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Nested Chain Movement of length 1 of Beta Number in James Abacus Diagram

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Abstract

James abacus diagram is a graphical representation for any partition μ of a positive integer t. One way of producing the diagram is by using beta number with the special number of even columns e, where $e \geqslant 2$. This paper constructs a new method for partition μ with a single motion of nested chain movement of length 1 in James abacus diagram. First, the establishment of an arithmetic sequence among the nested chains of the diagram position is considered. Then, for the movement, we select several beta numbers as the initial points in every chain. The location of the rest of the beta numbers in the James abacus diagram would be changed anticlockwise by length 1 accordingly when the positions of initial beta numbers are changed. Using these new nested chains, a new diagram A^{tc1} that displays a new partition, is constructed. Furthermore, guides, which are finite number of partitions that are obtained from the original partition after adding zeros, are developed. The number of common beta numbers among these developed guides are then determined. We have established rules to obtain new diagram using a single motion of nested chain movement of length 1. The new diagram can be used in areas of number theory and design. Finally, the proposed method is employed as a special type of James abacus diagram where the number of columns is an even integer smaller than the number of rows in the diagram.

AMS subject classification: 05A17.
Keywords: James abacus diagram, Beta number, Arithmetic sequence and Partition.

1. Introduction

James abacus diagram is a graphical representation of a special type of non-increasing sequence $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ called partition of t if $|\mu| = \sum_{i=1}^{b} \mu_i = t$ with $\mu_{(b+1)} = 0$ where b is the number of partition parts and t is any positive integer [6]. James abacus diagram came from the idea of the abacus diagram. For any number of column (runner) e, James abacus diagram is seen as an important component in modern algebra which plays a key role in Iwahoriâˆ'Hecke and q-Schur algebras [11], [1], [2], [9]. The James abacus diagram configuration for beta numbers, $\beta_1, \beta_2, \beta_3, \ldots, \beta_b$ can be created by rearranging them on the runners, where $\beta_i = \mu_i + b - i$ for $1 \leqslant i \leqslant b$ [8]. There are infinitely many James abacus diagrams since one or more zeros can be added to the partition. The diagram of b_s is obtained after adding $s - 1$ zeros to μ where James abacus diagram of b_s have $b + s - 1$ beta numbers [7]. Furthermore, several types of movement of beta numbers have been constructed by previous researchers such as moving all of them as high as possible in the runner [3], adding empty runner [4], adding full runner [2], removing runner [1], reflecting their positions in leading diagonal [12], scanning movement [9], using justified positions movement [10] and moving single-step bead [7]. Guides or main diagrams are finite number of partitions that are obtained after adding zeros. A study has proven that there are common beta numbers among guides [7]. Some studies have applied the properties of James abacus diagram in countless fields and constructed several diagrams by transforming the diagram position that preserves the original structure of the diagram [14], [15], [16], [18]. Among the transformation constructed are reflection in line x and y, rotation by 90^o, 180^o and 270^o as well as the composition of both geometric transformations of reflection and rotation by 90^o, 180^o and 270^o. A single chain movement was applied in case $e = 2$ of any length [19].

In this paper, a new diagram will be constructed by applying a single motion of nested chain movement of length 1 and is called diagram A^{tc1}. The idea of nested chain movement is adopted from graph theory [8]. A nested chain comprises one outer chain and several inner chains. This paper seeks to address several questions. Can James abacus diagram positions be divided into nested chains? How many positions are there in every chain? What are the relationships among the diagram positions? Is the movement of the beta number positions in the new main diagrams regular or not? If it is regular, can the new main diagrams of b_2 that depend on the new main diagram for b_1 be designed? In general, is the intersection of the new main diagrams equal to the intersection of the original main diagrams?

2. Preliminaries

This section briefly discusses some basic definitions and theorems. The partition of each James abacus diagram can be connected by e runners which are labelled from left to right as 0 to $e-1$, where $e = 2x$, $x \in Z^{+}$. Beta numbers will be referred to as bead positions. The bead positions on the James abacus diagram which are labelled from left to right and continues from top to bottom starting with 0 are located across the runners. The bead positions $me, me+1, \ldots, (m+1)e-1$ are located in row m of the diagram, as can be seen in Figure 1a for case $e = 4$.

