Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1873-1883

http://www.ripublication.com/gjpam.htm

A study of a class of a dynamic system of Fitzhugh-Nagumo type

Amine Sakri and Azzedine Benchettah

Laboratoire d'Analyse Numérique,
Optimisation et Statistique,
Université Badji Mokhtar,
23000 Annaba, BP12, Algérie.

Abstract

This work address the FitzHugh-Nagumo system which models the slow-fast dynamics of neuronal action potentials. We investigate global solution, stationary points, asymptotic and global stability, bifurcation and limit cycles.

AMS subject classification: $34C15$, $37D45$, $37G35$, $37N25$, $92C20$.
Keywords: Neuron model, FitzHugh-Nagumo, asymptotic dynamics, bifurcation, limit cycles.

1. Introduction

We study two particular cases of the system

$$\begin{cases} dx = \dfrac{1}{\varepsilon}(x - x^3 + y)dt, \\ dy = (a - bx - cy)dt. \end{cases} \tag{1}$$

In this system: x is the membrane potential, y is the exchange of ions across the neuron membrane, $a, b, c \in \mathbb{R}$ and ϵ is a small parameter which implies that x changes rapidly. This FitzHugh-Nagumo system is a simplified model of the four dimensional Hodgkin-Huxley system, which gives a description of the main ionic fluxes across the neuronal membrane creating the neuronal signal. For more details of the understanding of the biological meaning [7], [9], [10]. Results for this type of slow fast systems has been obtained in different ways by many authors, numerically by [6], and using perturbation techniques by [1]. Here, we investigate in another direction, qualitative way, where we prove the existence of a global solution, stationary points, asymptotic and global stability, bifurcation and limit cycles. More precisely, it is difficult to study this system, so we decompose it in two particular systems which give us some information for the study of the general system. In the first part, let $b = 0$, $c = 1$ in (1), we obtain the system

$$\begin{cases} d\mathbf{x} = \frac{1}{\varepsilon}(\mathbf{x} - \mathbf{x}^3 + \mathbf{y})dt, \\ d\mathbf{y} = (a - \mathbf{y})dt. \end{cases} \tag{2}$$

The study of this system allows us to find the equilibrium points as well as their asymptotic stability. We show that the solution of this system is bounded and, using Bendixon-Dulac criterion, we prove that there is no limit cycles.

In the second part, let $b = 1$, $c = 0$ in (1), we obtain the system

$$\begin{cases} d\mathbf{x} = \frac{1}{\varepsilon}(\mathbf{x} - \mathbf{x}^3 + \mathbf{y})dt, \\ d\mathbf{y} = (a - \mathbf{x})dt. \end{cases} \tag{3}$$

Here, besides the existence of the global solution, we also show using some results on Lienard systems together with the Normal Form Algorithm methods that there exists a unique equilibrium point which is a global attractor if it is stable or an hyperbolic stable limit cycle otherwise.

2. System 1

We consider the system

$$\begin{cases} dx = \frac{1}{\varepsilon}(x - x^3 + y)dt, \\ dy = (a - y)dt. \end{cases} \tag{4}$$

The question of existence of equilibrium points is given in the following theorem.

Theorem 2.1. Let

$$\Delta = 4 - 27a^2.$$

Then the system (4) has

1. a unique equilibrium point if $\Delta < 0$,
2. two equilibrium points if $\Delta = 0$,
3. three equilibrium points if $\Delta > 0$.

Proof. The existence of equilibrium points is given by solving of the following system

$$\begin{cases} x - x^3 + y = 0, \\ a - y = 0, \end{cases} \tag{5}$$

which yields the equation $x^3 - x - a = 0$. By using Cardan formulas, we obtain the expression

$$\Delta = 4 - 27a^2$$

which enables us to prove the theorem. ■

The stability of equilibrium points is given in the following theorem

Theorem 2.2. We have that:

1. If $|a| > \frac{2\sqrt{3}}{9}$, the unique equilibrium point is a stable node.
2. If $|a| < \frac{2\sqrt{3}}{9}$, the three equilibrium points are two stable nodes and the third one is a saddle point.
3. If $|a| = \frac{2\sqrt{3}}{9}$, the two equilibrium points are one stable node and the other a saddle-node which is instable.

