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Inverse Closed Domination in Graphs

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Abstract

In this paper, we introduce the inverse closed domination in graphs. Some interesting relationships are known between closed domination and inverse closed domination. In this paper, we also investigate the closed domination in the join of graphs.

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1. Introduction

Domination as a graph theoretic concept was first introduced by C. Berge in 1958 and O. Ore in 1962. It was O. Ore [8] who introduced the term dominating set and domination number. In 1977, E.J. Cockayne and S.T. Hedetniemi [2] presented a survey on published works in domination. Since a publication of the said survey, domination theory has been studied extensively. In their book, T.W. Haynes, S.T. Hedetniemi and P.J. Slater listed in [4] over 1200 references in this topic including over 75 variations. The paper of Kulli and Sigarkanti [7] in 1991 which initiated the study of inverse domination in graphs and further read in [3, 6, 10]. In this study we introduced a new domination parameter, the inverse closed domination in graphs and give some important results.

The graph G denotes a graph which is simple and undirected. The symbol $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We write uv to denote the edge joining the vertices u and v. The *order* of G refers to the cardinality $|V(G)|$ of $V(G)$, and by the size of G mean $|E(G)|$. If $E(G) = \emptyset$, G is called an *empty graph*.

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Any graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a non-empty $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph H of G for which $|E(H)|$ is the maximum size of a subgraph of G with vertex set S.

An edge e of G is said to be incident to vertex v whenever $e = uv$ for some $u \in V(G)$. The symbol $G - v$ denotes the resulting subgraph of G after removing v from G and all edges in G incident to v.If $u, v \in V(G)$, the symbol $G + uv$ denotes the graph obtained from G by adjoining to G the edge uv.

Two distinct vertices u and v of G are *neighbors* in G if $uv \in E(G)$. The *closed neighborhood* $N_G[v]$ of a vertex v of G is the set consisting of v and every neighbor of v in G. Any $S \subseteq V(G)$ is a *dominating set* in G if $\bigcup_{v \in S} N_G[v] = V(G)$. A dominating set in G is also called a γ-set in G. The minimum cardinality $\gamma(G)$ of a γ-set in G is the *domination number* of G. Any γ-set in G of cardinality $\gamma(G)$ is referred to as the minimum γ-set in G.

A dominating set is called a *closed dominating set* if given a graph G, choose $v_1 \in V(G)$ and put $S_1 = \{v_1\}$. if $N_G[S_1] \neq V(G)$, choose $v_2 \in V(G) \setminus S_1$ and put $S_2 = \{v_1, v_2\}$. Where possible, $k \geq 3$, choose $v_k \in V(G) \setminus N_G[S_{k-1}]$ and put $S_k = \{v_1, v_2, \ldots, v_k\}$. There exists a positive k such that $N_G[S_k] = V(G)$. The smallest cardinality of a closed dominating set is called the *closed domination number* of G, and denoted by $\overline{\gamma}(G)$. A close dominating set of cardinality $\overline{\gamma}(G)$ is called $\overline{\gamma}$-*set* of G. A closed dominating set S is said to be in its canonical form if it is written as $S = \{v_1, v_2, \ldots, v_k\}$, where the vertices v_j satisfy the properties given above.

Let D be a minimum dominating set in G. The dominating set $S \subseteq V(G) \setminus D$ is called an *inverse dominating set* with respect to D. The minimum cardinality of inverse dominating set is called an *inverse domination number* of G and is denoted by $\gamma^{-1}(G)$. An inverse dominating set of cardinality $\gamma^{-1}(G)$ is called γ^{-1}-*set* of G. Motivated by the definition of inverse domination in graph, we define a new domination parameter. Let C be a minimum closed dominating set in G. The closed dominating set $S \subseteq V(G) \setminus C$ is called an *inverse closed dominating set* with respect to C. The minimum cardinality of an inverse closed dominating set is called an *inverse closed domination number* of G and is denoted by $\overline{\gamma}^{-1}(G)$. An inverse closed dominating set of cardinality $\overline{\gamma}^{-1}(G)$ is called $\overline{\gamma}^{-1}$-*set* of G.

2. Results

A classical result in the domination theory which was introduced by Ore in 1962 state the following theorem:

Theorem 2.1. [8] Let G be a graph with no isolated vertex. If $S \subseteq V(G)$ is a γ-*set*, then $V(G) \setminus S$ is also a dominating set in G.

