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Revisit nonlinear differential equations associated with Bernoulli numbers of the second kind

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Abstract

In this paper, we revisit nonlinear differential equations which are derived from the generating function of the Bernoulli numbers of the second kind. In addition, we give explicit and new identities for the Bernoulli numbers and higher-order Bernoulli numbers of the second kind.

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1. Introduction

For $r \in \mathbb{N}$, it is well known that the higher order Bernoulli numbers of the second kind are defined by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{n=0}^{\infty} b_n^{(r)} \frac{t^n}{n!}, \quad \text{(see [1, 6, 11])}. \tag{1}$$

From (1), we note that

$$b_n^{(r)} = B_n^{(n-r+1)}(1), \quad (n \geq 0),$$

where $B_n^{(r)}(x)$ are Bernoulli polynomials of order r which are given by the generating function

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}, \quad \text{(see [2, 4, 7])}.$$

In particular, for $r = 1$, $b_n = b_n^{(1)}$ are called the Bernoulli numbers of the second kind.

In [2], Bayad and Kim studied the following nonlinear differential equations

$$y^N = \frac{1}{(N-1)!} \sum_{k=1}^{N} a_k(N) y^{k-1}, \quad (k \in \mathbb{N}), \tag{2}$$

where $y^{(k)} = \left(\frac{d}{dt}\right)^k y(t)$.

For $y = y(t) = \frac{1}{qe^t \pm 1}$, Bayad-Kim gave explicit formula for Apostol-Bernoulli and Apostol-Euler numbers and polynomials arising from (2).

Recently, Kim and Kim introduced the following nonlinear differential equations arising from the generating function of Bernoulli numbers of the second in order to obtain explicit identities and formulas of the Bernoulli numbers of the second kind as follows:

$$F^{(N)}(t) = \frac{(-1)^N}{(1+t)^N} \sum_{j=2}^{N+1} (j-1)!(N-1)! H_{N-1,j-2} F^j, \quad \text{(see [6])}, \tag{3}$$

where

$$H_{N,0} = 1 \quad \text{for all } N$$
$$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N},$$
$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-2}}{N-1} + \cdots + \frac{H_{0,j-1}}{1}, \quad H_{0,j-1} = 0 \quad (2 \leq j \leq N).$$

Several authors have studied nonlinear differential equations arising from the generating functions of special numbers and polynomials (see [1, 2, 3, 4, 5, 8, 6, 7, 11, 9, 10, 12, 13, 14]).

In this paper, we revisit nonlinear differential equations which are derived from the generating functions of the Bernoulli numbers of the second kind. In addition, we give explicit and new identities for the Bernoulli numbers of the second kind and higher-order Bernoulli numbers of the second kind.

2. Revisit nonlinear differential equations associated with Bernoulli numbers of the second kind

Let

$$F = F(t) = \frac{1}{\log(1+t)}. \tag{4}$$

Then, by (1), we get

$$F^{(1)} = \frac{dF(t)}{dt} = -\frac{1}{\log(1+t)^2}\frac{1}{1+t} = -\frac{1}{1+t}F^2. \tag{5}$$

From (5), we can derive the following equations:

$$F^2 = -(1+t)F^{(1)}, \tag{6}$$

$$2!F^3 = (-1)^2\left\{(1+t)F^{(1)} + (1+t)^2 F^{(2)}\right\}, \tag{7}$$

and

$$3!F^4 = (-1)^3\left\{(1+t)F^{(1)} + 3(1+t)^2 F^{(2)} + (1+t)^3 F^{(3)}\right\}. \tag{8}$$

Thus, we are led to put

$$N!F^{N+1} = (-1)^N \sum_{i=1}^{N} a_i(N)(1+t)^i F^{(i)}, \quad (N = 0, 1, 2, \dots), \tag{9}$$

where $F^{(i)} = \left(\frac{d}{dt}\right)^i F(t)$.

Taking derivatives on both sides of (9), we have

$$(N+1)!F^N F^{(1)} = (-1)^N \left\{\sum_{i=1}^{N} a_i(N)\, i\,(1+t)^{i-1} F^{(i)} + \sum_{i=1}^{N} a_i(N)(1+t)^i F^{(i+1)}\right\}. \tag{10}$$

Thus, by (5) and (10), we get

$$\begin{aligned}(N+1)!F^{N+2} &= (-1)^{N+1}\left\{\sum_{i=1}^{N} ia_i(N)(1+t)^i F^{(i)} + \sum_{i=1}^{N} a_i(N)(1+t)^{i+1} F^{(i+1)}\right\} \quad (11)\\ &= (-1)^{N+1}\left\{\sum_{i=1}^{N} ia_i(N)(1+t)^i F^{(i)} + \sum_{i=2}^{N+1} a_{i-1}(N)(1+t)^i F^{(i)}\right\}.\end{aligned}$$

On the other hand, by replacing N by $N+1$ in (9), we get

$$(N+1)!F^{N+2} = (-1)^{N+1}\sum_{i=1}^{N+1} a_i(N+1)(1+t)^i F^{(i)}. \quad (12)$$