Proof. Let

$$A = \begin{pmatrix} \frac{1-3x^2}{\varepsilon} & \frac{1}{\varepsilon} \\ 0 & -1 \end{pmatrix}$$

be the Jacobian matrix of the system (4). By [11], the nature of equilibrium points is determined by eigenvalues of this matrix which are the roots of the characteristic polynomial of the matrix A at the equilibrium points (x^*, y^*). As

$$\det(A(x^*) - \lambda I) = \lambda^2 - \operatorname{Tr} A(x^*).\lambda + \det A(x^*),$$

where

$$\operatorname{Tr} A(x^*) = \frac{1}{\varepsilon} - \frac{3x^{*2}}{\varepsilon} - 1$$

and

$$\det A(x^*) = \frac{3x^{*2} - 1}{\varepsilon},$$

so, the polynomials $\operatorname{Tr} A(x^*)$ and $\det A(x^*)$ have two real roots

$$x^*_{tr1} = +\frac{\sqrt{3(1-\varepsilon)}}{3}, \; x^*_{tr2} = -\frac{\sqrt{3(1-\varepsilon)}}{3},$$

and

$$x^*_{d1} = +\frac{\sqrt{3}}{3}, \; x^*_{d2} = -\frac{\sqrt{3}}{3},$$

respectively. On the other hand, the solutions of the characteristic polynomial are

$$\lambda_1 = \frac{1}{2}[\mathrm{Tr}\, A + \sqrt{(\mathrm{Tr}\, A)^2 - 4\det A}] \text{ and } \lambda_2 = \frac{1}{2}[\mathrm{Tr}\, A - \sqrt{(\mathrm{Tr}\, A)^2 - 4\det A}].$$

Thus we have

If $x \in \left]\frac{-\sqrt{3}}{3}, -\frac{\sqrt{3(1-\varepsilon)}}{3}\right] \cup \left[\frac{\sqrt{3(1-\varepsilon)}}{3}, \frac{\sqrt{3}}{3}\right[$ then $\mathrm{Tr}\, A \leq 0$ and $\det A < 0$, so λ_1 and λ_2 are two opposite reals, therefor we have a saddle point.

If $x \in \left[-\frac{\sqrt{3(1-\varepsilon)}}{3}, \frac{\sqrt{3(1-\varepsilon)}}{3}\right]$, we can show that $\mathrm{Tr}\, A \geq 0$ and $\det A < 0$, thus λ_1 and λ_2 are two opposite reals, thus we have a saddle point.

If $|x| > \frac{\sqrt{3}}{3}$, then $\mathrm{Tr}\, A < 0$, $\det A > 0$ and $(\mathrm{Tr}\, A)^2 > 4\det A$, thus λ_1 and λ_2 are two negative reals thus we have a stable node.

If $|x| = \frac{\sqrt{3}}{3}$, then $f(x, y) = 0$ and $\det A = 0$, by the definition 3.9, p. 77 in [4], the two points $\left(\frac{-\sqrt{3}}{3}, \frac{2\sqrt{3}}{9}\right)$ and $\left(\frac{\sqrt{3}}{3}, \frac{-2\sqrt{3}}{9}\right)$ are two saddle-node points since $\frac{\partial^2 f}{\partial x^2} \neq 0$ and $\frac{\partial f}{\partial y} \neq 0$ for these two points.

