{i\ell j} \left| \left\{ (m,n,k) \in I'_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k \left[|x_{mnk}|^{1/m+n+k}, \bar{0} \right] \right\} \geq \epsilon \right| = 0.$$

In this case we write

$$S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j} \right]} - P - lim x = 0.$$

Theorem 3.7. Let $\theta_{i,\ell,j} = \{m_i, n_\ell, k_j\}$ be a triple lacunary sequence. If $I^{'}_{i,\ell,j} \subseteq I_{i,\ell,j}$, then the inclusion

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right] \subset S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$$

is strict and

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right] - P - limx = S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - limx = 0.$$

Proof. Let

$$K_{Q_{i\ell j}}(\epsilon) = \left|\left\{(m,n,k) \in I^{'}_{i\ell j} : q_m \overline{q}_n \overline{\overline{q}}_k \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq \epsilon\right| \tag{3.2}$$

Suppose that

$$x \in \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right].$$

Then for each i, ℓ and j

$$P - lim_{i\ell j} \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] = 0.$$

Since

$$\begin{aligned}
&\frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] \\
&\geq \frac{1}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} \sum_{m\in I_{i\ell j}} \sum_{n\in I_{i\ell j}} \sum_{k\in I_{i\ell j}} q_m \overline{q}_n \overline{\overline{q}}_k \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] \\
&= \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}}
\end{aligned}$$

for all i, ℓ and j, we get

$$P - lim_{i,\ell,j} \frac{\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i \mu_\ell \gamma_j H_{i\ell j}} = 0$$

for each i, ℓ and j. This implies that

$$x \in S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}.$$

To show that this inclusion is strict, let $x = (x_{mnk})$ be defined as

$$(x_{mnk}) = \begin{bmatrix} 1 & 2 & 3 & \ldots\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(1)!} & 0 & \ldots \\ 1 & 2 & 3 & \ldots\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(1)!} & 0 & \ldots \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(1)!} & 2 & 3 & \ldots\frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}-1}{(1)!} & 0 & \ldots \\ . & & & & & \\ \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & \frac{\lambda_i\mu_\ell\gamma_j\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{(1)!} & 0 & \ldots & \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 0 & 0 & 0 & \ldots 0 & 0 & \ldots \\ . & & & & & \\ . & & & & & \\ . & & & & & \end{bmatrix};$$

and $q_m = 1; \overline{q}_n = 1; \overline{\overline{q}}_k = 1$ for all m, n and k. Clearly, x is unbounded sequence. For $\epsilon > 0$ and for all i, ℓ and j we have

$$\left|\left\{(m,n,k) \in I'_{i\ell j} : q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((1)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq \epsilon\right|$$

$$= P - lim_{i\ell j}\left(\frac{\lambda_i\mu_\ell\gamma_j\,(1)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(1)!}\right)^{1/m+n+k} = 0.$$

Therefore

$$x \in S_{\left[\chi^3_{R_{\lambda_{m_i}n_\ell k_j}}, \theta_{i\ell j}\right]}$$

with the $P - lim = 0$. Also note that

$$P - lim_{i\ell j}\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left((1)!\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]$$

$$= P - \frac{1}{2}\left(lim_{i\ell j}\left(\frac{\lambda_i\mu_\ell\gamma_j\,(1)!\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}}{\left[\sqrt[4]{H_{i,\ell,j}}\right]^{m+n+k}(1)!}\right)^{1/m+n+k} + 1\right)$$

$$= \frac{1}{2}.$$

Hence

$$x \notin \left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right].$$

■

Theorem 3.8. Let $I'_{i\ell j} \subseteq I_{i\ell j}$. If the following conditions hold, then

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu \subset S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]}$$

and

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}, q\right]_\mu - P - limx = S_{\left[\Gamma^3_{R_{\lambda_i\mu_\ell\gamma_j}}, \theta_{i\ell j}\right]} - P - limx = 0.$$

(1) $0 < \mu < 1$ and $0 \leq \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] < 1.$

(2) $1 < \mu < \infty$ and $1 \leq \left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] < \infty.$