Definition 2.1. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_b)$ represents the partition of a positive integer t such that $|\mu| = \sum_{i=1}^{b} \mu_i = t$ with $\mu_{(b+1)} = 0$ where b is the number of partition parts. The β-sequence of partition μ, $\beta_1^{\mu}, \beta_2^{\mu}, \ldots, \beta_b^{\mu}$, is defined by

$$\beta(\mu, b) = \{\mu_1 + b - 1, \mu_2 + b - 2, \ldots, \mu_b + b - b\}$$

where the construction is discussed in [8].

The abacus configuration for μ is called James abacus diagram of b_s where each b_s has $b + s - 1$ beads and s is a positive integer greater than or equal to one.

Example 2.2. Let $\mu = (8, 8, 6, 3, 2, 1, 1, 1, 1, 0, 0)$ be a partition of 31 where $b = 9$. If $s = 3$, then $\mu_{10} = 0$, $\mu_{11} = 0$ and the set of β-numbers is $\{18, 17, 14, 10, 8, 6, 5, 4, 3, 1, 0\}$.

Every beta number will be represented by (o) for a bead position in the diagram while the empty bead position will be represented by (-). Figure 1 shows a general diagram and final diagram of $e = 4$ displaying the arrangement of beads representing $\mu = (8, 8, 6, 3, 2, 1, 1, 1, 1)$.

Diagrams of b_s where $s = 1, 2, \ldots, e$ are defined as main diagrams. The intersection of the main diagrams is also examined.

Theorem 2.3. [13] Let

$$(\mu_1, \mu_2, \ldots, \mu_b) = (\mu_1^{\tau_1}, \mu_2^{\tau_2}, \ldots, \mu_k^{\tau_k}).$$

The numerical value of the resulting intersection of main James abacus diagrams is

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15
16	17	18	19

$\longrightarrow$

o	o	–	o
o	o	o	–
o	–	o	–
–	–	o	–
–	o	o	–

a b

Figure 1: The bead positions on the abacus with 4 runners (a) General diagram (b) Final diagram.

denoted by $\#\bigcap_{s=1}^{e} m \cdot d\cdot$ and it is equal to ϕ in the case of no existence of common bead positions.

1. $\#\bigcap_{s=1}^{e} m \cdot d\cdot = \phi$ if $\tau_k = 1, k \in \mathbb{N}$.

2. Let Ω be the number of parts of λ which satisfies the condition $\tau_k \geq e$ for some k, then $\#\bigcap_{s=1}^{e} m \cdot d\cdot = \sum_{k=1}^{\Omega} \tau_k - \Omega(e-1)$.

3. Method

The construction of the new diagram is developed by first converting the original James abacus diagram to matrix form. James abacus diagram positions are divided into several nested chains. Then, a new method is constructed for partitioning μ with a single motion of nested chain movement of length 1. Next, main diagrams are developed and the intersection points of the main diagrams are obtained.

3.1. Convert James abacus diagram to matrix form

In order to formulate this work easily, the James abacus diagram is converted to matrix form. If $(\beta_1, \beta_2, \ldots, \beta_b)$ is a set of beta numbers of a partition μ of a positive integer t and r is the number of rows in James abacus diagram of b_1 then

$$r = \left\lceil \frac{\beta_1}{e} \right\rceil .$$

The number of rows in a James abacus diagram of b_s is $r + s - 1$ where $s \in \mathbb{N}$.

Lemma 3.1. Suppose that $\{\beta_1, \beta_2, \cdots, \beta_b\}$ a set of beta numbers of a partition $\mu = (\mu_1, \mu_2, \cdots, \mu_b)$ of a positive integer t where $b \in \mathbb{Z}$, then every James abacus diagram position can be converted to a matrix $A_{r \times e}$ by

$$me + n \Rightarrow a_{(m+1)(n+1)}$$

Proof. In James abacus diagram, the bead positions in column n and row m are numbered as $(me + n)$ for $e \geqslant 2$. The row numbers are from 0 to $r - 1$ and column numbers are from 0 to $e - 1$ while every matrix $(m \times n)$ consists of m rows from 1 to r and n columns from 1 to e where m and n are positive integers; so any position $me + n$ in the James abacus diagram is an element $a_{(m+1)(n+1)}$ in the matrix $(r \times e)$. Then

$$\beta_b = me + n \Rightarrow a_{(m+1)(n+1)}$$

where r refers to the number of rows in the diagram. ■

3.2. Nested chains

Nested chains are formed from James abacus diagram position. These chains are numbered from 1 to i where i is a positive integer and chain 1 is the outer chain. There is no intersection between any two of the chains as shown in Figure 2 where there are three nested chains for $\mu = (35, 16, 8, 8, 6, 3, 2, 1, 1, 1, 1)$ and $e = 6$.