Comparing the coefficients on both sides of (11) and (12), we have

$$a_1(N+1) = a_1(N), \quad a_{N+1}(N+1) = a_N(N), \quad (13)$$

and

$$a_i(N+1) = ia_i(N) + a_{i-1}(N), \quad (2 \le i \le N). \quad (14)$$

In addition, by (6) and (9), we get

$$-(1+t)F^{(1)} = F^2 = -a_1(1)(1+t)F^{(1)}. \quad (15)$$

Thus, by (15), we see

$$a_1(1) = 1. \quad (16)$$

From (7) and (9), we can derive the following equation:

$$\begin{aligned}(-1)^2\left\{(1+t)F^{(1)} + (1+t)^2 F^{(2)}\right\} &= 2!F^3 \quad (17)\\ &= (-1)^2\left\{a_1(2)(1+t)F^{(1)} + a_2(2)(1+t)^2 F^{(2)}\right\}.\end{aligned}$$

By comparing the coefficients on the both sides of (17), we get

$$a_1(2) = 1, \quad a_2(2) = 1. \quad (18)$$

From (13), we note that

$$a_1(N+1) = a_1(N) = a_1(N-1) = \cdots = a_1(1) = 1, \quad (19)$$

and

$$a_{N+1}(N+1) = a_N(N) = a_{N-1}(N-1) = \cdots = a_1(1) = 1. \quad (20)$$

For $i = 2, 3, 4$ in (14), we have

$$a_2(N+1) = \sum_{k=0}^{N-1} 2^k a_1(N-k), \tag{21}$$

$$a_3(N+1) = \sum_{k=0}^{N-2} 3^k a_2(N-k), \tag{22}$$

and

$$a_4(N+1) = \sum_{k=0}^{N-3} 4^k a_3(N-k). \tag{23}$$

So, we can deduce that, for $2 \le i \le N$,

$$a_i(N+1) = \sum_{k=0}^{N-i+1} i^k a_{i-1}(N-k). \tag{24}$$

Now, we give explicit expressions for $a_i(N+1)$.
By (21), we get

$$a_2(N+1) = \sum_{k_1=0}^{N-1} 2^{k_1} = 2^N - 1, \tag{25}$$

$$a_3(N+1) = \sum_{k_2=0}^{N-2} 3^{k_2} a_2(N-k_2) = \sum_{k_2=0}^{N-2} \sum_{k_1=0}^{N-2-k_2} 3^{k_2} 2^{k_1}, \tag{26}$$

and

$$\begin{aligned} a_4(N+1) &= \sum_{k_3=0}^{N-3} 4^{k_3} a_3(N-k_3) \\ &= \sum_{k_3=0}^{N-3} \sum_{k_2=0}^{N-3-k_3} \sum_{k_1=0}^{N-3-k_3-k_2} 4^{k_3} 3^{k_2} 2^{k_1}. \end{aligned} \tag{27}$$

So, we deduce that, for $2 \le i \le N$,

$$a_i(N+1) = \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_2=0}^{N-i+1-k_{i-1}} \cdots \sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} i^{k_{i-1}} (i-1)^{k_{i-2}} \cdots 2^{k_1}. \tag{28}$$

Therefore, we obtain the following theorem.

Theorem 2.1. For $N \in \mathbb{N}$, the nonlinear differential equations

$$N!F^{N+1} = (-1)^N \sum_{i=1}^{N} a_i\,(N)\,(1+t)^i\, F^{(i)} \quad (N = 1, 2, \cdots)$$

have a solution $F = F\,(t) = \dfrac{1}{\log(1+t)}$, where $a_1\,(N) = 1$ and

$$a_i\,(N) = \sum_{k_{i-1}=0}^{N-i} \sum_{k_{i-2}=0}^{N-i-k_{i-1}} \cdots \sum_{k_1=0}^{N-i-k_{i-1}-\cdots-k_2} i^{k_{i-1}}\,(i-1)^{k_{i-2}} \cdots 2^{k_1}, \quad (2 \le i \le N)\,.$$

Now, we recall that the Bernoulli numbers of the second kind, $b_k (k \geq 0)$, are defined by the generating function

$$\frac{t}{\log(1+t)} = \sum_{k=0}^{\infty} b_k \frac{t^k}{k!}.$$

Also, the Bernoulli numbers of the second kind of order r, $b_k^{(r)} (r \in \mathbb{N})$, are given by the generating function

$$\left(\frac{t}{\log(1+t)}\right)^r = \sum_{k=0}^{\infty} b_k^{(r)} \frac{t^k}{k!}.$$

From (4), we note that

$$\begin{aligned} F &= F\,(t) && (29)\\ &= \frac{t}{\log(1+t)} \cdot \frac{1}{t} \\ &= \sum_{k=0}^{\infty} b_k \frac{t^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty} b_k \frac{t^{k-1}}{k!} + \frac{1}{t} \\ &= \sum_{k=0}^{\infty} \frac{b_{k+1}}{k+1} \frac{t^k}{k!} + \frac{1}{t}. \end{aligned}$$