This motivate a new domination parameter, the inverse closed domination in graphs. Theorem 2.1 guarantees the existence of $\overline{\gamma}^{-1}$-*set* in some graph G. Since the inverse closed dominating set of any graph G of order n cannot be $V(G)$, it follows that $\overline{\gamma}^{-1}(G) \neq$

n and hence $\overline{\gamma}^{-1}(G) < n$.

Since $\overline{\gamma}^{-1}(G)$ does not always exists in a connected nontrivial graph G, we denote by $\mathcal{G}_c^{-1}$ be a family of all graphs with inverse closed dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{G}_c^{-1}$. From the definitions, the following result is immediate.

Remark 2.2. Let G be a connected graph of order $n \geq 2$. Then

(i) $1 \leq \overline{\gamma}^{-1}(G) < n$;

(ii) $\gamma(G) \leq \overline{\gamma}^{-1}(G) \leq \gamma^{-1}(G)$.

Consider, for example, the graph G in Figure 1. We have the set {a,b,c} is the minimum dominating set, thus $\gamma(G) = 3$. The set {a,b, j,k} is the minimum closed dominating set, thus $\overline{\gamma}(G) = 4$. The set {c,d,e,f,g,h,i} is the minimum inverse closed dominating set, thus $\overline{\gamma}^{-1}(G) = 7$ and the set {d,e,f,g,h,i,j,k} is the minimum inverse dominating set, thus $\gamma^{-1}(G) = 8$.

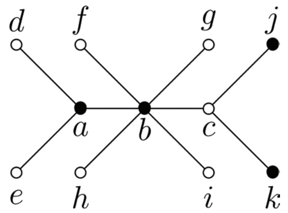

Figure 1: Graph G where $\gamma(G) \leq \overline{\gamma}^{-1}(G) \leq \gamma^{-1}(G)$.

Since $\overline{\gamma}(G)$ is the order of the minimum closed dominating set of G, it follows that $\overline{\gamma}(G) \leq \overline{\gamma}^{-1}(G)$. The following remark holds.

Remark 2.3. Let G be a connected nontrivial graph of order $n \geq 2$. Then $\overline{\gamma}(G) \leq \overline{\gamma}^{-1}(G)$.

Theorem 2.4. Let G be a connected nontrivial graph of order $n \geq 2$. Then $\overline{\gamma}^{-1}(G) = 1$ if and only if either $G = K_2$ or $G = K_2 + G^*$, for some graph G^*.

Proof. If $G = K_2$ or $G = K_2 + G^*$ for some graph G^*, then $\overline{\gamma}^{-1}(G) = 1$. Suppose that $\overline{\gamma}^{-1}(G) = 1$, then $\overline{\gamma}(G) = 1$ by Remark 2.3, thus, G contains two distinct vertices u and v such that $\{u\}$ and $\{v\}$ are closed dominating sets in G. ■

Theorem 2.5. Let G be a connected nontrivial graph of order $n \geq 2$. Then $\overline{\gamma}^{-1}(G) = n - 1$ if and only if $G = K_{1,n-1}$.

Proof. Suppose that $\overline{\gamma}^{-1}(G) = n - 1$, and let $S \subseteq V(G)$ be a $\overline{\gamma}^{-1}$-*set* in G. Let $v \in V(G) \subseteq S$. Then $N_G[v] = V(G)$, that is, $vx \in E(G)$ for all $x \in V(G) \setminus \{v\}$. Now we claim that $xy \notin E(G)$ for all $x, y \in V(G) \setminus \{v\}$. Suppose that there exists

$x, y \in V(G) \setminus \{v\}$ such that $xy \in E(G)$. In particular,

$$v, y \in N_G[x] \subseteq N_G[S \setminus \{y\}].$$

Thus, $S \setminus \{y\}$ is a $\overline{\gamma}^{-1}$-*set* in G. This is a contradiction. Therefore, $G = K_{1,n-1}$.

For the converse, suppose that $G = K_{1,n-1}$, then $\{v\}$ is a $\overline{\gamma}$-*set* in G. Let $S = V(G) \setminus \{v\}$. Then S is a $\overline{\gamma}^{-1}$-*set*, and $\overline{\gamma}^{-1}(G) = n - 1$. ■

3. Join of graphs

The *Join* of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H) = V(G) \overset{\bullet}{\cup} V(H)$ and edge-set $E(G+H) = E(G) \overset{\bullet}{\cup} E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Clearly, $\overline{\gamma}^{-1}(G + K_1) = \overline{\gamma}(G)$. We consider $G + H$ with nontrivial graphs G and H. For any $u \in V(G)$ and $v \in V(H)$, the set $\{u, v\}$ is a closed dominating set in $G + H$. Thus, $\overline{\gamma}(G + H) \leq 2$.