Now, from previous results, we can prove the theorem. Indeed, it is easy to determine the equilibrium points. For the assertion 1, since $|a| > \frac{2\sqrt{3}}{9}$, the equilibrium point is given by $\left(\left(\frac{a+\sqrt{\frac{27a^2-4}{27}}}{2}\right)^{\frac{1}{3}} + \left(\frac{a-\sqrt{\frac{27a^2-4}{27}}}{2}\right)^{\frac{1}{3}}, a\right)$, and since $\left(\left(\frac{a+\sqrt{\frac{27a^2-4}{27}}}{2}\right)^{\frac{1}{3}} + \left(\frac{a-\sqrt{\frac{27a^2-4}{27}}}{2}\right)^{\frac{1}{3}} > \frac{\sqrt{3}}{3}\right)$, the equilibrium point is a stable node. The assertions 2 and 3 of the theorem are obtained similarly. ∎

Theorem 2.3. Assume that the system (4) with the initial condition $(x(0), y(0))$ has a unique solution. Then there exist $\beta > 0$ and $T > 0$ depending on the initial condition, such that, for all $t > T$, $|(x(t), y(t))| < \beta$.

Proof. Let $\Phi = \varepsilon x^2 + y^2$. The system (4) yields

$$\frac{d\Phi}{dt} = -x^4 + x^2 + xy + ay - y^2.$$

The Cauchy-Schwarz inequality gives

$$\frac{d\Phi}{dt} \leq -x^4 + \frac{3}{2}x^2 + ay - \frac{1}{2}y^2,$$

and by Young inequality, we get

$$\frac{d\Phi}{dt} \leq -x^4 + \frac{3}{2}\left(\frac{h}{2}x^4 + \frac{1}{2h}\right) + a\left(\frac{k}{2}y^2 + \frac{1}{2k}\right) - \frac{1}{2}y^2, \text{ for all } h, k > 0.$$

For h and k sufficiently small, we can find some constants α, β, $\gamma > 0$, such that

$$\frac{d\Phi}{dt} \leq -\alpha x^4 - \beta y^2 + \gamma, \tag{*}$$

and since

$$x^2 \leq \frac{x^4}{2} + \frac{1}{2},$$

we have

$$-\alpha x^4 \leq -2\alpha x^2 + \alpha,$$

then the inequality (*) becomes

$$\frac{d\Phi}{dt} \leq -2\alpha x^2 + \alpha - \beta y^2 + \gamma.$$

Thus, we can find two constants $C_0, C_1 > 0$, such that

$$\frac{d\Phi}{dt} \leq -C_0\Phi + C_1.$$

Multiplying the inequality by $\exp(C_0 t)$, we obtain after integration

$$(\varepsilon x^2 + y^2)(t) \leq (\exp(-C_0 t))(\varepsilon x^2(0) + y^2(0)) + \frac{C_1}{C_0}(1 - \exp(-C_0 t)),$$

which proves the theorem. ■

Theorem 2.4. The system (4) has no limit cycle.

Proof. Let be $S = (f, g)^T$ where $(.,.)^T$ is the transposed vector,

$$f(x, y) = \frac{1}{\varepsilon}(x - x^3 + y) \text{ and } g(x, y) = a - y.$$

As

$$divS = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y},$$

we have

$$divS = -\frac{3}{\varepsilon}x^2 - 1 + \frac{1}{\varepsilon},$$

thus

$$divS < 0 \quad \text{if } |x| > \frac{\sqrt{3(1-\varepsilon)}}{3},$$

and

$$divS > 0 \quad \text{if } |x| < \frac{\sqrt{3(1-\varepsilon)}}{3}.$$

In both cases, the Bendixon-Dulac criterion allows us to conclude that we have not a limit cycle in these two regions. As, in the case where $|x| = \frac{\sqrt{3(1-\varepsilon)}}{3}$, we have proved that the equilibrium point is a saddle point and then there is no limit cycle; thus the theorem is proved. ■

