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An Isomorphism of Partitioning Hamiltonian Circuits in Complete Graphs

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Abstract

This paper presents a cutting-edge method to partition isomorphic Hamiltonian circuits in complete graphs. The conceptual of matrix theory will be employed to construct the isomorphism classes. The crux of this method is classifying the Hamiltonian circuits based on the theory of matrix transposition. In illustrating this theory a special reference for complete graph of order five will be decomposed and partitioned. The novel result is formulated and proved throughout developing this method.

Keywords: Complete graphs, decomposition, Hamiltonian circuits, isomorphism.

INTRODUCTION

An isomorphism of a graph is an overwhelmingly interesting problem due to the fact that it can be adopted in the field of organic chemistry to determine two identical molecules (Milan (1977), Jean-Loup (1998)). In graph theory, a complete graph K_n

is known to have $n!$ Hamiltonian circuits (HC) and $\frac{(n-1)!}{2}$ distinct HC (Riaz and Khiyal (2006), Douglas (2001)). The isomorphic HC among those $n!$ circuits need to be classified to produce the distinct HC. Thus, in this paper, we aim to construct a cutting-edge method to partition isomorphic HC in K_n. The idea of adjacency matrix and matrix transposition are considered in order to classify the isomorphism classes of the HC. The definitions needed along this paper are given below.

Definition 1 A *complete graph* K_n is a simple graph with n vertices whose vertices are pairwise adjacent.

Definition 2 A *Hamiltonian circuit* is a circuit that starts and ends at the same vertex, and visits each vertex exactly once.

Definition 3 Let $C_1{}^* = (V_1, E_1)$ and $C_2{}^* = (V_2, E_2)$ be two Hamiltonian circuits. $C_1{}^* \cong C_2{}^*$ if there is a one-to-one function $f: V(C_1{}^*) \to V(C_2{}^*)$ such that $uv \in E$ $(C_1{}^*)$ if and only if $f(u)f(v) \in (C_2{}^*)$.

Definition 4 Suppose $G = (V, E)$ where $v_1, v_2, v_3, \ldots, v_n \in V$. The *adjacency matrix* $\mathbf{A}$ of G (or $\mathbf{A}_G$), with respect to this listing of vertices, is the $n \times n$ matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, 0 as its (i, j)th entry when they are not adjacent.

Definition 5 If $\mathbf{M}$ is a $m \times n$ matrix, then the *transpose matrix* of $\mathbf{M}$ denoted by $\mathbf{M}^T$ is an $n \times m$ matrix, where the columns of $\mathbf{M}$ be the rows of $\mathbf{M}^T$ and the rows of $\mathbf{M}$ be the columns of $\mathbf{M}^T$.

In linear algebra, a symmetric matrix is a square matrix that is equal to its transpose. Formally, a matrix $\mathbf{M}$ is symmetric if $\mathbf{M} = \mathbf{M}^T$.

Definition 6 Let ${C_1}^*$ be a circuit with direction $(x_1, x_2, x_3, \dots, x_{n-1}, x_n, x_1)$. Then, a circuit ${C_2}^*$ is a *mirror image* to circuit ${C_1}^*$ if the direction of ${C_2}^*$ is $(x_1, x_n, x_{n-1}, \dots, x_3, x_2, x_1)$.

Definition 7 Let ${C_1}^*$ and ${C_2}^*$ be two circuits with n vertices. If ${C_2}^*$ is the mirror image of ${C_1}^*$, then ${C_1}^* \cong {C_2}^*$.

Definition 8 If the mapping of ${C_1}^*$ and ${C_2}^*$ is $(1, a)(2, b)(3, c) \dots (n, z)$ and $(z, n) \dots (c, 3)(b, 2)(a, 1)$ respectively, then ${C_1}^*$ and ${C_2}^*$ has an *opposite mapping*, where $a, b, c, \dots, z$ are the images.

Definition 9 Suppose the sets of vertices $\{x_1, x_2, x_3, \dots, x_n\} \in {C_1}^*$ and $\{x_1a, x_2b, x_3c, \dots, x_nz\} \in {C_2}^*$. A function $g = \begin{pmatrix} x_1 & x_2 & x_3 \cdots & x_n \\ x_1a & x_2b & x_3c \cdots & x_nz \end{pmatrix}$ maps the vertices $\{x_1, x_2, x_3, \dots, x_n\}$ of ${C_1}^*$ to other vertices $\{x_1a, x_2b, x_3c, \dots, x_nz\}$ of ${C_2}^*$ where $\{x_1a, x_2b, x_3c, \dots, x_nz\}$ are the images. That is, $x_1 \longmapsto x_2b$, $x_2 \longmapsto x_3c$, …, $x_n \longmapsto x_nz$ for $n \in \mathbb{Z}^+$. Then, the mapping is written as a *product of transposition* $(x_1, x_1a)(x_2, x_2b) \dots (x_n, x_nz)$.

METHOD

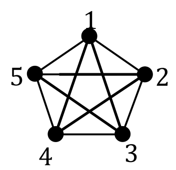

Figure 1: A complete graph, K_5.

