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The Countably Lipschitz Integral

Ch. Rini Indrati

Department of Mathematics,
Faculty of Mathematics and natural Sciences,
Universitas Gadjah Mada.
Sekip Utara, Yogyakarta,
Indonesia, 55281.

Lina Aryati

Department of Mathematics,
Faculty of Mathematics and natural Sciences,
Universitas Gadjah Mada.
Sekip Utara, Yogyakarta,
Indonesia, 55281.

Abstract

In this paper, we discuss an integral involving a countably Lipschitz condition. It is called CL-integral. Some characteristics of the integral will be given. The existence of the integral will be done by proving every Lebesgue integrable function on $[a, b]$ is CL-integrable on $[a, b]$. The prove will be done by using globally small Riemann sums property. Furthermore, we prove that every CL-integrable function on $[a, b]$ is Denjoy integrable in the wide sense.

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1. Introduction

In 1912, Denjoy generalized the concept of AC (absolutely continuous) to be ACG (generalized absolutely continuous) in the Lebesgue integral to have Denjoy integral in the wide sense. Denjoy also generalized the concept of AC^* (strongly absolutely continuous) to be ACG^* (generalized strongly absolutely continuous) in the Lebesgue integral to have Denjoy integral in the restricted sense. The Denjoy integral in the restricted sense is equivalent with the Henstock-Kurzweil integral on $[a, b]$ [7, 8]. Those three integrals are nonabsolute, in the sense that if f is integrable, its absolute function may not be integrable. Many reseachers work on the Henstock-Kurzweil integral [1, 4, 5, 6, 7, 8, 11].

The primitive of Denjoy integrable function is ACG and the primitive of an Henstock-Kurzweil integrable function is ACG^*. In a practical situation it is not always easy to verify that a function is Henstock-Kurzweil integrable or its primitive is ACG^*. It is a same problem for Denjoy integrable function or its primitive is ACG. In fact, in applications the Henstock-Kurzweil integral appears to be over powerful [6, 11].

Let consider an initial value problem (IVP)

$$\frac{dy}{dx} = f(x, y), \; y(x_o) = y_o. \tag{1}$$

It is well-known that if the function f is continuous and satisfies Lipschitz condition with respect to y on a domain D that contains (x_o, y_o) as its interior point, the IVP in (1) has a unique solution on $[x_o - h, x_o + h]$ for some $h > 0$ (Picard's Theorem).

In 2015, Indrati generalized the Lipschitz condition to be countably Lipschitz condition and gave some characteristics of that condition [3]. In this paper, it will be constructed an integral involving countably Lipschitz condition. It is CL-integral. It will be proved that the integral is more general than Lebesgue integral but less general than Denjoy integral in the wide sense.

2. Preliminary Notes

We restate some concepts that will be used in the discussion of the research, including absolutely continuous function and the Henstock–Kurzweil integral.

Definition 2.1.

(i) A function $F : X \subseteq \mathbb{R} \to \mathbb{R}$ is said to be absolutely continuous on X, if for every $\epsilon > 0$, there exists $\eta > 0$, such that for every finite or infinite collection intervals $\{I_n = [a_n, b_n]\}$, $a_n, b_n \in X$, where $I_n^o \cap I_m^o = \emptyset$, $m \neq n$, $\sum |b_n - a_n| < \eta$, we have

$$\sum |F(b_n) - F(a_n)| < \epsilon.$$

Collection of all absolutely continuous functions is denoted by $AC(X)$.

(ii) The function F is said to be generalized absolutely continuous on $[a, b]$, if there exists a countable collection of sets $\{X_n\}$, with $\cup X_n = [a, b]$, and $F \in AC(X_n)$, for every n. Collection of all generalized absolutely continuous functions is denoted by $ACG(X)$.

Definition 2.2. Let δ be a positive function on $[a, b]$. A collection of interval-point $\mathcal{D} = \{([u_i, v_i], x_i) : i = 1, 2, 3, \ldots, p)\}$ is called a δ-fine partition of $[a.b]$, if $\cup_{i=1}^{p}[u_i, v_i] = [a, b]$, $(u_m, v_m) \cap (u_n, v_n) = \emptyset$, for every $1 \leq n \neq m \leq p$, $x_i \in I_i \subset (x_i - \delta(x_i), x_i + \delta(x_i))$, $i = 1, 2, 3, \ldots, p$.

In short, the collection $\mathcal{D} = \{([u_1, v_i], x_1), \ldots, ([u_p, v_p], x_p)\}$ will be written by $\mathcal{D} = \{([u, v], x)\}$.