Lemma 3.1. For nontrivial graphs G and H, $\overline{\gamma}^{-1}(G + H) \leq 2$.

Proof. By the preceding remark, $\overline{\gamma}(G + H) \leq 2$. First, we consider the case where $\overline{\gamma}(G + H) = 1$, and suppose that $S = \{v\}$ is a closed dominating set in $G + H$. Assume $v \in V(G)$. Take $u \in V(G) \setminus \{v\}$ and $w \in V(H)$. Then $D = \{u, w\} \subseteq V(G + H) \setminus S$ and D is a closed dominating set in $G + H$. Thus $\overline{\gamma}^{-1}(G + H) \leq |D| = 2$. Next, we assume that $\overline{\gamma}(G + H) = 2$. Pick any $u \in V(G)$ and $v \in V(H)$. Then $S = \{u, v\}$ is a $\overline{\gamma}$-*set* set in $G + H$. Thus, for any $x \in V(G) \setminus S$ and $y \in V(H) \setminus S$, the set $D = \{x, y\}$ is a $\overline{\gamma}^{-1}$-*set* in $G + H$. Since G and H are nontrivial graphs, such D exists. Thus $\overline{\gamma}^{-1}(G + H) = |D| = 2$. ■

Proposition 3.2. Let G and H be nontrivial graphs. If $\overline{\gamma}^{-1}(G+H) = 1$, then $\overline{\gamma}(G) = 1$ or $\overline{\gamma}(H) = 1$ The converse, however, is not necessarily true.

Proof. The assumption implies that $\overline{\gamma}(G+H) = 1$. Therefore, $\overline{\gamma}(G) = 1$ or $\overline{\gamma}(H) = 1$. To prove the second statement, consider the graph $K_{1,5} + P_7$. Note that $\overline{\gamma}(K_{1,5}) = 1$ but $\overline{\gamma}^{-1}(K_{1,5} + P_7) = 2$. ■

Theorem 3.3. Let G and H be nontrivial graphs. Then $\overline{\gamma}^{-1}(G + H) = 1$ if and only if one of the following is true:

(i) $\overline{\gamma}(G) = 1$ and $\overline{\gamma}(H) = 1$;

(ii) $\overline{\gamma}(G) = 1$ and G has at least two minimum $\overline{\gamma}$-*sets*;

(iii) $\overline{\gamma}(H) = 1$ and H has at least two minimum $\overline{\gamma}$-sets.

Proof. Suppose that (i) holds and $\{v\} \subseteq V(G)$ and $\{w\} \subseteq V(H)$ are closed dominating sets in G and H, respectively. Then $\{v\}$ and $\{w\}$ are minimum closed dominating sets in

$G + H$. The conclusion follows from the fact that since $\{v\} \subseteq V(G + H) \setminus \{w\}$, $\{v\}$ is a $\overline{\gamma}^{-1}$-*set* in G. Now, suppose that (ii) holds and let $\{u\}$ and $\{v\}$ be closed dominating sets in G. Then $\{u\}$ and $\{v\}$ are closed dominating sets in $G+H$. Since $\{u\} \subseteq V(G+H)\setminus\{v\}$, $\overline{\gamma}^{-1}(G + H) = 1$. Similarly, if (iii) holds, then $\overline{\gamma}^{-1}(G + H) = 1$.

Conversely, suppose that $\overline{\gamma}^{-1}(G + H) = 1$. By Proposition 3.2, $\overline{\gamma}(G) = 1$ or $\overline{\gamma}(H) = 1$. If $\overline{\gamma}(G) = 1 = \overline{\gamma}(H)$, then we are done. Suppose that $\overline{\gamma}(H) \neq 1$. Then $\overline{\gamma}(G) = 1$. Now, let $\{v\}$ be a minimum $\overline{\gamma}^{-1}$-*set* in $G + H$. Then, in particular, $V(H) \subseteq N_{G+H}[v]$. Since $\overline{\gamma}(H) \geq 2$, $v \notin V(H)$. Thus $v \in V(G)$. Necessarily, $\{v\}$ is a $\overline{\gamma}$-*set* in G. Therefore, G has at least two $\overline{\gamma}$-*sets* and (ii) holds. Similarly, if $\overline{\gamma}(G) \neq 1$, then (iii) holds. ■

Corollary 3.4. Let G be any graph with no isolated vertex. Then $\overline{\gamma}^{-1}(G + H) = 1$. if and only if $G = K_p$, $p \geq 2$, or $G = H + K$ for some nontrivial graphs H and K satisfying one of the following:

(i) $\overline{\gamma}(H) = 1$ and $\overline{\gamma}(K) = 1$

(ii) $\overline{\gamma}(H) = 1$ and H has at least two minimum $\overline{\gamma}$-*sets*;

(iii) $\overline{\gamma}(K) = 1$ and K has at least two minimum $\overline{\gamma}$-*sets*.