3. System 2

We consider the system

$$\begin{cases} d\mathbf{x} = \frac{1}{\varepsilon}(\mathbf{x} - \mathbf{x}^3 + \mathbf{y})dt, \\ d\mathbf{y} = (a - \mathbf{x})dt. \end{cases} \tag{6}$$

By the Cauchy-Lipschitz theorem, one can easily prove that the system (6) has a unique local solution. It is clear that the point $(a, a^3 - a)$ is the unique stationary point. The transformation $x = \mathbf{x} - a$ and $y = \mathbf{y} - a^3 + a$ leads to

$$\begin{cases} dx = \frac{1}{\varepsilon}(g_a(x) + y)dt, \\ dy = -xdt, \end{cases} \tag{7}$$

where $g_a(x) = (1 - 3a^2)x - 3ax^2 - x^3$ and $(0, 0)$ is the equilibrium point. We have the following result given the nature of the equilibrium point.

Theorem 3.1. Let the system (7) be given. Then the equilibrium point $(0, 0)$ is

1. a focus if $a^2 \in]\frac{1}{3}(1 - 2\sqrt{\varepsilon}), \frac{1}{3}(1 + 2\sqrt{\varepsilon})[$ and a node elsewhere,
2. asymptotically instable if $|a| < \frac{\sqrt{3}}{3}$,
3. asymptotically stable if $|a| > \frac{\sqrt{3}}{3}$,
4. a Hopf bifurcation point if $|a| = \frac{\sqrt{3}}{3}$.

Proof. It suffices to consider the linearized system of (7),

$$\begin{cases} dx = \frac{1}{\varepsilon}((1-3a^2)x + y)dt, \\ dy = -xdt. \end{cases} \tag{8}$$

The eigenvalues of the associated matrix

$$\begin{pmatrix} \frac{1-3a^2}{\varepsilon} & \frac{1}{\varepsilon} \\ -1 & 0 \end{pmatrix}$$

are

$$\lambda_{1,2} = \begin{cases} \frac{1}{2\varepsilon}((1-3a^2) \pm \sqrt{\Delta_\varepsilon}), \text{ if } \Delta_\varepsilon \geq 0 \\ \frac{1}{2\varepsilon}((1-3a^2) \pm i\sqrt{-\Delta_\varepsilon}), \text{ if } \Delta_\varepsilon < 0 \end{cases},$$

where $\Delta_\varepsilon = (1-3a^2)^2 - 4\varepsilon$. Using the classic theory of dynamical systems ([8], [11]), we obtain:

For $a^2 > \frac{1}{3}$, we have $Re(\lambda_{1,2}) < 0$ and thus the equilibrium point (0, 0) is stable.

For $a^2 < \frac{1}{3}$, we have $Re(\lambda_{1,2}) > 0$ and thus the equilibrium point is instable.

For $a^2 \in]\frac{1}{3}(1-2\sqrt{\varepsilon}), \frac{1}{3}(1+2\sqrt{\varepsilon})[$, the two eigenvalues are complex conjugated, thus the equilibrium point is a focus, and for $a^2 \notin]\frac{1}{3}(1-2\sqrt{\varepsilon}), \frac{1}{3}(1+2\sqrt{\varepsilon})[$, the two eigenvalues are reals, and thus the equilibrium point is a node.

For $|a| = \frac{\sqrt{3}}{3}$, let $\lambda_{1,2} = z(a) \pm iw(a)$ where $z(a) = \frac{1}{2\varepsilon}((1-3a^2)$ and $w(a) = \sqrt{4\varepsilon - (1-3a^2)^2}$. Since $z\left(\pm\frac{\sqrt{3}}{3}\right) = 0$, $w\left(\pm\frac{\sqrt{3}}{3}\right) = \sqrt{4\varepsilon}$ and $\frac{\partial z}{\partial a}\left(\pm\frac{\sqrt{3}}{3}\right) = \pm\frac{\sqrt{3}}{\varepsilon}$, using the theorem 36 p. 61 in [8], the system undergoes so-called singular Hopf bifurcations [2], [3], [5] at $a = \pm\frac{\sqrt{3}}{3}$. So the theorem is proved. ■

We will discuss the global stability of the equilibrium point in the next theorem, which allows us to show the existence and uniqueness of a global solution of the system (7).