Proof. Let $x = (x_{mnk})$ be strongly $\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{n_j}}}, \theta_{i\ell j}, q\right]_\mu$ − almost P-convergent to the limit 0. Since

$$q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]^\mu \geq q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]$$

for (1) and (2), for all i, ℓ and j, we have

$$\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]^\mu$$
$$\geq \frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]$$
$$\geq \frac{\epsilon\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}$$

where $K_{Q_{i\ell j}}(\epsilon)$ is as in (3.1). Taking limit $i, \ell, j \to \infty$ in both sides of the above inequality, we conclude that

$$S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - limx = 0.$$

■

Definition 3.9. A triple sequence $x = (x_{mnk})$ is said to be Riesz lacunary of Γ almost P-convergent 0 if $P - lim_{i,\ell,j} w^{i\ell j}_{mnk}(x) = 0$, uniformly in i, ℓ and j, where

$$w^{i\ell j}_{mnk}(x) = w^{i\ell j}_{mnk} = \frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}$$
$$\times\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right].$$

Definition 3.10. A triple sequence (x_{mnk}) is said to be Riesz lacunary Γ almost statistically summable to 0 if for every $\epsilon > 0$ the set

$$K_\epsilon = \left\{(i, \ell, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \left|w^{i\ell j}_{mnk}, \bar{0}\right| \geq \epsilon\right\}$$

has triple natural density zero, (i.e) $\delta_3(K_\epsilon) = 0$. In this we write

$$\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]_{st_2} - P - limx = 0.$$

That is, for every $\epsilon > 0$,

$$P - lim_{rst}\frac{1}{rst}\left|\left\{i \leq r, \ell \leq s, j \leq t : \left|w^{i\ell j}_{mnk}, \bar{0}\right| \geq \epsilon\right\}\right| = 0,$$

uniformly in i, ℓ and j.

Theorem 3.11. Let $I'_{i\ell j} \subseteq I_{i\ell j}$. and

$$q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right] \leq M$$

for all $m, n, k \in \mathbb{N}$ and for each i, ℓ and j. Let $x = (x_{mnk})$ be

$$S_{\left[\Gamma^3_{R_{\lambda_{m_i}\mu_{n_\ell}\gamma_{k_j}}}, \theta_{i\ell j}\right]} - P - limx = 0.$$

Let

$$K_{Q_{i\ell j}}(\epsilon) = \left|\left\{(m, n, k) \in I'_{i\ell j} : q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right\} \geq \epsilon\right|.$$

Then

$$\begin{aligned}\left|w^{i\ell j}_{mnk}, \bar{0}\right| = &\left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I_{i\ell j}}\sum_{n\in I_{i\ell j}}\right.\\ &\left.\times\sum_{k\in I_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right|\\ \leq &\left|\frac{1}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}}\sum_{m\in I'_{i\ell j}}\sum_{n\in I'_{i\ell j}}\sum_{k\in I'_{i\ell j}} q_m\overline{q}_n\overline{\overline{q}}_k\left[\left(\left|x_{m+i,n+\ell,k+j}\right|\right)^{1/m+n+k}, \bar{0}\right]\right|\\ \leq &\frac{M\left|K_{Q_{i\ell j}}(\epsilon)\right|}{\lambda_i\mu_\ell\gamma_j H_{i\ell j}} + \epsilon\end{aligned}$$

for each i, ℓ and j, which implies that $P - lim_{i,\ell,j} w^{i\ell j}_{mnk}(x) = 0$, uniformly i, ℓ and j. Hence, $St_2 - P - lim_{i\ell j} w^{i\ell j}_{mnk} = 0$ uniformly in i, ℓ, j. Hence $\left[\Gamma^3_{R_{\lambda_i \mu_\ell \gamma_j}}, \theta_{i\ell j}\right]_{st_2} - P - limx = 0$.

To see that the converse is not true, consider the triple lacunary sequence

$$\theta_{i\ell j}\left\{\left(2^{i-1}3^{\ell-1}4^{j-1}\right)\right\}, q_m = 1, \overline{q}_n = 1, \overline{\overline{q}}_k = 1$$

for all m, n and k, and the triple sequence $x = (x_{mnk})$ defined by $x_{mnk} = (-1)^{m+n+k}$ for all m, n and k.

References

[1] T. Apostol, Mathematical Analysis, *Addison-Wesley , London*, 1978.

[2] A. Esi, On some triple almost lacunary sequence spaces defined by Orlicz functions, *Research and Reviews: Discrete Mathematical Structures*, **1(2)**, (2014), 16–25.

[3] A. Esi and M. Necdet Catalbas,Almost convergence of triple sequences, *Global Journal of Mathematical Analysis*, **2(1)**, (2014), 6–10.