Now we consider K_5 as shown in Figure 1, it can be decomposed into $(5-1)! = 24$ HC from K_5 (Riyaz and Khiyal,2006). Since there are $\frac{(n-1)!}{2}$ distinct HC in K_5 (Douglas,2001), we use the idea of adjacency matrix and transpose matrix to partition the isomorphism HC to get the distinct HC. Among the twenty four circuits, as an example, we provide several HC from K_5 in Figure 2. Then, the adjacency matrix as well as its transpose are presented in Table 1. Then, we investigate which matrices are symmetric.

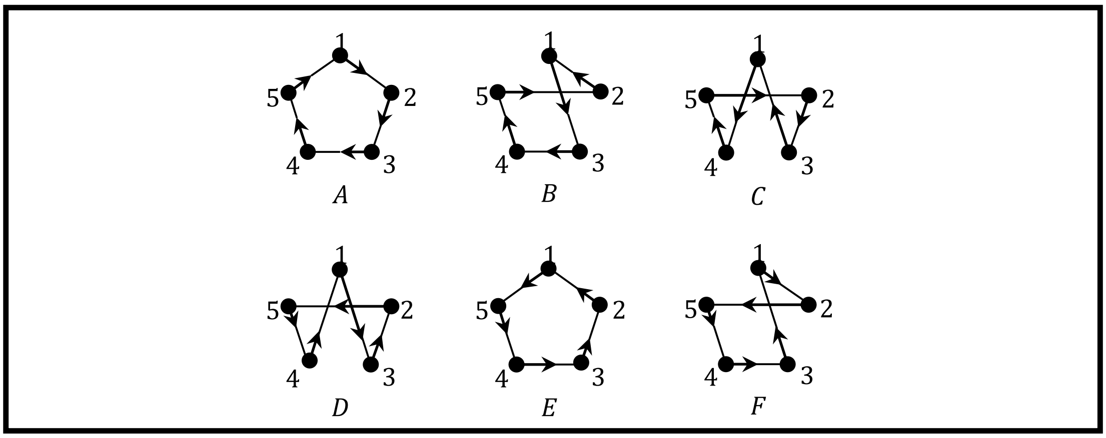

Figure 2: Several HC from K_5.

Table 1: Adjacency matrix and its transpose for K_5

Adjacency matrix

$A =$

	1	2	3	4	5
1	0	1	0	0	0
2	0	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1
5	1	0	0	0	0

The transpose

$A^{T} =$

	1	2	3	4	5
1	0	0	0	0	1
2	1	0	0	0	0
3	0	1	0	0	0
4	0	0	1	0	0
5	0	0	0	1	0

Adjacency matrix

$B =$

	1	2	3	4	5
1	0	0	1	0	0
2	1	0	0	0	0
3	0	0	0	1	0
4	0	0	0	0	1
5	0	1	0	0	0

The transpose

$B^{T} =$

	1	2	3	4	5
1	0	1	0	0	0
2	0	0	0	0	1
3	1	0	0	0	0
4	0	0	1	0	0
5	0	0	0	1	0

$C =$

	1	2	3	4	5
1	0	0	0	1	0
2	0	0	1	0	0
3	1	0	0	0	0
4	0	0	0	0	1
5	0	1	0	0	0

$C^{T} =$

	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	0	1
3	0	1	0	0	0
4	1	0	0	0	0
5	0	0	0	1	0

$D =$

	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	0	1
3	0	1	0	0	0
4	1	0	0	0	0
5	0	0	0	1	0

$D^{T} =$

	1	2	3	4	5
1	0	0	0	1	0
2	0	0	1	0	0
3	1	0	0	0	0
4	0	0	0	0	1
5	0	1	0	0	0

$E =$

	1	2	3	4	5
1	0	0	0	0	1
2	1	0	0	0	0
3	0	1	0	0	0
4	0	0	1	0	0
5	0	0	0	1	0

$E^{T} =$

	1	2	3	4	5
1	0	1	0	0	0
2	0	0	1	0	0
3	0	0	0	1	0
4	0	0	0	0	1
5	1	0	0	0	0

$F =$

	1	2	3	4	5
1	0	1	0	0	0
2	0	0	0	0	1
3	1	0	0	0	0
4	0	0	1	0	0
5	0	0	0	1	0

$F^{T} =$

	1	2	3	4	5
1	0	0	1	0	0
2	1	0	0	0	0
3	0	0	0	1	0
4	0	0	0	0	1
5	0	1	0	0	0

There are symmetric matrices produced in Table 1, i.e. $A^T = E$, $B^T = F$, $C^T = D$, $D^T = C$, $E^T = A$, and $F^T = B$. Without loss of generality, since K_5 has twenty four HC, thus K_5 can be partitioned into twelve distinct HC as shown in Appendix A.

RESULT

From the case K_5 discussed in previous section, a theorem is produced as shown in the next paragraph.

A complete graph K_n is known can be partitioned into $(n-1)!$ HC (Riyaz and Khiyal,2006). Suppose a HC as shown in Figure 3. To partition the isomorphic classes of the circuits, we use the idea of adjacency matrices and transpose matrices as discussed below.

