Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 5 (2016), pp. 3873–3883

http://www.ripublication.com/gjpam.htm

Existence of solution for an elliptic problem involving $p(x)$-Laplacian in $\mathbb{R}^N$.

Abdelmalek Brahim

*Laboratory of Mathematics,
Dynamics and Modelization, Univ Annaba,
Fac. of Sciences, B. P. 12. 23000 Annaba, Algeria.*

Djellit Ali

*Laboratory of Mathematics,
Dynamics and Modelization, Univ Annaba,
Fac. of Sciences, B. P. 12. 23000 Annaba, Algeria.*

Ghannem Lahcen

*Labo. MIP. P. Sabatier university,
Toulouse 3, France.*

Abstract

In this paper we study a class of nonlinear elliptic problems involving the $p(x)$-Laplacian operator. Under some additional assumptions on the nonlinearities, the corresponding functional verifies the Palais-Smale condition. So, we can use the Mountain Pass Theorem to prove the existence of nontrivial solution.

AMS subject classification: 35J60, 35B30, 35B40.
Keywords: $p(x)$-Laplacian equations, Nonlinear elliptic problems, Mountain pass theorem, Palais-Smale condition.

1. Introduction

The aim of this paper is to prove some existence results for nonlinear elliptic problem

$$\begin{cases} -\Delta_{p(x)}u = \lambda V(x)|u|^{q(x)-2}u + f(x,u), \quad x \in \mathbb{R}^N \\ u \geq 0, u \neq 0, u \in W \end{cases} \tag{1.1}$$

$\Delta_{p(x)}$ is so-called $p(x)$-Laplacian operator i.e. $\Delta_{p(x)}u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. In the case $p(x) = p$, then $\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is well-known p-Laplacian and the problem is the usual p-Laplacian equation. f is real-valued function with domain $\mathbb{R}^N \times \mathbb{R}$; u is unknown real valued function defined in $\mathbb{R}^N$ and belonging to appropriate function spaces; λ is positive parameter; p and q are reals functions satisfying $p(x), q(x) \in C_+\left(\mathbb{R}^n\right)$.

Problems involving the $p(x)$-Laplacian operator arise from many branches of mathematics as in the study of elastic mechanics (see [22]), electrorheological fluids (see [1], [7]), (see [17]) or image restoration (see [6]).

Let the eigenvalue problem involving variable exponent growth conditions intensively studied is the following

$$-\Delta_{p(x)}u = \lambda V(x)|u|^{q(x)-2}u, \text{ in } \Omega. \tag{1.2}$$

where Ω is bounded domain in $\mathbb{R}^N$, $n \geq 3$, with smooth boundary $\partial\Omega$,

In [21] the author studied the problem (1.2) in bounded domain where $V(x) = 1$, under the assumption $1 < \min_{\overline{\Omega}} q(x) < \min_{\overline{\Omega}} p(x) < \max_{\overline{\Omega}} q(x)$, the continuous spectrum is proved.

However [18] the author established in bounded domain, using the simple variational arguments based on the Ekeland's principle, that there exists $\lambda^* > 0$ such that for any $\lambda \in \left(0, \lambda^*\right)$ is an eigenvalue for the above problem.

This paper is organized as follows. In Section 1 we recall some previous results. In Section 2, we state some basic results for the variable exponent Lebesgue-Sobolev spaces, which are given in Fan and Zhao (see [11]), O. Kovăcĭk, J. Răkosnĭk (see [19]). In Section 3, we give sufficient conditions on V and f to obtain the existence of solution for the problem (1.1) above.

2. Preliminary results

We recall some background facts concerning the generalized Lebesgue-Sobolev spaces and introduce some notations used below.

Let

$$C_+(\Omega) = \{p \in C(\Omega) : p(x) > 1, \text{ for every } x \in \Omega\}$$

$p^+ = \max\{p(x) \in \Omega\}$ et $p^- = \min\{p(x) \in \Omega\}$ for every $p \in C_+(\Omega)$.

Denote by $\mathcal{M}(\Omega)$ the set of measurable real-valued functions defined on Ω.