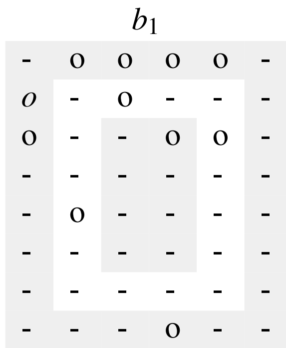

Figure 2: Nested chain diagram.

From Figures 1b and 2, we have the following cases.

i. If $e = 2$, the diagram has only one chain which consists of all diagram positions.

ii. If $e = 4$ and $r \geqslant 4$, this implies that the diagram has two chains.

- Chain 1 = $\{a_{m1}, a_{m4}, a_{1n}, a_{rn} : 1 \leqslant m \leqslant 10, 1 < n < 4\}$.
- Chain 2 = $\{a_{m2}, a_{m3} : 2 \leqslant m \leqslant 9\}$

1. If $e = 6$ and $r \geq 6$, this implies that the diagram has three chains.

- Chain 1 = $\{a_{m1}, a_{m6}, a_{1n}, a_{rn} : 1 \leqslant m \leqslant 7, 1 < n < 6\}$.

- Chain 2 = $\{a_{m2}, a_{m5}, a_{2n}, a_{((r-1)n)} : 2 \leqslant m \leqslant 6, 2 < n < 5\}$.
- Chain 3 = $\{a_{m3}, a_{m4} : 3 \leqslant m \leqslant 5\}$

Finally, we can make a generalization for any even number e.

Definition 3.2. Let e be any even column number and r the number of rows in James abacus diagram where $e < r$. For any partition μ of a positive integer t,

$$\text{chain } i = \{a_{mi}, a_{m(e-i+1)}, a_{in}, a_{(r-i+1)n} : i \leqslant m \leqslant (r-i+1), i < n < e-i+1\}$$

where

1. Every chain is derived from two columns, i and $e-i+1$.
2. The last chain is derived from two consecutive columns.

The generalization of our main result is stated in the following theorem.

Theorem 3.3. Let e be any even column number and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$. Then, for any partition μ of a positive integer t, then the number of chains in the diagram is $\frac{e}{2}$.

Proof. By Definition 3.2, every chain is derived from two columns, which are i and $e-i+1$. Since the last chain is derived from two consecutive columns, the difference between these two column numbers is

$$e - i + 1 - i = 1.$$

Thus,

$$i = \frac{e}{2}.$$

■

Hence, for Example 2.2, if $e = 4$ we have two chains as shown in Figure 1.b. The next theorem shows that the number of positions in every chain is $2r + 2e - 4(2i - 1)$.

Theorem 3.4. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$. Then, for any partition μ of a positive integer t, the number of positions in each chain is

$$2r + 2e - 4(2i - 1).$$

Proof. Since the chains form a rectangle then the length of chain i is

$$(r - i + 1)i = r - 2i + 1.$$

The width of chain i is

$$(e - i + 1) - i = e - 2i + 1.$$

Then the perimeter of chain i is given by

$$2[(r - 2i + 1) + (e - 2i + 1)] = 2r + 2e - 4(2i - 1).$$

■

Consider Example 2.2 and the partition $\mu = (8, 8, 6, 3, 2, 1, 1, 1, 1)$. If $e = 4$, the diagram has two chains and chain 1 has 14 positions while chain 2 has 6 positions.

An arithmetic sequence among the nested chains can be obtained as given in the following Theorem 3.5.

Theorem 3.5. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$. Then, for any partition μ of a positive integer t

$$\ll V_i \gg = \ll V_1, V_2, V_3, \ldots, V_{\frac{e}{2}} \gg$$

is the arithmetic sequence for the number of positions in the nested chains with -8 as the common difference of successive terms where V_i is the number of positions in chain i.

Proof. Let V_{i+1}, V_i represent the number of positions in chain $i + 1$ and chain i respectively where $i = 1, 2, \ldots, \dfrac{e-2}{2}$ by Theorem 3.3. Thus

$$[2r + 2e - 4(2(i - 1) - 1)][2r + 2e - 4(2i - 1)] = -8.$$

■

In the next section, the application of a single motion of the nested chain movement on James abacus diagram of length $[1, 1, \ldots, 1]$ will be examined.