Step 1 : Find the adjacency matrices of each HC.

Step 2 : Find the transpose matrices of each adjacency matrix obtained in Step 1.

Step 3 : Investigate which matrices are symmetric to partition the isomorphic circuits.

Step 4 : Determine the distinct HC.

Without loss of generality, we have the following theorem.

Theorem 1. Let P and Q be two Hamiltonian circuits with opposite direction. If the adjacency matrix of P equals to the transpose of adjacency matrix of Q $(\mathbf{P} = \mathbf{Q}^T)$, then circuit $P \cong Q$.

Proof. Suppose P and Q are two Hamiltonian circuits with n vertices as shown in Figure 3.

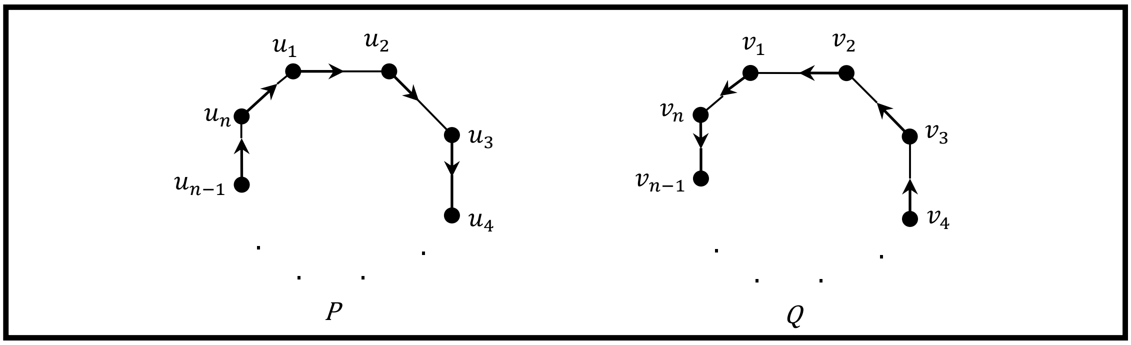

Figure 3: A complete graph K_n

Both P and Q have n vertices, n edges, and vertices of degree two. Because P and Q agree with respect to these invariants, we define a function f to investigate the one-to-one function. Since all vertices in both P and Q have degree two, then we have $f(u_n) = v_3$, $f(u_1) = v_2$, $f(u_2) = v_1$, $f(u_3) = v_n$, $f(u_4) = v_{n-1}$, ..., $f(u_{n-1}) = v_4$. To examine whether f preserves edges, we examine the adjacency matrices of P and Q as well as their transpose, with the rows and columns labeled by the images of their corresponding vertices.

$$\mathbf{P} = \begin{array}{c|ccccccc|}
 & u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \\
\hline
u_1 & 0 & 1 & 0 & 0 & & 0 & 0 \\
u_2 & 0 & 0 & 1 & 0 & & 0 & 0 \\
u_3 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
u_4 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
u_{n-1} & 0 & 0 & 0 & 0 & & 0 & 1 \\
u_n & 1 & 0 & 0 & 0 & & 0 & 0 \\
\hline
\end{array}$$

$$\mathbf{P}^T = \begin{array}{c|ccccccc|}
 & u_1 & u_2 & u_3 & u_4 & \cdots & u_{n-1} & u_n \\
\hline
u_1 & 0 & 0 & 0 & 0 & & 0 & 1 \\
u_2 & 1 & 0 & 0 & 0 & & 0 & 0 \\
u_3 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
u_4 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
u_{n-1} & 0 & 0 & 0 & 0 & & 0 & 0 \\
u_n & 0 & 0 & 0 & 0 & & 1 & 0 \\
\hline
\end{array}$$

$$\mathbf{Q} = \begin{array}{c|ccccccc|}
 & v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \\
\hline
v_1 & 0 & 0 & 0 & 0 & & 0 & 1 \\
v_2 & 1 & 0 & 0 & 0 & & 0 & 0 \\
v_3 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
v_4 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
v_{n-1} & 0 & 0 & 0 & 0 & & 0 & 0 \\
v_n & 0 & 0 & 0 & 0 & & 1 & 0 \\
\hline
\end{array}$$

$$\mathbf{Q}^T = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ \vdots \\ v_{n-1} \\ v_n \end{array} \begin{array}{c} \begin{array}{ccccccc} v_1 & v_2 & v_3 & v_4 & \cdots & v_{n-1} & v_n \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 \end{bmatrix} \end{array}$$

From the above matrices, we have adjacency matrices $\mathbf{P} = \mathbf{Q}^T$ and $\mathbf{Q} = \mathbf{P}^T$ which shows that f preserves the edges. Thus, we conclude that P and Q are isomorphic.

DISCUSSIONS

We have developed a new approach in partitioning the isomorphic classes of HC in K_n. A case of $n = 5$ is discussed as a basis to find the isomorphism among the HC. A theorem has been produced to prove that two circuits are isomorphic if both circuits share the same edges.

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