Proof. First, note that $\overline{\gamma}^{-1}(K_p) = 1$ for all $p \geq 2$. Suppose that G is a noncomplete graph. Suppose, further, that $\overline{\gamma}^{-1}(K_p) = 1$. Then there exist two distinct vertices u and v of G such that $\{u\}$ and $\{v\}$ are $\overline{\gamma}$-*sets* in G. Moreover, $uv \in E(G)$. Put $H = \langle\{u, v\}\rangle$ and $K = \langle G - \{u, v\}\rangle$. Then $G = H + K$. Furthermore, $\{u\}$ and $\{v\}$ are two distinct $\overline{\gamma}$-*sets* in H. Consequently, (ii) holds.

The converse follows immediately from Theorem 3.3. ■

Proposition 3.5. Let G and H be nontrivial graphs. Then $\overline{\gamma}^{-1}(G + H) = 2$ if and only if any of the following is true:

(i) $\overline{\gamma}(G) \geq 2$ and $\overline{\gamma}(H) \geq 2$.

(ii) $\overline{\gamma}(G) = 1$ and $\overline{\gamma}(H) \geq 2$ but $G \neq K_1 + (K_1 + \bigcup_j G_j)$ for any graphs G_j.

Proof. Suppose that $\overline{\gamma}^{-1}(G + H) = 2$. Then either $\overline{\gamma}(G + H) = 1$ or $\overline{\gamma}(G + H) = 2$. It is clear that if $\overline{\gamma}(G + H) = 2$, then $\overline{\gamma}(G) \geq 2$ and $\overline{\gamma}(H) \geq 2$. Suppose that $\overline{\gamma}^{-1}(G + H) = 1$. Then $\overline{\gamma}(G) = 1$ or $\overline{\gamma}(H) = 1$. Assume that $\overline{\gamma}(G) = 1$. Then $G = \{v\} + \bigcup_j G_j$ for some components G_j of G. Thus,

$$\overline{\gamma}^{-1}(G + H) = \overline{\gamma}(H + \bigcup_j G_j) = 2.$$

Necessarily, $\overline{\gamma}(H) \geq 2$ and $\overline{\gamma}(\bigcup_j G_j) \geq 2$. This means that, in particular, $G \neq K_1 + (K_1 + \bigcup_j G_j)$.

To prove the converse, we first consider the case where $\overline{\gamma}(G) \geq 2$ and $\overline{\gamma}(H) \geq 2$. Then $\overline{\gamma}(G+H) = 2$. Since $\overline{\gamma}(G+H) \leq \overline{\gamma}^{-1}(G+H)$, then $\overline{\gamma}^{-1}(G+H) \geq 2$. Now pick $u \in V(G)$ and $v \in V(H)$, and let $x \in V(G) \setminus \{u\}$ and $y \in V(H) \setminus \{v\}$. Then $S = \{u, v\}$ is a minimum dominating set in $G + H$ so that $D = \{x, y\}$ is a $\overline{\gamma}^{-1}$-*set* in $G + H$. Thus $\overline{\gamma}^{-1}(G+H) \leq 2$. Accordingly, $\overline{\gamma}^{-1}(G+H) = 2$.

Next, we proceed with the case where $\overline{\gamma}(G) = 1$ and $\overline{\gamma}(H) \geq 2$ but $G \neq K_1 + [K_1 + \bigcup_j G_j]$. Let $S = \{u\} \subseteq V(G)$ be a closed dominating set in G. Then S is a closed dominating set in $G + H$. We consider

$$(G+H) - u = (G-u) + H.$$

The condition for G implies that $G - u \neq K_1 + \bigcup_j G_j$ for any components G_j of G. Thus, $\overline{\gamma}(G-u) \geq 2$. If $\overline{\gamma}(G-u) \geq 2$ and $\overline{\gamma}(H) \geq 2$, then $\overline{\gamma}^{-1}(G+H) = \overline{\gamma}((G-u)+H) = 2$. ■

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