3.3. Nested chain movement of length $[1, 1, \ldots, 1]$

This section describes the new method for partitioning μ with a single motion of nested chain movement of length 1 in James abacus diagram where e is an even column number and $e^2 \leqslant \beta_1$. First, a beta number on every chain in the James diagram is selected randomly. The bead that represents the selected beta number is denoted as the initial point of every chain. As all the initial bead positions are simultaneously moved anticlockwise by length 1 in a single motion to the next positions in the chains that they belong to, the location of the remaining beads in the diagram will be changed accordingly. A new diagram A^{tc1} that displays a new partition is constructed. Applying such chain nested movement of length 1 yields the following results where the notation $\rightarrow$ as in $j \rightarrow k$ means that j represents the original location and k represents the new location.

Rule 3.6. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$. Then

$$a_{mn}^{b_1} \rightarrow \begin{cases} a_{(m-1)n}^{A^{tc1}b_1} & if \quad i+1 \leqslant m \leqslant (r-i+1), n = e-i+1 \\ a_{(m+1)n}^{A^{tc1}b_1} & if \quad i \leqslant m \leqslant (r-i), n = i \\ a_{m(n-1)}^{A^{tc1}b_1} & if \quad m = i, i+1 < n \leqslant e-i+1 \\ a_{m(n+1)}^{A^{tc1}b_1} & if \quad m = (r-i+1), i \leqslant n < e-i \end{cases}$$

Fig. 3 illustrates the above rule for $\mu = (35, 16, 8, 8, 6, 3, 2, 1, 1, 1, 1)$ and $e = 6$.

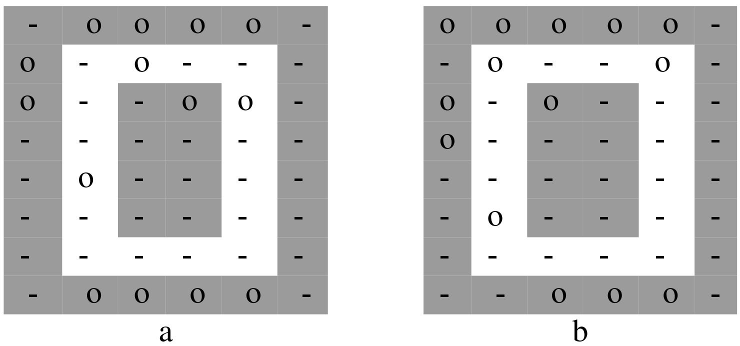

Figure 3: (a) James abacus diagram (b) Diagram A^{tc1}.

3.4. Development of new main diagrams

We have obtained diagram A^{tc1} of b_1 from James abacus diagram using a single motion of nested chain movement of length 1. Next, we can find the diagram of b_2 from the diagram of b_1 after applying the similar chain movement of length 1 again. The following two rules apply for the case of $e = 2$ and $e = 4x$.

Rule 3.7. Let $e = 2$ be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$ then

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{(m-2)2}^{A^{tc1}b_2} & if \quad \leq m \leq r, n = 1 \\ a_{(m+3)1}^{A^{tc1}b_2} & if \quad m = 1, 2, \cdots, r-2, n = 2 \\ a_{r2}^{A^{tc1}b_2} & if \quad m = r-1, n = 2 \\ a_{(r-1)2}^{A^{tc1}b_2} & if \quad m = r, n = 2 \\ a_{11}^{A^{tc1}b_2} & if \quad m = 2, n = 1 \\ a_{31}^{A^{tc1}b_2} & if \quad m = 1, n = 1 \end{cases} .$$

where $a_{mn}^{b_1}$ is any position in the diagram A^{tc1} of b_1 in column n, rows m and $m = 1, 2, \ldots, r-1$.

Rule 3.8. Let $e = 4x$ where x is any positive integer, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$. Applying a single motion of nested chain movement of length 1 will give the following results:

1. If $m = i, i \leqslant n \leqslant e - i + 1$ and $i \neq \dfrac{e}{2}$ then

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{m(n+1)}^{A^{tc1}b_2} & if \quad m = i, n+1 < e - i \\ a_{(m-1)(n+2)}^{A^{tc1}b_2} & if \quad m = i, n = e - i, i \neq 1 \\ a_{m(n+1)}^{A^{tc1}b_2} & if \quad m = i, n = e - i + 1, i \neq 1 \\ a_{31}^{A^{tc1}b_2} & if \quad m = 1, n = e - i \\ a_{41}^{A^{tc1}b_2} & if \quad m = 1, n = e \end{cases}$$