We introduce for $p \in C_+(\Omega)$, the space

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{M}(\Omega) \text{ such that, } \int_\Omega |u(x)|^{p(x)} dx < +\infty \right\}$$

equipped with the so called Luxemburg norm

$$|u|_{p(x),\Omega} = \inf \left\{ t > 0 : \int_\Omega \left| \frac{u(x)}{t} \right|^{p(x)} dx \le 1 \right\}.$$

In what follow $|u|_{p(x)}$ will denote $|u|_{p(x),\mathbb{R}^N}$. It is well-know that this norm confers a reflexive Banach structure.

Define the variable exponent Sobolev space W the closure of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|_{p(x)} = |\nabla u|_{p(x)}.$$

Moreover, we recall some previous results.

Proposition 2.1. ([8]) If $p \in C_+(\mathbb{R}^N)$, then $L^{p(x)}(\mathbb{R}^N)$ and $W^{1,p(x)}(\mathbb{R}^N)$ are separable and reflexive Banach spaces.

Proposition 2.2. ([8]) The topological dual space of $L^{p(x)}(\mathbb{R}^N)$ is $L^{p'(x)}(\mathbb{R}^N)$, where

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Moreover for any $(u, v) \in L^{p(x)}(\mathbb{R}^N) \times L^{p'(x)}(\mathbb{R}^N)$, we have

$$\left| \int_{\mathbb{R}^N} uv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \le 2 |u|_{p(x)} |v|_{p'(x)}.$$

Let us now define the modular corresponding to the norm $|.|_{p(x)}$ by

$$\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)} dx.$$

Proposition 2.3. ([11],[19]) For all $u \in L^{p(x)}(\mathbb{R}^N)$, we have

$$\min \left\{ |u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right\} \le \rho(u) \le \max \left\{ |u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+} \right\}.$$

In addition, we have

(i) $|u|_{p(x)} < 1$ (resp. $= 1; > 1$) $\Leftrightarrow \rho(u) < 1$ (resp. $= 1; > 1$),

(ii) $|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$,

(iii) $|u|_{p(x)} > 1 \Longrightarrow |u|_{p(x)}^{p^+} \leq \rho(u) \leq |u|_{p(x)}^{p^-}$,

(iv) $\rho\left(\frac{u}{|u|_{p(x)}}\right) = 1.$

Proposition 2.4. **([8])** Let $p(x)$ and $s(x)$ be measurable functions such that $p(x) \in L^{\infty}(\mathbb{R}^N)$ and $1 \leq p(x)s(x) \leq \infty$ almost every where in $\mathbb{R}^N$. If $u \in L^{s(x)}(\mathbb{R}^N)$, $u \neq 0$, then

$$|u|_{p(x)s(x)} \leq 1 \Longrightarrow |u|_{p(x)s(x)}^{p^-} \leq \left||u|^{p(x)}\right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^+},$$

$$|u|_{p(x)s(x)} \geq 1 \Longrightarrow |u|_{p(x)s(x)}^{p^+} \leq \left||u|^{p(x)}\right|_{s(x)} \leq |u|_{p(x)s(x)}^{p^-}.$$

In particular, if $p(x) = p$ is a constant, then

$$\left||u|^{p}\right|_{s(x)} = |u|_{ps(x)}^{p}.$$

Proposition 2.5. **([11])** If $u, u_n \in L^{p(x)}(\mathbb{R}^N)$, $n = 1, 2, \ldots$, then the following statements are mutually equivalent:

(1) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,

(2) $\lim_{n\to\infty} \rho(u_n - u) = 0$,

(3) $u_n \to u$ in measure in $\mathbb{R}^N$ and $\lim_{n\to\infty} \rho(u_n) = \rho(u)$.

Let $p^*(x)$ be the critical Sobolev exponent of $p(x)$ defined by

$$p^*(x) = \begin{cases} \dfrac{Np(x)}{N - p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \geq N \end{cases},$$

and let $C^{0,1}(\mathbb{R}^N)$ be the Lipschitz-continuous functions space.

Proposition 2.6. **([11],[9])** If $p(x) \in C_+^{0,1}(\mathbb{R}^N)$, then there exists a positive constant c such that

$$|u|_{p^*(x)} \leq c_{p(x)} |\nabla u|_{p(x)}, \qquad \text{for all } u \in W^{1,p(x)}(\mathbb{R}^N).$$

Proposition 2.7. **([9])** 1) If $s \in L_+^{\infty}(\mathbb{R}^N)$ and $p(x) \leq s(x) \ll p^*(x)$, $\forall x \in \mathbb{R}^N$, then the embedding

$$W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{s(x)}(\mathbb{R}^N)$$

is continuous but not compact.