Definition 2.3. A function f is said to be Henstock–Kurzweil integrable on $[a, b]$, if there exists a real number A, such that for every $\epsilon > 0$, there is a positive function δ on $[a, b]$, such that for every δ-fine partition $\mathcal{D} = \{([u, v], x)\} = \{([u_i, v_i], x_i) : i = 1, 2, \ldots, p\}$ of $[a, b]$, we have

$$|(\mathcal{D}) \sum f(x)(v - u) - A| < \epsilon,$$

where $(\mathcal{D}) \sum f(x)(v - u) = \sum_{i=1}^{p} f(x_i)(v_i - u_i)$.

The real number A in Definition 2.3 is unique and will be defined as the Henstock–Kurzweil integral value of f on $[a, b]$, shortly, it is written

$$A = (HK) \int_a^b f \qquad \text{or} \qquad A = (HK) \int_a^b f(x)\, dx.$$

We define a function $F : [a, b] \to \mathbb{R}$, by

$$F(x) = (HK) \int_a^x f(t)\, dt, \tag{2}$$

for every $x \in [a, b]$.

The function F in (2) is called HK-primitive of f on $[a, b]$. By this primitive, the definition of the Henstock-Kurzweil integral could be restated as follows: a function f is Henstock-Kurzweil integrable on $[a, b]$ with its HK-primitive F if and only if for every $\epsilon > 0$, there exists a positive function δ on $[a, b]$ such that for every δ-fine partition $\mathcal{D} = \{([u, v], x)\}$ of $[a, b]$ we have

$$(\mathcal{D}) \sum |f(x)(v - u) - F(u, v)| < \epsilon,$$

where $F(u, v) = F(v) - F(u)$.

If a function f is HK-integrable on $[a, b]$ with its HK-primitive F on $[a, b]$, then F is continuous on $[a, b]$, $F' = f$ almost everywhere in $[a, b]$, and $F \in ACG^*[a, b]$.

The Henstock integrability of a measurable function f depends on it behaves on the set of x in which $|f(x)|$ is large, i.e. $|f(x)| > N$ for some N. This behavior inspired Lu, et.al. to develope locally small Riemann sums (LSRS) property to be globally small Riemman sums. Firstly, LSRS was developed for the Lebesgue integral by Schurle [9]. It was also generalized by Schurle for the Perron integral which is equivalent to the Henstock-Kurzweil integral [10]. Finally, the LSRS has been formulated for the Henstock-Kurzweil in the real system [7] and in the n-dimensional space [2].

Definition 2.4. A measurable function f defined on $[a, b]$ is said to have GSRS property, if for every $\epsilon > 0$, there is a positive integer N such that for every $n \geq N$, there is a positive function δ_n on $[a, b]$ such that for every δ_n-fine partition $\mathcal{D} = \{([u, v], x)\}$ of $[a, b]$ we have

$$\left| (\mathcal{D}) \sum_{|f(x)|>n} f(x)(v-u) \right| < \epsilon,$$

with the summation $\mathcal{D}$ is taken over $|f(x)| > n$.

Based on GSRS property we have a characteristic of Henstock integrable function as stated in Theorem 2.5.

Theorem 2.5. A measurable function f has GSRS property on $[a, b]$ if and only if f is Henstock integrable on $[a, b]$ and the sequence $\{F_n\}$ converges to F, where F and F_n are Henstock primitives of f and f_n on $[a, b]$, respectively, where

$$f_n(x) = \begin{cases} f(x), & |f(x)| \leq n \\ 0, & |f(x)| > n, \end{cases}$$

for every n [7].

Definition 2.6. A measurable function $f : [a, b] \to \mathbb{R}$ is said to be HL-integrable on $[a, b]$, if it has GSRS property on $[a, b]$ [7].

Theorem 2.7. A measurable function $f : [a.b] \to \mathbb{R}$ is HL-integrable if and only if there exists a real number A such that for every $\epsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for every $n \geq n_o$, there is a positive function δ_n on $[a, b]$ so for every δ_n-fine $\mathcal{D} = \{([u, v], x)\}$, we have

$$|(\mathcal{D}_1) \sum f(x)(v-u) - A| < \epsilon$$

and

$$|(\mathcal{D}_2) \sum f(x)(v-u)| < \epsilon,$$

where $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$, $\mathcal{D}_1 \subseteq \mathcal{D}$ with $x \notin X = \{x \in [a, b] : |f(x)| > n\}$ and $\mathcal{D}_2 \subseteq \mathcal{D}$ with $x \in X$ [7].