2. If $n = e - i + 1, i < m < r - i + 1$ and $i \neq \dfrac{e}{2}$ then

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{m(n+1)}^{A^{tc1}b_2} & if \quad i < m \leqslant r - i + 1, n = e - i + 1, i \neq 1 \\ a_{(m+3)1}^{A^{tc1}b_2} & if \quad 1 < m \leqslant r - 2, n = e \\ a_{(r+1)2}^{A^{tc1}b_2} & if \quad m = r - 1, n = e \end{cases}$$

3. If $m = r - i + 1, i \leqslant n \leqslant e - i + 1$ and $i \neq \dfrac{e}{2}$ then

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{m(n+1)}^{A^{tc1}b_2} & if \quad m = r - i + 1, i \leqslant n < e - i + 2 \\ a_{(m-1)n}^{A^{tc1}b_2} & if \quad m = r - i + 1, e - i \leqslant n \leqslant e - i + 1 \end{cases}$$

4. If $n = i, i \leqslant m \leqslant e - i + 1$ and $i \neq \dfrac{e}{2}$ then

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{m(n+1)}^{A^{tc1}b_2} & if \quad n = i, r - i + 1 \leqslant m - 1 \leqslant i \\ a_{m(n+1)}^{A^{tc1}b_2} & if \quad n = i, m = i + 1 \\ a_{(m-1)n}^{A^{tc1}b_2} & if \quad m = r - i + 1, n = i \end{cases}$$

5. $i \leqslant n \leqslant i + 1, i \leqslant m \leqslant r - i + 1$ and $i = \dfrac{e}{2}$

$$a_{mn}^{A^{tc1}b_1} \rightarrow \begin{cases} a_{(m-1)(n+2)}^{A^{tc1}b_1} & if \quad n = \frac{e}{2}, m = \frac{e}{2} \\ a_{(m-1)n}^{A^{tc1}b_1} & if \quad n = \frac{e}{2}, m = \frac{e+2}{2} \\ a_{(m-2)(n+1)}^{A^{tc1}b_1} & if \quad n = \frac{e}{2}, \frac{e+4}{2} \leq m \leq \frac{2r-e+2}{2} \\ a_{m(n+1)}^{A^{tc1}b_1} & if \quad n = \frac{e+2}{2}, \frac{e}{2} \leq m < \frac{2r-e+2}{2} \\ a_{(m-1)n}^{A^{tc1}b_1} & if \quad n = \frac{e+2}{2}, m = \frac{2r-e+2}{2} \end{cases}$$

6. $\left\{ a_{mn}^{A_{b_2}^{tc1}}, a_{re}^{A_{b_2}^{tc1}} | m = r + 1, 2 < n \leqslant e \right\}$ are bead positions.

Figure 4(a) depicts the original James abacus diagram for b_1, b_2, b_3 and b_4 while Figure 4(b) shows all results according to ules 3.6 and 3.8 for $\mu = (8, 8, 6, 3, 2, 1, 1, 1, 1)$ and $e = 4$.

In general, Rules 3.7 and 3.8 can be further applied to find the guides or the main diagrams A^{tc1} of $b_3, b_4, \ldots$.

b_1				b_2				b_3				b_4			
-	o	o	o	o	-	o	o	o	o	-	o	o	o	o	-
o	-	o	-	o	o	-	o	o	o	o	-	o	o	o	o
o	-	-	-	-	o	-	-	o	-	o	-	-	o	-	o
o	-	-	o	-	o	-	-	-	-	o	-	-	-	-	o
o	-	-	-	o	o	-	-	-	o	o	-	-	-	o	o
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

a

A^{tc1} of b_1				A^{tc1} of b_2				A^{tc1} of b_3				A^{tc1} of b_4			
o	o	o	-	-	o	o	o	o	-	o	-	o	o	-	o
-	o	-	-	o	-	-	-	o	o	o	-	o	o	-	o
o	-	-	o	o	o	-	-	o	o	o	-	o	o	-	o
o	-	-	-	-	o	-	-	o	-	o	-	-	o	o	o
o	o	-	-	-	o	o	-	-	-	-	-	-	-	-	-
-	-	-	-	o	-	-	-	-		-	-	-	-	-	-
-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-

b

Figure 4: (a) Original James abacus abacus of (b) Diagram A^{tc1}.