Theorem 3.2. The system (7) has a unique global solution and we have

1. if $|a| \geqslant \frac{\sqrt{3}}{3}$, then the equilibrium point is a global attractor.

2. if $|a| < \frac{\sqrt{3}}{3}$, then there exists a unique hyperbolic stable limit cycle.

To prove this theorem, we need these two technical results.

Lemma 3.3. Let us consider the system

$$\begin{cases} dx = \frac{1}{\varepsilon}(-3ax^2 + y)dt, \\ dy = -xdt, \end{cases} \tag{9}$$

where $|a| > \frac{\sqrt{3}}{3}$. Then the point $(0, 0)$ is a center for the system (9), and all the trajectories passing through y^- axis ($x = 0$, $y < 0$) are periodic.

Proof of Lemma 3.3. We consider only the case $a < -\frac{\sqrt{3}}{3}$, the other case will be deduced in the same way. It is clear that the trajectories of (9) are symmetric with respect to the axis $x = 0$. So we only prove that any trajectory starting from y^- attains y^+ in finite time. To do that, we remark that, for any initial condition $(0, y_0)$, $y_0 < 0$, the trajectory attains the branch $(x < 0, 3ax^2)$ in finite time since $\frac{dx}{dt} < 0$ and $\frac{dy}{dt} > 0$. Next, the trajectory attains the semi-axis $(x < 0, y = 0)$ in finite time since $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} > 0$ and finally, it attains the semi-axis $y > 0$, $x = 0$ in finite time since $\frac{dx}{dt} > 0$ and $\frac{dy}{dt} < 0$. ■

Lemma 3.4. [12] Let us consider the system

$$\begin{cases} \frac{dx}{dt} = \alpha(y) - \beta(y)F(x)), \\ \frac{dy}{dt} = -g(x), \end{cases} \tag{10}$$

where α, β, F and g are continuous functions and we assume that the system (10) has one and only one solution. Let $G(x) = \int_0^x g(s)ds$, and let us assume the following conditions.

1. $\alpha(0) = 0$, α is strictly increasing and $\alpha(\pm\infty) = \pm\infty$.
2. $xg(x) > 0$ when $x \neq 0$ and $G(\pm\infty) = \infty$.
3. $\beta(y) > 0$ for $y \in \mathbb{R}$, and β is non-increasing.
4. There exist constants x_1, x_2 with $x_1 < 0 < x_2$ such that $F(x_1) = F(0) = F(x_2) = 0$ and $xF(x) < 0$ for $x \in]x_1, x_2[\setminus \{0\}$.

5. $F(x)$ is non-decreasing for $x \in]-\infty, x_1[\cup]x_2, \infty[$.

6. There exist constants $M > 0, k, k_0$ with $k > k_0$, such that $F(x) < k_0$ for $x \leq -M$ and $F(x) > k$ for $x \geq M$.

7. $G(-x) \geq G(x)$ for $x > 0$.

Assume that the conditions 1-7 are satisfied, then the system (10) has exactly one closed orbit, a hyperbolic stable limit.

Proof of the Theorem 3.2. First, we begin with the case where $|a| > \frac{\sqrt{3}}{3}$. By Lemma 3.3, we know that the trajectories of system (9) passing through y^- are periodic. We show now that the equilibrium point is a global attractor. Indeed, the vector field of (7) is directed towards the "interior" of the trajectories of (9), since $(1-3a^2)x - x^3$ is strictly positive if $x < 0$, and strictly negative if $x > 0$. Finally we can prove in the same way as in Lemma 3.3 that, for any initial condition, the trajectories of (7) hit y^- in finite time.