[4] A. Esi and E. Sava's, On lacunary statistically convergent triple sequences in probabilistic normed space, *Appl. Math. and Inf. Sci.*, **9(5)**, (2015), 2529–2534.

[5] A. Esi, Statistical convergence of triple sequences in topological groups, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **40(1)**, (2013), 29–33.

[6] E. Savas and A. Esi, Statistical convergence of triple sequences on probabilistic normed space , *Annals of the University of Craiova, Mathematics and Computer Science Series*, **39(2)**, (2012), 226–236.

[7] G.H. Hardy, On the convergence of certain multiple series, *Proc. Camb. Phil. Soc.*, **19** (1917), 86–95.

[8] N. Subramanian and U.K. Misra, Characterization of gai sequences via double Orlicz space, *Southeast Asian Bulletin of Mathematics*, **35**, (2011), 687–697.

[9] N. Subramanian, C. Priya and N. Saivaraju, The $\int \chi^{2I}$ of real numbers over Musielak metric space, *Southeast Asian Bulletin of Mathematics*, **39(1)**, (2015), 133–148.

[10] N. Subramanian, P. Anbalagan and P. Thirunavukkarasu, The Ideal Convergence of Strongly of Γ^2 in p-Metric Spaces Defined by Modulus, *Southeast Asian Bulletin of Mathematics*, **37**, (2013), 919–930.

[11] N. Subramanian, The Semi Normed space defined by Modulus function, *Southeast Asian Bulletin of Mathematics*, **32**, (2008), 1161–1166.

[12] Deepmala, N. Subramanian and Vishnu Narayan Misra, Double almost $(\lambda_m \mu_n)$ in χ^2-Riesz space, *Southeast Asian Bulletin of Mathematics*, **35**(2016), 1–11.

[13] N. Subramanian, B.C. Tripathy and C. Murugesan, The double sequence space of Γ^2, *Fasciculi Math.*, **40**, (2008), 91–103.

[14] N. Subramanian, B.C. Tripathy and C. Murugesan, The Cesáro of double entire sequences, *International Mathematical Forum*, **4(2)**(2009), 49–59.

[15] A. Sahiner, M. Gurdal and F.K. Duden, Triple sequences and their statistical convergence, *Selcuk J. Appl. Math.*, **8(2)**, (2007), 49–55.

[16] N. Subramanian and A. Esi, The generalized triple difference of χ^3 sequence spaces , *Global Journal of Mathematical Analysis*, **3(2)** (2015), 54–60.

[17] N. Subramanian and A. Esi, The study on χ^3 sequence spaces, *Songklanakarin Journal of Science and Technology*, under review.

[18] N. Subramanian and A. Esi, Characterization of Triple χ^3 sequence spaces , *Mathematica Moravica*, in press.

[19] N. Subramanian and A. Esi, Some New Semi-Normed Triple Sequence Spaces Defined By A Sequence Of Moduli, *Journal of Analysis & Number Theory*, **3(2)** (2015), 79–88.

[20] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The Triple Almost Lacunary Γ^3 sequence spaces defined by a modulus function, *International Journal of Applied Engineering Research*, **Vol. 10, No. 80** (2015), 94–99.

[21] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The triple entire sequence defined by Musielak Orlicz functions over p-metric space, *Asian Journal of Mathematics and Computer Research, International Press*, **5(4)** (2015), 196–203.

[22] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The Random of Lacunary statistical on Γ^3 over metric spaces defined by Musielak Orlicz functions, *Modern Applied Science*, **Vol. 10, No. 1** (2016), 171–183.

[23] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The Triple Γ^3 of tensor products in Orlicz sequence spaces, *Mathematical Sciences International Research Journal*, **Vol. 4, No. 2** (2015), 162–166.

[24] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, The strongly generalized triple difference Γ^3 sequence spaces defined by a modulus, *Mathematica Moravica*, in press.

[25] T.V.G. Shri Prakash, M. Chandramouleeswaran and N. Subramanian, Lacunary Triple sequence Γ^3 of Fibonacci numbers over probabilistic p-metric spaces, *International Organization of Scientific Research*, **Vol. 12, Issue-1, Version-IV** (2016), 10–16.

[26] H. Nakano 1953. Concave modulars, *Journal of the Mathematical society of Japan*, **5**, 29–49.