3.5. Intersection of the new main diagrams

Consider the original partition with the addition of $e-1$ zeros. The common bead locations can be obtained if any part of the partitions repeats e times or more. The numerical value of the resulting intersection of main James abacus diagrams is denoted by $\#\bigcap_{s=1}^{e} m \cdot d \cdot$ and it is equal to ϕ in the case of no existence of common bead positions. The reason for observing the intersection of the main James abacus diagrams is not only to know the common bead locations but also to identify whether the intersection points exist or not. In this section, we introduce an important theorem to find the common bead positions in the main diagrams. We will first introduce the following definition, rules and necessary remarks.

Definition 3.9. Set $\rho = \{\beta_k | 1 \leqslant k \leqslant b\}$ is a set of common beta numbers in the main diagrams if $\{\beta_{k-1}, \beta_{k-2}, \cdots \beta_{k+e-1}\} \subseteq$ set of beta numbers.

Remark 3.10. For any $e < r$, let τ be the number of redundant parts of the partition μ of t, where $\mu = (\mu_1, \mu_2, \cdots, \mu_\eta) = (\mu_1^{\tau_1}, \mu_2^{\tau_2}, \cdots, \mu_f^{\tau_f})$. Let z be the smallest number which achieves

$$er - \left[\sum_{i=1}^{z} \tau_i + (\mu_i - \mu_{i+1})\right] + p \geqslant R_1$$

where $1 \leqslant z \leqslant f$. Then

1. If $\left[\left[\sum_{i=1}^{z} \tau_i + (\mu_i - \mu_{i+1})\right] + p - (\mu_z - \mu_{z+1})\right] < R_1$ then $\mu' = (\mu_1^{\tau_1'}, \mu_2^{\tau_2'}, \ldots, \mu_z^{\tau_z'})$ where $\tau_i' = \tau_i$ for $i = 1, 2, \ldots, z$.

2. If $\left[\left[\sum_{i=1}^{z} \tau_i + (\mu_i - \mu_{i+1})\right] + p - (\mu_z - \mu_{z+1})\right] \geq R_1$ then $\mu' = (\mu_1^{\tau_1'}, \mu_2^{\tau_2'}, \ldots, \mu_z^{\tau_z'})$ where $\tau_z' = \tau_z - R_2$ and $\tau_w' = \tau_w$ for $w = 1, 2, \ldots, z-1$.

3. $\mu'' = (\mu_z^{\tau_z''}, \mu_{z+1}^{\tau_{z+1}''}, \ldots, \mu_f^{\tau_f''})$ where $\tau_z'' = \tau_z - \tau_z'$, $\tau_{z+\delta}'' = \tau_{z+\delta}$ for $\delta = 1, 2, \ldots, f - z$.

such that

$$R_2 = \sum_{i=1}^{z} \tau_i + \left[\sum_{i=1}^{z} \tau_i(\mu_i - \mu_{i+1}) + p - (\mu_z - \mu_{z+1})\right] + 1 - R_1,$$

$R_1 = re - R$ and $R = \dfrac{2er - e^2 - 2}{2}$ where

$$p = \begin{cases} 0 & if \quad [\mu_1 + \sum_{i=1}^{f} \tau_i] = er \\ re - [\mu_1 + \sum_{i=1}^{f} \tau_i] & if \quad [\mu_1 + \sum_{i=1}^{f} \tau_i] < er \end{cases}.$$

Rule 3.11. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$, then for any position $a_{mn}^{b_s}$ in μ''

$$a_{mn}^{b_s} \to \begin{cases} a_{(m-1)n}^{A^{tc1} b_s} & if \quad i < m \leqslant r - i + 1, n = e - i + 1 \\ a_{(m+1)n}^{A^{tc1} b_s} & if \quad i \leqslant m \leqslant r - i, n = i \\ a_{m(n-1)}^{A^{tc1} b_s} & if \quad i < n \leqslant e - i + 1, m = i \end{cases}$$

where $1 \leq m < \dfrac{2r - e + 2}{2}$ and $1 \leq n \leq e$.