2) If p is continuous on $\overline{\Omega}$ and s is a measurable function on Ω, with $p(x) \leq s(x) < p^*(x)$, $\forall x \in \Omega$, then the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{s(x)}(\Omega)$$

is compact.

3. Main result and proof

Definition 3.1. *$u \in W$ is a weak solution of* (1.1) *if for all $v \in W$,*

$$\int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla v dx - \lambda \int_{\mathbb{R}^N} V(x) |u|^{q(x)-2} uv dx - \int_{\mathbb{R}^N} f(x,u) v dx = 0,$$

The present paper is studied under the following hypotheses. Put $F(x,u) = \int_0^u f(x,t)\,dt$.

(H1) We suppose that the functions p, q are continuous and satisfy $p(x) < N$, along with $1 < p^- < p^+ < q^- < q^+ \leq p^*(x)$. In particular, p verifies the weak Lipschitz condition, namely, $|p(x) - p(y)| \leq \frac{c}{|\log|x-y||}$ holds for $|x-y| \leq \frac{1}{2}$ and $x, y \in \mathbb{R}^N$.

(H2) We assume $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a the Caratheodory function and satisfies $f \in C^1(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and

$$|f(x,u)| \leq a(x) |u|^{\frac{p(x)}{\alpha(x)}}, \ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Here $a \in L^{\alpha(x)}(\mathbb{R}^N)$, is nonnegative mesurable function, along with $\frac{1}{\alpha(x)} + \frac{1}{p(x)} = 1$.

(H3) Suppose that $0 \leq \theta F(x,u) \leq uf(x,u)$, such that $p^+ < \theta < q^-$, $x \in \mathbb{R}^N$.

(H4) The potential $V \in L^\infty(\mathbb{R}^N) \cap L^{r(x)}(\mathbb{R}^N)$ is nonnegative, and $\frac{1}{r(x)} + \frac{1}{q(x)} = 1$.

Remark 3.2. As in [3] the hypothesis (H3) implies that, for all $t > 1$, $F(x,tu) \geq t^\theta F(x,u)$. Moreover, in vew of (H1), $W = W^{1,p(x)}$.

The main result for this paper is given by the following theorem.

Theorem 3.3. If the hypotheses (H1)–(H4) fulfilled, then the problem (1.1) has a non-trivial weak solution for all $\lambda > 0$.

We need some lemmas to prove main result. The energy functional corresponding to problem (1.1) is defined by

$$J_\lambda(u) = \int_{\mathbb{R}^n} \frac{1}{p(x)} |\nabla u|^{p(x)} dx - \int_{\mathbb{R}^n} \lambda \frac{V(x)}{q(x)} |u|^{q(x)} dx - \int_{\mathbb{R}^n} F(x,u)\, dx$$

and we put

$$\begin{aligned} \varphi(u) &= \int_{\mathbb{R}^n} \frac{1}{p(x)} |\nabla u|^{p(x)} dx, \\ \psi(u) &= \int_{\mathbb{R}^n} \frac{V(x)}{q(x)} |u|^{q(x)} dx, \\ K(u) &= \int_{\mathbb{R}^n} F(x,u)\, dx. \end{aligned}$$

Lemma 3.4. The functional J_λ is well defined and $C^1(W,\mathbb{R})$. Moreover

$$\left\langle J'_\lambda(u), v\right\rangle = \int_{\mathbb{R}^n} \left(|\nabla u|^{p(x)-2} \nabla u \nabla v - \lambda V(x) |u|^{q(x)-2} uv \right) dx - \int_{\mathbb{R}^n} f(x,u)\, v dx.$$

By (H2) togheter with (H4), it is easy to see that J'_λ belongs to the topological dual of W.

Lemma 3.5. There exists positives constants R and ρ such that $J_\lambda(u) \geq \rho$ on $\|u\|_{p(x)} = R$.

