Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 3933–3940

http://www.ripublication.com/gjpam.htm

Multiple (h, q)-tangent polynomials and (h, q)-tangent zeta functions

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Abstract

In this paper, we introduce the (h, q)-tangent polynomials $T_{n,q}^{(h,k)}(x)$ of order k. Finally we construct (h, q)-tangent zeta function of order k which interpolates higher order (h, q)-tangent numbers at negative integer.

AMS subject classification:
Keywords: Tangent numbers, tangent polynomials, the (h, q)-tangent numbers and polynomials, (h, q)-tangent zeta function.

1. Introduction

Several mathematicians have studied Bernoulli numbers, Bernoulli polynomials, Euler numbers, Euler polynomials, Genocchi numbers, Genocchi polynomials, tangent numbers, and tangent polynomials (see [1,2,3,4]). These numbers and polynomials posses many interesting properties and arising in many areas of mathematics and physics. In this paper, we introduce the (h, q)-tangent numbers $T_{n,q}^{(h,k)}$ and polynomials $T_{n,q}^{(h,k)}(x)$ of higher order. Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of rational numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$ and $\mathbb{Z}_+ = \mathbb{Z} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N, we set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \quad X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$, $z \in \mathbb{C}_p$, set

$$\mu_z(a + dp^N\mathbb{Z}_p) = \frac{z^a}{[dp^N : z]}$$

and this can be extended to a a distribution on X. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{0\leq x<p^N} g(x)(-1)^x, \quad \text{(see[1])}. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0). \tag{1.2}$$

We assume that $q \in \mathbb{C}_p$ and $h \in \mathbb{Z}$. In [4], we studied the (h, q)-tangent numbers $T_{n,q}^{(h)}$ and polynomials $T_{n,q}^{(h)}(x)$. The (h, q)-tangent numbers $T_{n,q}^{(h)}$ are defined by the generating function:

$$F_q^{(h)}(t) = \frac{2}{q^h e^{2t} + 1} = \sum_{n=0}^{\infty} T_{n,q}^{(h)} \frac{t^n}{n!}. \tag{1.3}$$

We introduce the (h, q)-tangent polynomials $T_{n,q}^{(h)}(x)$ as follows:

$$F_q^{(h)}(x,t) = \left(\frac{2}{q^h e^{2t} + 1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{1.4}$$

Theorem 1.1. For $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} q^{hx}(2x)^n d\mu_{-1}(x) = T_{n,q}^{(h)},$$

$$\int_{\mathbb{Z}_p} q^{hy}(x+2y)^n d\mu_{-1}(y) = T_{n,q}^{(h)}(x).$$

2. The (h, q)-tangent polynomials of higher order

In this section, we assume that $a_1, \ldots, a_k \in \mathbb{Z}$. We introduce the (h, q)-tangent polynomials of higher order, $T_{n,q}^{(h,k)}(x)$. We use the notation

$$\sum_{k_1=0}^{m} \cdots \sum_{k_n=0}^{m} = \sum_{k_1 \cdots k_n=0}^{m} .$$

The binomial formulae are known as

$$(1-a)^n = \sum_{i=0}^{n} \binom{n}{i} (-a)^i, \text{ where } \binom{n}{i} = \frac{n(n-1)\ldots(n-i+1)}{i!},$$

and

$$\frac{1}{(1-a)^n} = (1-a)^{-n} \sum_{i=0}^{n} \binom{-n}{i} (-a)^i = \sum_{i=0}^{n} \binom{n+i-1}{i} a^i$$

Now, using multiple of p-adic integral, we introduce the (h, q)-tangent polynomials of higher order $T_{n,q}^{(h,k)}(x)$: For $k \in \mathbb{N}$ and $h \in \mathbb{Z}$, we define

$$\sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x) \frac{t^n}{n!} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} q^{hx_1+hx_2+\cdots+hx_k} e^{(x+2x_1+2x_2+\cdots+2x_k)t} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.1}$$

By using Taylor series of e^{xt} in the above equation, we obtain

$$\sum_{n=0}^{\infty} \left(\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{hx_1+\cdots+hx_k} (x + 2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k) \right) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x) \frac{t^n}{n!}$$

By comparing coefficients $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.1. For positive integers n, k, and $h \in \mathbb{Z}$, we have

$$T_{n,q}^{(h,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{hx_1+\cdots+hx_k} (x + 2x_1 + \cdots + 2x_k)^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_k). \tag{2.2}$$

By (1.4), the (h, q)-tangent polynomials of higher order, $T_{n,q}^{(h,k)}(x)$ are defined by means of the following generating function

$$F_q^{(h,k)}(x,t) = \left(\frac{2}{q^h e^{2t} + 1} \right)^k e^{xt} = \sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x) \frac{t^n}{n!}. \tag{2.3}$$