Next, in the case where $|a| < \frac{\sqrt{3}}{3}$, we use Lemma 3.4, with

$$\alpha(y) = \frac{y}{\varepsilon}, \beta(y) = 1, F(x) = \frac{(-1+3a^2)x + 3ax^2 + x^3}{\varepsilon} \text{ and } g(x) = x.$$

We show that the conditions of Lemma 3.4 are satisfied. As we have: $\alpha(0) = \frac{0}{\varepsilon} = 0$, the function α is strictly increasing, and $\alpha(\pm\infty) = \pm\infty$. As $xg(x) = x^2$, we have $xg(x) > 0$ when $x \neq 0$, and $G(\pm\infty) = \infty$. β is non-increasing function. As the solutions of the equation $F(x) = 0$ are

$$x_1 = \frac{-3a - \sqrt{4-3a^2}}{2}, x_2 = \frac{-3a + \sqrt{4-3a^2}}{2} \text{ and } x_3 = 0$$

and the roots of the function $F'(x) = 3x^2 + 6ax + 3a^2 - 1$ are

$$x_1^* = -a - \frac{\sqrt{3}}{3} \text{ and } x_2^* = -a + \frac{\sqrt{3}}{3},$$

we can deduce that $xF(x) < 0$ for $x \in]x_1, x_2[\setminus\{0\}$, and that the function F is increasing for $x \in]-\infty, x_1[\cup]x_2, \infty[$. We can always find $\gamma > 0$ such that, if $a > 0$, we can choose

$$M = x_2 + \gamma, k = F(x_1^*) \text{ and } k_0 = F(x_2^*),$$

and if $a < 0$, we can take

$$M = x_2 + \gamma, k = F(x_2^*) \text{ and } k_0 = F(x_1^*).$$

We have also $G(-x) = G(x)$. Thus, by Lemma 3.4, the system (7) has a unique stable hyperbolic limit cycle. Finally, in the case where $|a| = \frac{\sqrt{3}}{3}$, we prove the result if $a = \frac{\sqrt{3}}{3}$. The case $a = -\frac{\sqrt{3}}{3}$ is obtained similarly. Thus, the system (7) becomes

$$\begin{cases} dx = \frac{1}{\varepsilon}(-\sqrt{3}x^2 - x^3 + y)dt, \\ dy = -xdt. \end{cases} \tag{11}$$

In this case, the previous methods cannot be applied since the characteristic polynomial of the Jacobian matrix of the system (11) has two purely imaginary eigenvalues. Therefore we use the Normal Form Algorithm method to deduce the nature of the equilibrium point. In order to do that, we need that the eigenvalues of the Jacobian matrix to be equal at i and $-i$. For this, by the following judicious transformation

$$X = (\varepsilon)^{\frac{1}{4}}x,\ Y = \frac{y}{(\varepsilon)^{\frac{1}{4}}} \text{ and } \tau = \frac{t}{\sqrt{\varepsilon}},$$

the system (11) becomes

$$\begin{cases} dX = (-\frac{\sqrt{3}}{(\varepsilon)^{\frac{5}{4}}}X^2 - \frac{X^3}{(\varepsilon)^{\frac{3}{2}}} + Y)d\tau, \\ dY = -Xd\tau. \end{cases} \tag{12}$$

The Jacobian matrix for the system (12) is

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Using the Normal Form Algorithm [13], after some computation, we find that the second Lyaponov coefficient is negative. Thus, we can deduce that the equilibrium point is a spiral sink. ■

4. Conclusion

In this paper, a theoretical analysis of the two particular cases of the FitzHugh-Nagumo system has been done. For the two systems, we have discussed the global stability of the equilibrium points which enable us to prove the existence of a global solution and the existence of a hyperbolic stable limit cycle. These results reflect the biological aspect of these systems and give us useful informations to analysis the general system.

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