Rule 3.12. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$ then for any position $a_{mn}^{b_s}$ in μ'

1. If $i \leqslant n \leqslant \frac{e}{2}$ and $m = r - i + 1$,

$$a_{mn}^{b_s} \to \begin{cases} a_{m(n+1)}^{A^{tc1}b_s} & if \quad 1 \leqslant s \leqslant n - i + 1 \\ a_{(m+1)n}^{A^{tc1}b_s} & if \quad n - i + 1 < s \leqslant e \end{cases}$$

2. If $\frac{e}{2} < n \leqslant e - i$ and $m = r - i + 1$,

$$a_{mn}^{b_s} \to \begin{cases} a_{m(n+1)}^{A^{tc1}b_s} & if \quad 1 \leqslant s \leqslant e - n - i + 1 \\ a_{(m-1)n}^{A^{tc1}b_s} & if \quad e - n - i + 1 < s \leqslant e \end{cases}$$

This rule has been clarified in Figure 4 where $\mu = (8, 8, 6, 3, 2, 1, 1, 1, 1)$ and $e = 4$.

Lemma 3.13. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$ then for any position $a_{mn}^{b_s}$ in μ'

1. If $a_{(r-i+1)n}$ is an intersection point in original James abacus diagram and

$$S = \left\{ a_{(r-i+2)(n-n')}^{b_1} | n' = 1, 2, \ldots, n - i + 1 \right\}$$

is a set of bead positions, then $a_{(r-i+2)n}^{A^{tc1}}$ is an intersection point in diagram A^{tc1} where $i \leqslant n \leqslant \frac{e}{2}$ and $i \neq 1$.

2. If $a_{(r-i+1)n}$ is an intersection point in original James abacus diagram and

$$S = \left\{ a_{((r-i)(n-n''))}^{b_1} | n'' = 1, 2, \ldots, e - i - n + 1 \right\}$$

is a set of bead positions, then $a_{(r-i)n}^{A^{tc1}}$ is an intersection point in diagram A^{tc1} such that $n = \frac{e + 2u}{2}$ where $1 \leqslant u \leqslant \frac{e}{2}$.

Proof.

1. By Rule 3.12, for $i > 1$

$$a_{(r-i+1)n}^{b_s} \to \begin{cases} a_{(r-i+1)(n+1)}^{A^{tc1}b_s} & if \quad n = i, s = 1 \\ a_{(r-i+2)n}^{A^{tc1}b_s} & if \quad n = i, 1 < s \leqslant e \\ a_{(r-i+1)(n+1)}^{A^{tc1}b_s} & if \quad n = i + d, 1 \leqslant s \leqslant d + 1 \\ a_{(r-i+2)n}^{A^{tc1}b_s} & if \quad n = i + d, d + 1 < s \leqslant e \end{cases}$$

and

$$a^{b_s}_{(r-i+2)(n-n')} \rightarrow \begin{cases} a^{A^{tc1}b_s}_{(r-i+2)n} & if \quad n = i, s = 1 \\ a^{A^{tc1}b_s}_{(r-i+3)(n-1)} & if \quad n = i, 1 < s \leqslant e \\ a^{A^{tc1}b_s}_{(r-i+2)(n-n'+1)} & if \quad n = i + d, 1 \leqslant s \leqslant d + 1 \\ a^{A^{tc1}b_s}_{(r-i+3)(n-n')} & if \quad n = i + d, d + 1 < s \leqslant e \end{cases}$$

where $d \geqslant 1$. Hence $a^{A^{tc1}}_{(r-i+2)n}$ is a bead position in b_s.

2. By Rule 3.12

- If $n - n'' = \dfrac{e-2d+2}{2}$

$$a^{b_s}_{(r-i+1)n} \rightarrow \begin{cases} a^{A^{tc1}b_s}_{(r-i+1)(n+1)} & if \quad 1 \leqslant s < \dfrac{e-2d+2}{2} \\ a^{A^{tc1}b_s}_{(r-i)n} & if \quad \dfrac{e-2d+2}{2} \leqslant s \leqslant e \end{cases}$$

and

$$a^{b_s}_{(r-i)(n-n'')} \rightarrow \begin{cases} a^{A^{tc1}b_s}_{(r-i)(n-n''+1)} & if \quad 1 \leqslant s < \dfrac{e-2d}{2} \\ a^{b_s}_{(r-i+1)(n-n'')} & if \quad \dfrac{e-2d}{2} \leqslant s \leqslant e \end{cases}$$

- If $n - n'' = \dfrac{e+2d}{2}$

$$a^{b_s}_{(r-i)(n-n'')} \rightarrow \begin{cases} a^{A^{tc1}b_s}_{(r-i)(n-n''+1)} & if \quad 1 \leqslant s < \dfrac{e-2d+2}{2} \\ a^{A^{tc1}b_s}_{(r-i-1)(n-n''+1)} & if \quad \dfrac{e-2d+2}{2} \leqslant s \leqslant e \end{cases}$$