3. Main Results

In this part, it will be given the construction of the countably integral and its some properties. It will be given also the relationship between the constructed integral with the Lebesgue integral. We start by stating the countably Lipschitz condition.

3.1. Countably Lipschitz Condition

The countably Lipschitz condition is a generalization of Lipschitz condition.

Definition 3.1. Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ be a function.

(i) The function f is said to have Lipschitz condition on X, if there exists a real number $M \geq 0$, such that for every $x, y \in X$ we have

$$|f(x) - f(y)| \leq M|x - y|.$$

(ii) The function f is said to have countably Lipschitz condition on X, if there exists a countable collection $\{X_n\}$ with $\cup_n X_n = X$ and f satisfies Lipschitz condition on X_n for every n.

Based on the Definition 3.1, it is clear every function satisfies Lipschitz condition on $X \subseteq \mathbb{R}$ has countably Lipschitz confition on X. However, there is a function satisfies countably Lipschitz condition on $X \subseteq \mathbb{R}$ but it does not satisfy Lipschitz condition on X. For example, the function $f : [0, 2] \to \mathbb{R}$, where

$$f(x) = \begin{cases} 4x, & x \in [0, 1] \\ x + 1, & x \in (1, 2]. \end{cases} \tag{3}$$

As corollary, a function with countably Lipschitz condition on X may not be continuous on X. From the definition, it is easy to prove that the number of discontinuity of a function that satisfies countably Lipschitz condition is at most countable.

For the next discussion, $CLC(X)$ denotes a collection of all functions that satisfy countably Lipschitz condition on $X \subseteq \mathbb{R}$.

3.2. The Countably Lipschitz integral

The countably Lipschitz integral will be constructed by considering a fact that a countably Lipschitz function may not be continuous. For simplicity, countably Lipschitz integral will be written by CL-integral.

Definition 3.2. A function $f : [a, b] \to \mathbb{R}$ is said to be CL-integrable on $[a, b]$, if there exists a continuous function $F : [a, b] \to \mathbb{R}$ such that $F' = f$ almost everywhere in $[a, b]$ and $F \in CLC[a, b]$.

The function F in Definition 3.2 is called the CL-primitive of f on $[a, b]$.

Based on the properties of continuous functions and properties of derivative, it is easy to prove that collection of all CL-integrable functions is a linear space, i.e., if f and g are CL-integrable functions on $[a, b]$, then

(i) αf is CL-integrable for every constant α.

(ii) $f + g$ is CL-integrable on $[a, b]$.

Furthermore, the properties of continuous functions and properties of derivative also imply:

(i) The integrability of f on $[a, b]$ implies the integrability of f on each sub-interval $[c, d] \subseteq [a, b]$.

(ii) If a function f is CL-integrable on $[a, c]$ and on $[c, b]$, then f is CL-integrable on $[a, b]$.

It will be shown that the CL-integrable function is measurable on $[a, b]$ by proving that the function is Denjoy integrable function on $[a, b]$. The prove will be done by Theorem 3.3.

Theorem 3.3. If $F \in CLC[a, b]$ then $F \in ACG[a, b]$.

Proof. Since $F \in CLC[a, b]$, there exists a countable collection $\{X_n\}$ with $\cup_n X_n = [a, b]$ and F satisfies Lipschitz condition on X_n for every n. Let n be fixed. There is $M_n \geq 0$, such that for everyp $x, y \in X_n$, we have

$$|F(x) - F(y)| \leq M_n|x - y|.$$

Let $\epsilon > 0$ be given. We define $\delta = \dfrac{\epsilon}{2(1 + M_n)}$. Therefore, for every collection of closed intervals $\{I_{nk} = [x_{nk}, y_{nk}]\}$, where $x_{nk}, y_{nk} \in X_n$, if

$$\sum_k |I_k| = \sum_k |x_{nk} - y_{nk}| < \delta,$$

then

$$|\sum_k [F(x_{nk}) - F(y_{nk})]| \leq \sum_k M_n|x_{nk} - y_{nk}| < M_n\delta < \epsilon.$$

■

Theorem 3.4. If a function f is CL-integrable on $[a, b]$, then f is Denjoy integrable in the wide sense on $[a, b]$.

Proof. The proof complete by Theorem 3.3. ■

The existence of the CL-integrable function will be given by proving that every Lebesgue integrable function on $[a, b]$ is CL-integrable on $[a, b]$. The proof will be done by using globally small Riemann sums (GSRS) property that has been used to define an HL-integral, stand for Henstock-Lebesgue integral. It is an integral that is more general than the Lebesgue integral but it is less general than the Henstock integral [7].