By using (2,1), the (h, q)-tangent numbers of higher order, $T_{n,q}^{(h,k)}$ are defined by the following generating function

$$\left(\frac{2}{q^h e^{2t}+1}\right)^k = \sum_{n=0}^{\infty} T_{n,q}^{(h,k)} \frac{t^n}{n!}. \tag{2.4}$$

When $k = 1$, above (2.3) and (2.4) will become the corresponding definitions of the (h, q)-tangent polynomials $T_{n,q}^{(h)}(x)$ and the (h, q)-tangent numbers $T_{n,q}^{(h)}$(see [4]). Observe that for $x = 0$, the equation (2.4) reduces to (2.3). Note that when $k = 1$, then we have (1.2), when $q \to 1$, then we have

$$F^{(k)}(x,t) = \left(\frac{2}{e^{2t}+1}\right)^k e^{xt} = \sum_{n=0}^{\infty} T_n^{(k)}(x) \frac{t^n}{n!},$$

where $T_n^{(k)}(x)$ denote the tangent polynomials of higher order (see [5]).

Corollary 2.2. For positive integers n, k, and $h \in \mathbb{Z}$, we have

$$T_{n,q}^{(h,k)} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{hx_1+\cdots+hx_k} (2x_1+\cdots+2x_k)^n d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

By using binomial expansion in (2.2), we obtain

$$T_{n,q}^{(h,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{hx_1+\cdots+hx_k} (2x_1+\cdots+2x_k)^l d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_k).$$

By Corollary 2.2, we arrive at the following theorem.

Theorem 2.3. For positive integers n, k, and $h \in \mathbb{Z}$, we have

$$T_{n,q}^{(h,k)}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,q}^{(h,k)} x^{n-l}.$$

We define distribution relation of the (h, q)-tangent polynomials of higher order as follows: For $m \in \mathbb{N}$ with $m \equiv 1(\bmod 2)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x) \frac{t^n}{n!} \\
&= \left(\frac{2}{q^h e^{2t}+1}\right)\left(\frac{2}{q^h e^{2t}+1}\right)\cdots\left(\frac{2}{q^h e^{2t}+1}\right) e^{xt} \\
&= \left(\frac{2}{q^{mh} e^{2mt}+1}\right)^k \sum_{a_1,\cdots,a_k=0}^{m-1} (-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)} e^{\left(\frac{2a_1+\cdots+2a_k+x}{m}\right)(mt)}.
\end{aligned}$$

From the above, we obtain

$$\sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!} = \sum_{a_1,\cdots,a_k=0}^{m-1} (-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)} \times \sum_{n=0}^{\infty} T_{n,q^m}^{(h,k)}\left(\frac{2a_1+\cdots+2a_k+x}{m}\right)\frac{(mt)^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.4. (Distribution relation of the (h, q)-tangent polynomials of higher order). For $m \in \mathbb{N}$ with $m \equiv 1(\bmod 2)$, we have

$$T_{n,q}^{(h,k)}(x) = m^n \sum_{a_1,\cdots,a_k=0}^{m-1} (-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)} T_{n,q^m}^{(h,k)}\left(\frac{2a_1+\cdots+2a_k+x}{m}\right).$$

By (2.3), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!} &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} (-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)} e^{(2a_1+\cdots+2a_k+x)t} \\ &= 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m q^{hm} e^{(2m+k+x)t}.\end{aligned} \tag{2.5}$$

From the above, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\left(2^k \sum_{a_1,\cdots,a_k=0}^{\infty} (-q^h)^{a_1+\cdots+a_k}(x+2a_1+\cdots+2a_k)^n\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-q^h)^m(2m+x)^n\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we arrive at the following theorem.

Theorem 2.5. For positive integers $n, k,$ and $h \in \mathbb{Z}$, we have

$$\begin{aligned}T_{n,q}^{(h,k)}(x) &= 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} (-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)}(2a_1+\cdots+2a_k+x)^n \\ &= 2^k \sum_{m=0}^{\infty}\binom{m+k-1}{m}(-1)^m q^{hm}(2m+x)^n.\end{aligned} \tag{2.6}$$

Since

$$\begin{aligned}\sum_{l=0}^{\infty} T_{l,q}^{(h,k)}(x+y)\frac{t^l}{l!} &= \left(\frac{2}{q^h e^{2t}+1}\right)^k e^{(x+y)t}\\ &= \sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty} y^m \frac{t^m}{m!}\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!} y^{l-n}\frac{t^{l-n}}{(l-n)!}\right)\\ &= \sum_{l=0}^{\infty}\left(\sum_{n=0}^{l}\binom{l}{n} T_{n,q}^{(h,k)}(x) y^{l-n}\right)\frac{t^l}{l!},\end{aligned}$$

we have the following addition theorem.