Hence $a^{A^{tc1}}_{(r-i+2)n}$ is a common bead position. ■

Theorem 3.14. Let e be any even column number, r the number of rows and i the number of chains in the James abacus diagram where $e^2 \leqslant \beta_1$ and $\mu = (\mu_1, \mu_2, \cdots, \mu_\eta) = (\mu_1^{\tau_1}, \mu_2^{\tau_2}, \cdots, \mu_f^{\tau_f})$ then

1. If $\tau_w < e$ then in diagram A^{tc1}, $\#\bigcap_{s=1}^{e} m.d. = \varnothing$ where $1 \leqslant w \leqslant f$.

2. If $\tau_z^{''} < e$ then in diagram A^{tc1},
$$\#\bigcap_{s=1}^{e} m.d. = \sum_{z=1}^{\omega} \tau_z^{'} - \omega(e-1)).$$

3. If $S = \{\beta_k^{\mu} : \beta_k^{\mu} = (r-i)e + (n-1)\} \subseteq \rho$ then in diagram A^{tc1},
$$\#\bigcap_{s=1}^{e} m.d. = \sum_{z=1}^{\omega} \tau_z^{'} - \omega(e-1)) - S.$$

4. In diagram A^{tc1} for $\mu^{'}$,
$$\#\bigcap_{s=1}^{e} m.d_{b_s}$$
$$= \sum_{z=1}^{\omega}[\tau_z^{'} - \omega(e-1)] + \sum_{z=1}^{\omega}[\tau_z^{''} - \omega(e-1)] - S - \rho + S^{'} + \rho^{'}.$$
where
$$\rho' = \left\{(r-i-1)\,e + \left(n - n^{'} - 1\right)\right\} \subseteq \beta^{\mu}$$
and
$$\rho'' = \left\{(r-i-1)\,e + \left(n - n^{''} - 1\right)\right\} \subseteq \beta^{\mu}.$$

Proof.

1. Since every part of μ has repeated just for u times where $1 \leqslant u < e$ then in diagram A^{tc1}, $\#\bigcap_{s=1}^{e} m.d_{b_s} = \varnothing$.

2. Since $\tau_z^{''} < e$ then $\#\bigcap_{s=1}^{e} m.d_{b_s}\mu^{''} = \varnothing$. Then the common bead positions are located in $\mu^{'}$. By Rule 3.11, the diagram positions will be moved in the same way as in b_s where $s = 1, 2, \ldots, e$. Hence, in diagram A^{tc1},
$$\#\bigcap_{s=1}^{e} m.d_{b_s} = \sum_{t=1}^{\Omega} \tau^{t} - \Omega(e-1).$$

3. Since β_k^μ and β_q^μ are intersection points in the original James abacus diagram, by Rule 3.12 and Remark **??**, β_k^μ and β_q^μ are not intersection points in diagram A^{tc1}. Hence in diagram A^{tc1},

$$\#\bigcap_{s=1}^{e} m.d_{b_s}\mu^{'} = \left[\sum_{t=1}^{\Omega}\tau^t - \Omega(e-1)\right] - s - \rho.$$

4. By lemma 3.13, $(r-i+1)e+(n-1)$ and $(r-i-1)e+(n-1)$ are intersection points. By Rule 3.12, in diagram A^{tc1},

$$\#\bigcap_{s=1}^{e} m.d_{b_s}\mu^{'}$$

$$= \sum_{z=1}^{\omega}[\tau_z^{'} - \omega(e-1)] + \sum_{z=1}^{\omega}[\tau_z^{''} - \omega(e-1)] - S - \rho + S^{'} + \rho^{'}.$$

■

4. Conclusion

In this paper, a new method is proposed to partition μ of a positive integer t with a single motion of nested chain movement of length 1 in James abacus diagram. Using these new nested chain movement, we created a diagram called diagram A^{tc1} that displays the new partition. Furthermore, we also construct rules to find the main diagram of b_s from the main diagram of b_{s-1}. Common bead positions among the new main diagrams are determined. We introduced remarks and established lemmas and theorem to find the common bead positions in the new main diagrams A^{tc1} which in turn yields the common beta numbers. Finally, the proposed method is employed as a special type of James abacus diagram where $e^2 \leqslant \beta_1$.

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