Theorem 3.5. If a measurable function f is HL-integrable on $[a, b]$, then f is CL-integrable on $[a, b]$.

Proof. From the hypothesis, f has GSRS property on $[a, b]$. Therefore, by Theorem 2.5, $f \in HK[a, b]$ and the sequence of functions $\{F_n(a, b)\}$ converges to $F(a, b)$ on $[a, b]$, where F and F_n denote the primitive-HK of f and f_n, respectively, on $[a, b]$, with

$$f_n(x) = \begin{cases} f(x), & |f(x)| \leq n \\ 0, & |f(x)| > n, \end{cases}$$

for every n.

That means, for every $\epsilon > 0$ there exists $n_o \in N$, such that for every $n \in \mathbb{N}$, $n \geq n_o$, there is a positive function δ_n on $[a, b]$ such that for every δ_n-fine partition $\mathfrak{D}$ of $[a, b]$, we have

$$|(\mathfrak{D}) \sum_{|f(x)|>n} f(x)(v-u)| < \epsilon.$$

For every $k \in \mathbb{N}$, we define

$$X_k = \{x \in [a, b] : |f(x)| \leq k\}.$$

We have $\cup_{n=1}^{\infty} X_k = [a, b]$.

Let $n \in \mathbb{N}$ be fixed. Put $K = \max\{n, n_o\}$. For every $x, y \in X_n$, without lost of generalization, we assume $x < y$. Since f is HK-integrable on $[x, y]$, there exists a positive function δ_* on $[a, b]$, such that for every δ_*-fine partition $\mathfrak{D}$ of $[x, y]$, we have

$$|(\mathfrak{D}) \sum F(x, y) - (\mathfrak{D}) \sum f(t)(v-u)| < \epsilon.$$

Let define a positive function δ on $[a, b]$, with $\delta(x) = \min\{\delta_*(x), \delta_n(x), \delta_{n_o}(x)\}$, for every $x \in [a, b]$. As corollary, for every δ-fine partition $\mathfrak{D}$ of $[x, y]$, we have

$$\begin{aligned} |F(x) - F(y)| &\leq |F(x, y) - \sum f(t)(v-u)| + |\sum_{|f(t)|\leq n} f(t)(v-u)| \\ &\quad + |\sum_{|f(t)|>n} f(t)(v-u)| \\ &< \epsilon + K|x-y| + |\sum_{|f(t)|>n_o} f(t)(v-u)| \\ &< 2\epsilon + K|x-y|. \end{aligned}$$

That is F satisfies Lipschitz condition on X_n. Therefore, $F \in CLC[a, b]$. ■

Corollary 3.6. Every Lebesgue integrable function on $[a, b]$ is CL-integrable on $[a, b]$.

Finally, we develop uniform convergence theorem for CL-integral.

Definition 3.7. A sequence of functions (F_k) is said to have uniformly countably Lipschitz condition on $[a, b]$, if there exists a collection $\{X_n\}$ with $\cup_n X_n = [a, b]$ such that for every $n \in \mathbb{N}$, there exists $M_n \geq 0$ such that for every $x, y \in X_n$,

$$|F_k(x) - F_k(y)| \leq M_n|x-y|,$$

for every k.

Theorem 3.8. Let (f_k) be a sequence of CL-integrable function on $[a, b]$ with F_k as in the definition of CL-integral of f_k for every k. If (f_k) converges uniformly to f on $[a, b]$ and (F_k) has uniformly countably Lipschitz condition on $[a, b]$, then f is CL-integrable on $[a, b]$.

Proof. The sequence $(F_k(a))$ converges. Since for every k, f_k is CL-integrable on $[a, b]$, the F_k is continuous on $[a, b]$, CLC on $[a, b]$ and $F_k' = f_k$ a.e. in $[a, b]$. Since (f_k) converges uniformly to f on $[a, b]$, then (F_k) converges uniformly to a function F with $F' = f$. The function F is continuous on $[a, b]$. Furthermore, let n be fixed. For every $x, y \in X_n$,

$$|F(x) - F(y)| = \lim_{k\to\infty} |F_k(x) - F_k(y)| \leq \lim_{k\to\infty} M_n|x - y| = M_n|x - y|.$$

That means, f is CL-integrable on $[a, b]$. ■

4. Concluding Remarks

The CL-integral is a linear integral. It is not an absolute integral (its primitive is ACG). From Theorem 3.5 we have CL-integral is more general than the Lebesgue integral.

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