Proof. By the Hölder inequality, we get

$$\begin{aligned} \int_{\mathbb{R}^n} |F(x,u)|\, dx &\leq \int_{\mathbb{R}^n} \left| \frac{a(x)}{q(x)} |u|^{q(x)} \right| dx \\ &\leq \frac{2}{q^-} |a|_{\alpha(x)} \left| |u|^{q(x)} \right|_{p(x)} \\ &\leq \frac{2c_1}{q^-} |a|_{\alpha(x)} \|u\|_{p(x)}^{q^i}, \\ i &= + \text{ if } \|u\|_{p(x)} > 1, \text{ and } i = - \text{ if } \|u\|_{p(x)} < 1 \end{aligned}$$

and we are

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{V(x)}{q(x)} |u|^{q(x)} dx &\leq \frac{2}{q^-} |V|_{r(x)} \left| |u|^{q(x)} \right|_{r'(x)} \\ &\leq \frac{2}{q^-} |V|_{r(x)} |u|_{q(x)r'(x)}^{q^i} \\ &\leq \frac{2c_2}{q^-} |V|_{r(x)} \|u\|_{p(x)}^{q^i}, \\ i &= + \text{ if } \|u\|_{p(x)} > 1, \text{ and } i = - \text{ if } \|u\|_{p(x)} < 1 \end{aligned}$$

$$\begin{aligned}
J_\lambda(u) &= \int_{\mathbb{R}^n}\left(\frac{1}{p(x)}|\nabla u|^{p(x)} - \lambda\frac{V(x)}{q(x)}|u|^{q(x)}\right)dx - \int_{\mathbb{R}^n}F(x,u)\,dx \\
&\geq \frac{1}{p^+}\int_{\mathbb{R}^n}|\nabla u|^{p(x)}\,dx - \frac{2\lambda c_2}{q^-}|V|_{r(x)}\|u\|_{p(x)}^{q^i} - \frac{2c_1}{q^-}|a|_{\alpha(x)}\|u\|_{p(x)}^{q^i} \\
&\geq \frac{1}{p^+}\|u\|_{p(x)}^{p^i} - \frac{2\lambda c_2}{q^-}|V|_{r(x)}\|u\|_{p(x)}^{q^i} - \frac{2c_1}{q^-}|a|_{\alpha(x)}\|u\|_{p(x)}^{q^i} \\
&\geq \frac{1}{p^+}\|u\|_{p(x)}^{p^i} - \left(\frac{2\lambda c_2}{q^-}|V|_{r(x)} + \frac{2c_1}{q^-}|a|_{\alpha(x)}\right)\|u\|_{p(x)}^{q^i}
\end{aligned}$$

where c_1, c_2 are positives constants. So, for all $\lambda > 0$, and $u \in W$ with $\|u\|_{p(x)} = R$ sufficiently small, there exists $\rho > 0$ such that

$$J_\lambda(u) \geq \rho > 0$$

■

Lemma 3.6. There exists $e \in W$ with $\|e\|_{p(x)} > R$ such that $J_\lambda(e) < 0$.

Proof. Choose $u_0 \in W$, $\|u_0\|_{p(x)} > 1$. For t large enough we obtain

$$\begin{aligned}
J_\lambda(tu_0) &= \int_{\mathbb{R}^n}\left(\frac{1}{p(x)}|\nabla tu_0|^{p(x)} - \lambda\frac{V(x)}{q(x)}|tu_0|^{q(x)}\right)dx - \int_{\mathbb{R}^n}F(x,tu_0)\,dx \\
&\leq \frac{1}{p^-}\int_{\mathbb{R}^n}|\nabla tu_0|^{p(x)}\,dx - \lambda\frac{1}{q^+}\int_{\mathbb{R}^n}V(x)|tu_0|^{q(x)}\,dx \\
&\leq \frac{t^{p+}}{p^-}\|u_0\|_{p(x)}^{p^+} - \frac{2\lambda c t^{q^-}}{q^+}\int_{\mathbb{R}^n}V(x)|u_0|^{q(x)}\,dx.
\end{aligned}$$

This yields $J_\lambda(tu_0) \to -\infty$, as $t \to +\infty$ since

$$0 \leq \int_{\mathbb{R}^n}V(x)|u_0|^{q(x)}\,dx \leq 2c_2|V|_{r(x)}\|u_0\|_{p(x)}^{q^+}.$$

■

Lemma 3.7. The functional J_λ satisfies the Palais-Smale condition $(\mathrm{PS})_c$, for any $c \in \mathbb{R}$.

Proof. Let (u_n) be a $(\mathrm{PS})_c$ sequence for the functional J_λ in W i.e. $J_\lambda(u_n)$ is bounded and $J'_\lambda(u_n) \to 0$. Then the sequence u_n is bounded in W.