Theorem 2.6. The (h, q)-tangent polynomials $T_{n,q}^{(h,k)}(x)$ of higher order satisfies the following relation:

$$T_{n,q}^{(h,k)}(x+y) = \sum_{l=0}^{n}\binom{n}{l} T_{l,q}^{(h,k)}(x) y^{n-l}.$$

3. Multiple (h, q)-tangent zeta function

In this section, we assume that $q \in \mathbb{C}$ with $|q| < 1$. We define multiple (h, q)-tangent zeta function. This function interpolates the (h, q)-tangent polynomials of higher order at negative integers.

By using (2.5), we have

$$F_q^{(h,k)}(x,t) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} (-q^h)^{a_1+\cdots+a_k} e^{(2a_1+\cdots+2a_k+x)t} = \sum_{n=0}^{\infty} T_{n,q}^{(h,k)}(x)\frac{t^n}{n!}.$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we can derive the following Eq.(3.1) form the Mellin transformation of $F_q^{(h,k)}(x,t)$.

$$\frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-1} F_q^{(h,k)}(x,-t)dt = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)}}{(2a_1+\cdots+2a_k+x)^s} \tag{3.1}$$

For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$, we define multiple (h, q)-tangent zeta function as follows:

Definition 3.1. For $s, x \in \mathbb{C}$ with $\mathcal{R}(x) > 0$ and $h \in \mathbb{Z}$, we define

$$\zeta_q^{(h,k)}(s,x) = 2^k \sum_{a_1,\cdots,a_k=0}^{\infty} \frac{(-1)^{a_1+\cdots+a_k} q^{h(a_1+\cdots+a_k)}}{(2a_1+\cdots+2a_k+x)^s}. \tag{3.2}$$

For $s = -l$ in (3.2) and using (2.6), we arrive at the following theorem.

Theorem 3.2. For positive integer l, we have

$$\zeta_q^{(h,k)}(-l, x) = T_{l,q}^{(h,k)}(x).$$

By (2.4), we have

$$\sum_{n=0}^{\infty} T_{n,q}^{(h,k)} \frac{t^n}{n!} = \left(\frac{2}{q^h e^{2t} + 1}\right)^k = 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-q^h)^m e^{(2m)t}.$$

By using Taylor series of $e^{(2m)t}$ in the above, we have

$$\sum_{n=0}^{\infty} T_{n,q}^{(h,k)} \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left(2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m q^{hm} (2m)^n\right) \frac{t^n}{n!}.$$

By comparing coefficients $\frac{t^n}{n!}$ in the above equation, we have

$$T_{n,q}^{(h,k)} = 2^k \sum_{m=0}^{\infty} \binom{m+k-1}{m} (-1)^m q^{mh} (2m)^n. \tag{3.3}$$

By using (3.3), we define multiple (h, q)-tangent zeta function as follows:

Definition 3.3. For $s \in \mathbb{C}$ and $h \in \mathbb{Z}$, we define

$$\zeta_q^{(h,k)}(s) = 2^k \sum_{m=1}^{\infty} \binom{m+k-1}{m} \frac{(-1)^m q^{mh}}{(2m)^s}. \tag{3.4}$$

The function $\zeta_q^{(h,k)}(s)$ interpolates the number $T_{n,q}^{(h,k)}$ at negative integers. Substituting $s = -n$ with $n \in \mathbb{Z}_+$ into (3.4), and using (3.3), we obtain the following theorem:

Theorem 3.4. Let $n \in \mathbb{Z}_+$, We have

$$\zeta_q^{(h,k)}(-n) = T_{n,q}^{(h,k)}.$$

References

[1] T. Kim, *q-Volkenborn integration*, Russ. J. Math. Phys. **9**(2002), 288–299.

[2] T. Kim, *Euler numbers and polynomials associated with zeta function, Abstract and Applied Analysis*, Art. ID 581582 (2008).

[3] Liu, G., (2006). *Congruences for higher-order Euler numbers*, *Proc. Japan Acad.*, **82 A**(2006), 30–33.

[4] Ryoo, C. S., (2014). A Note on the (h, q)-Tangent Numbers and Polynomials Associated with the p-Adic Integral on $\mathbb{Z}_p$, *Adv. Studies Theor. Phys.*, **8(3)**(2014), 143–149.

[5] Ryoo, C. S., (2014). Multiple tangent zeta functions and tangent polynomials of higher order, *Adv. Studies Theor. Phys.*, **8(14)**(2014), 457–462.