For a positive integer i, we have

$$\begin{aligned} F^{(i)} &= \left(\frac{d}{dt}\right)^i \left(\frac{1}{\log(1+t)}\right) && (30)\\ &= \sum_{k=i}^{\infty} \frac{b_{k+1}}{k+1} \frac{t^{k-i}}{(k-i)!} + (-1)^i\, i!t^{-i-1} \\ &= \sum_{k=0}^{\infty} \frac{b_{k+i+1}}{k+i+1} \frac{t^k}{k!} + (-1)^i\, i!t^{-i-1}. \end{aligned}$$

From Theorem 2.1, we note that

$$
\begin{aligned}
N!t^{N+1}F^{N+1} &= N!\left(\frac{t}{\log(1+t)}\right)^{N+1} \\
&= N!\sum_{n=0}^{\infty} b_n^{(N+1)}\frac{t^n}{n!}.
\end{aligned}
\tag{31}
$$

From Theorem 2.1, we can derive the following equation:

$$
\begin{aligned}
&(-1)^N\, t^{N+1}\sum_{i=1}^{N} a_i\,(N)\,(1+t)^i\,F^{(i)} \\
&= (-1)^N\, t^{N+1}\sum_{i=1}^{N} a_i\,(N)\sum_{l=0}^{\infty}(i)_l\,\frac{t^l}{l!}\left(\sum_{k=0}^{\infty}\frac{b_{k+i+1}}{k+i+1}\frac{t^k}{k!}+(-1)^i\,i!t^{-i-1}\right).
\end{aligned}
\tag{32}
$$

Now, we observe that

$$
\begin{aligned}
&(-1)^N\, t^{N+1}\sum_{i=1}^{N} a_i\,(N)\sum_{l=0}^{\infty}(i)_l\,\frac{t^l}{l!}\sum_{k=0}^{\infty}\frac{b_{k+i+1}}{k+i+1}\frac{t^k}{k!} \\
&= (-1)^N\, t^{N+1}\sum_{i=1}^{N} a_i\,(N)\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}(i)_{n-k}\,\frac{b_{k+i+1}}{k+i+1}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=1}^{N} a_i\,(N)\,(-1)^N\binom{n}{k}(i)_{n-k}\,\frac{b_{k+i+1}}{k+i+1}\frac{t^{n+N+1}}{n!} \\
&= \sum_{n=N+1}^{\infty}\sum_{k=0}^{n-N-1}\sum_{i=1}^{N} a_i\,(N)\,(-1)^N\binom{n-N-1}{k}(i)_{n-N-k-1}\,\frac{b_{k+i+1}}{k+i+1}\frac{t^n}{(n-N-1)!} \\
&= \sum_{n=N+1}^{\infty}\sum_{k=0}^{n-N-1}\sum_{i=1}^{N}(-1)^N\binom{n-N-1}{k}(i)_{n-N-k-1}\,(n)_{N+1}\,a_i\,(N)\,\frac{b_{k+i+1}}{k+i+1}\frac{t^n}{n!},
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
&(-1)^{N} t^{N+1} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{\infty} (i)_l \frac{t^l}{l!} (-1)^i \, i! t^{-i-1} \qquad (34)\\
&= (-1)^{N} \sum_{i=1}^{N} a_i(N) \sum_{l=0}^{\infty} (i)_l \frac{1}{l!} (-1)^i \, i! t^{N-i+l}\\
&= (-1)^{N} \sum_{i=0}^{N-1} a_{N-i}(N) \sum_{l=0}^{\infty} (N-i)_l \frac{1}{l!} (-1)^{N-i} (N-i)! t^{i+l}\\
&= (-1)^{N} \sum_{n=0}^{\infty} \sum_{i=0}^{\min\{N-1,n\}} a_{N-i}(N) (N-i)_{n-i} \frac{1}{(n-i)!} (-1)^{N-i} (N-i)! t^{n}\\
&= \sum_{n=0}^{\infty} \sum_{i=0}^{\min\{N-1,n\}} (-1)^{i} (N-i)_{n-i} (n)_i (N-i)! a_{N-i}(N) \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, by Theorem 2.1, (31), (32), (33) and (34), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, $N \geq 1$, we have

$$
b_n^{(N+1)} = \begin{cases}
\displaystyle\sum_{i=0}^{\min\{N-1,n\}} (-1)^i (N-i)_{n-i} \binom{n}{i} \binom{N}{i}^{-1} a_{N-i}(N), & \text{if } 0 \leq n \leq N,\\
\displaystyle\sum_{i=0}^{N-1} (-1)^i (N-i)_{n-i} \binom{n}{i} \binom{N}{i}^{-1} a_{N-i}(N) & \\
\displaystyle + (-1)^N \binom{n}{N+1} (N+1) & \\
\displaystyle \times \sum_{k=0}^{n-N-1} \sum_{i=1}^{N} \binom{n-N-1}{k} (i)_{n-N-k-1} a_i(N) \frac{b_{k+i+1}}{k+i+1} & \text{if } n \geq N+1.
\end{cases}
$$

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