Indeed, since $J_\lambda(u_n)$ is bounded, we have

$$\begin{aligned}
C_1 \geq J_\lambda(u_n) &= \int_{\mathbb{R}^n}\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)} - \lambda\frac{V(x)}{q(x)}|u_n|^{q(x)}\right)dx - \int_{\mathbb{R}^n}F(x,u_n)\,dx \\
&\geq \int_{\mathbb{R}^n}\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)} - \lambda\frac{V(x)}{q(x)}|u_n|^{q(x)}\right)dx - \int_{\mathbb{R}^n}F(x,u_n)\,dx \\
&\geq \int_{\mathbb{R}^n}\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx - \lambda\frac{V(x)}{q(x)}|u_n|^{q(x)}\right)dx - \int_{\mathbb{R}^n}\frac{u_n}{\theta}f(x,u_n)\,dx.
\end{aligned}$$

Furthermore

$$\left\langle J_\lambda'(u_n), u_n\right\rangle = \int_{\mathbb{R}^n} |\nabla u_n|^{p(x)} - \lambda V(x)|u_n|^{q(x)}\,dx - \int_{\mathbb{R}^n} f(x,u_n)\,u_n dx$$

Then

$$\begin{aligned} C_1 &\geq \frac{1}{p^+}\int_{\mathbb{R}^n} |\nabla u_n|^{p(x)}\,dx - \frac{1}{q^-}\int_{\mathbb{R}^n} \lambda V(x)|u_n|^{q(x)}\,dx + \frac{1}{\theta}\left\langle J_\lambda'(u_n), u_n\right\rangle \\ &\quad -\frac{1}{\theta}\int_{\mathbb{R}^n} |\nabla u_n|^{p(x)}\,dx + \frac{1}{\theta}\int_{\mathbb{R}^n} \lambda V(x)|u_n|^{q(x)}\,dx \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\int_{\mathbb{R}^n} |\nabla u_n|^{p(x)}\,dx \\ &\quad + \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\int_{\mathbb{R}^n} \lambda V(x)|u_n|^{q(x)}\,dx + \frac{1}{\theta}\left\langle J_\lambda'(u_n), u_n\right\rangle \end{aligned}$$

Arguing by contradiction, we assume that (u_n) is unbounded in W. In particular we can choose $\|u_n\| \geq 1$ for n sufficiently large. Hence, there exists $C_3 > 0$ such that

$$-C_3\|u_n\|_{p(x)} \leq \left\langle J_\lambda'(u_n), u_n\right\rangle \leq C_3\|u_n\|_{p(x)}$$

since $J_\lambda'(u_n) \to 0$. To this end,

$$\begin{aligned} C_1 &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|_{p(x)}^{p^+} + \left(\frac{1}{\theta} - \frac{1}{q^-}\right)\int_{\mathbb{R}^n} \lambda V(x)|u_n|^{q(x)}\,dx - \frac{1}{\theta}C_3\|u_n\|_{p(x)} \\ &\geq \left(\frac{1}{p^+} - \frac{1}{\theta}\right)\|u_n\|_{p(x)}^{p^+} - \frac{1}{\theta}C_3\|u_n\|_{p(x)}. \end{aligned}$$

This implies a contradiction.

Hence the sequence (u_n) is bounded in W. Thus, there exists a subsequence, again denoted (u_n), weakly convergent to u in W. We prove that (u_n) is strongly convergent to u in W.

To this end, we consider the following equality

$$\left\langle J_\lambda'(u_n) - J_\lambda'(u), u_n - u\right\rangle = \tag{1.3}$$

$$\left\langle \varphi'(u_n) - \varphi'(u), u_n - u\right\rangle - \left\langle \psi'(u_n) - \psi'(u), u_n - u\right\rangle - \left\langle K'(u_n) - K'(u), u_n - u\right\rangle.$$

Obviously, the term in the left hand side tends to zero for n large enough. First, we show that $\left\langle K'(u_n) - K'(u), u_n - u\right\rangle \to 0 \quad \text{as} \quad n \to \infty$.

Let B_R be the ball in $\mathbb{R}^N$ of radius R centered at the origin and $B_R' = \mathbb{R}^N - B_R$. We use well-know compacteness argument in unbounded domains. Roughly speaking, we write

$$\begin{aligned} \left|\left\langle K'(u_n) - K'(u), u_n - u\right\rangle\right| &= \left|\int_{\mathbb{R}^n} (f(x,u_n) - f(x,u))(u_n - u)\,dx\right| \\ &\leq \int_{B_R} |f(x,u_n) - f(x,u)|\,|u_n - u|\,dx \\ &\quad + \int_{B_R'} |f(x,u_n) - f(x,u)|\,|u_n - u|\,dx \end{aligned}$$

Taking into account Proposition 2.7 togheter with the compact embedding $W^{1,p(x)}(B_R) \hookrightarrow L^{p(x)}(B_R)$, the first term in the right hand side of the above inequality vanishes as $n \to \infty$. Contrariwise, the second term vanishes as $R \to \infty$. In fact, we have

$$\int_{B_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq 2 |f(x,u_n) - f(x,u)|_{\alpha(x)} |u_n - u|_{p(x),B_R}.$$

In virtue of (H2) the Nemyckii operator is bounded. Hence, we obtain

$$\int_{B_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq \frac{\varepsilon}{2}.$$

On the other hand, we have

$$\int_{B'_R} |f(x,u_n) - f(x,u)| |u_n - u| \, dx \leq$$

$$\int_{B'_R} a(x) |u_n|^{p(x)} + a(x) |u_n|^{p(x)-1} |u| + a(x) |u|^{p(x)} + a(x) |u|^{p(x)-1} |u_n| \, dx \leq \frac{\varepsilon}{2},$$

for R sufficiently $l\arg e$. Indeed,

$$\int_{B'_R} a(x) |u_n|^{p(x)} \, dx \leq 2 |a|_{\alpha(x)} \left| |u_n|^{p(x)} \right|_{p(x)} \leq \frac{\varepsilon}{8},$$

for R sufficiently $l\arg e$. Using the Young inequality, we get

$$\begin{aligned} \int_{B'_R} a(x) |u_n|^{p(x)-1} |u| \, dx &\leq \int_{B'_R} a(x) \left(|u_n|^{p(x)} + |u|^{p(x)} \right) dx \\ &\leq 2 |a|_{\alpha(x)} \left(\left| |u_n|^{p(x)} \right|_{p(x)} + \left| |u|^{p(x)} \right|_{p(x)} \right) \leq \frac{\varepsilon}{8}, \end{aligned}$$

for R sufficiently $l\arg e$.

In the same way, according to R, we show that both the two last terms are less than $\frac{\varepsilon}{8}$.

Similarly, using the same arguments, the following holds

$$\begin{aligned} &\langle \psi'(u_n) - \psi'(u), u_n - u \rangle \\ &\leq \lambda \int_{B_R} \left| V(x) \left(|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) \right| |u_n - u| \, dx \\ &\quad + \lambda \int_{B'_R} V(x) \left(|u_n|^{q(x)} + |u|^{q(x)-2} u_n u + |u|^{q(x)} + |u_n|^{q(x)-2} u_n u \right) dx \\ &\leq c_1 \left| V(x) \left(|u_n|^{q(x)-2} u_n - |u|^{q(x)-2} u \right) \right|_{r(x)} |u_n - u|_{q(x)} \\ &\quad + c_2 |V(x)|_{r(x)} \left(\left| |u_n|^{q(x)} \right|_{q(x)} + \left| |u|^{q(x)} \right|_{q(x)} \right) \leq \varepsilon. \end{aligned}$$

for n, R large enough.

It appears from (1.3) that $\left\langle \varphi'(u_n) - \varphi'(u), u_n - u \right\rangle \to 0$ as $n \to \infty$. Now, with the aid of an elementary inequality in $\mathbb{R}^N$, we get if $p(x) \geq 2$

$$2^{2-p^+} \int_{\mathbb{R}^N} \left| |\nabla u_n| - |\nabla u| \right|^{p(x)} dx \quad \leq$$

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^{p(x)-2} \nabla u_n - |\nabla u|^{p(x)-2} \nabla u \right) (\nabla u_n - \nabla u) \, dx \quad \to \quad 0 \quad \text{as} \quad n \to \infty.$$

Otherwise, use the following inequality in $\mathbb{R}^N$

$$(p-1) |\zeta - \eta|^2 (|\zeta| + |\eta|)^{p-2} \leq \left(|\zeta|^{p-2} \zeta - |\eta|^{p-2} \eta \right) (\zeta - \eta) \quad \text{if} \quad 1 < p < 2$$

and consider the following sets

$$U_p = \left\{ x \in \mathbb{R}^N, p(x) \geq 2 \right\}; \quad V_p = \left\{ x \in \mathbb{R}^N, 1 < p(x) < 2 \right\}$$

■

Proof [Proof of theorem 3.3]. Set

$$\Gamma = \{ \gamma \in C([0,1], W) : \gamma(0) = 0, \gamma(1) = e \}$$

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\lambda(\gamma(t)).$$

According to lemma 3.5 and lemma 3.6, the energy functional J_λ satisfies the geometrical conditions of the Mountain pass theorem. Hence c is a critical value of J_λ associated with a critical point $u \in W$, which is precisely one solution of (1.1). The proof is complete.

■

References

[1] E. Acerbi and G. Mingione, *Gradient estimates for the $p(x)$-Laplacean system*, J. Reine Angew. Math. 584(2005), 117–148.

[2] J.P.G. Azorero, I.P. Alonso, *Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues*, Comm. Partial Differential Equations 12(1987), 1389–1430.

[3] M. M. Boureanu, Existence of solutions for an elliptic equation involving the $p(x)$-Laplace operator, Electronic Journal of Differential Equations, Vol. 2006 (2006), No. 97, pp. 1–10.

[4] B. Cekic, R. Mashiyev and G.T. Alisoy, *On The Sobolev-type Inequality for Lebesgue Spaces with a Variable Exponent*, International Mathematical Forum, 1, 2006, no. 27, 1313–1323.

[5] J. Chabrowski, Y. Fu, Existence of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl. 198 (2004) 149–175.

[6] Y. Chen, S. Levine and R. Rao, *Variable exponent, linear growth functionals in image processing*, SIAM J. Appl. Math. 66 (2006), 1383–1406.

[7] L. Diening, *Theorical and numerical results for electrorheological fluids*, Ph.D. thesis, University of Frieburg, Germany, 2002.

[8] A. Djellit, Z. Youbi and S. Tas, Existence of solution for elliptic systems in $\mathbb{R}^N$ involving the $p\,(x)$-Laplacian, Electronic Journal of Differential Equations, Vol. 2012 (2012), No. 131, pp. 1–10.

[9] D. E. Edmunds and J. Rázkosnflk, "*Sobolev embedding with variable exponent*" Studia Mathematics 143 (2000), 267–293.

[10] X. Fan, J.S. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{k,p(x)}\,(\Omega)$, J. Math. Anal. Appl. 262 (2001) 749–760.

[11] X. L. Fan et D. Zhao, "On the spaces $L^{p(x)}$ and $W^{1,p(x)}$", Journal of Mathematical Analysis and Applications, 263 (2001), 424–446.

[12] X. Fan, A constrained minimization problem involving the $p(x)$-Laplacian in $\mathbb{R}^N$, Nonlinear anal 69(2008), 3661–3670.

[13] X. Fan, Eigenvalues of the $p(x)$-Laplacian Neumann problems, Nonlinear Anal. 67 (2007), 2982–2992.

[14] X. Fan, Q.H. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl. 302 (2005), 306–317.

[15] X. Fan, X.Y. Han, Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$, Nonlinear Anal. 59 (2004), 173–188.

[16] Y. Fu and X. Zhang, A multiplicity result for $p(x)$-Laplacian problem in $\mathbb{R}^N$, Nonlinear Anal. 70 (2009), 2261–2269.

[17] T. C. Halsey, Electrorheological fluids, Science, vol. 258, n ° 5083, pp. 761–766, 1992.

[18] K. Kefi, $p(x)$-Laplacian with indefinite weight, Proceedings of the AMS 139, 12 (2011), 4351–4360.

[19] O. Kovăcĩk, J. Răkosnĩk, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. Czechoslov. Math. J., 41 (1991), 592–618.

[20] A. Lē, Eigenvalue pproblems for the p-Laplacian, Nonlinear Analysis, TMA, (2005).

[21] M. Mihailescu and V. Raduescu, On a Nonhomogenous Quasilinear Eigenvalue Problem in Sobolev Spaces with Variable Exponent, Proceedings of the AMS, 135 (2007), 416–423.

[22] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. 29 (1987), 33–66.