

Limit Cycles of a Class of Generalized Liénard Polynomial Equations

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Abstract

In this paper we study the maximum number of limit cycles of the following generalized Liénard polynomial differential system of the first order

$$\begin{aligned}\dot{x} &= y^{2p-1} \\ \dot{y} &= -x^{2q-1} - \varepsilon f(x, y)\end{aligned}$$

where p and q are positive integers, ε is a small parameter and $f(x, y)$ is a polynomial of degree m . We prove that this maximum number depends on p , q and m .

AMS subject classification:

Keywords:

1. Introduction and statement of the main results

In 1900 Hilbert [9] in the second part of his 16th problem proposed to find a uniform upper bound for the number of limit cycles of all polynomial differential systems of a given degree and also to study their distribution or configuration in the plane. The generalized polynomial Liénard differential equation

$$\ddot{x} + f(x)\dot{x} + g(x) = 0 \tag{1}$$

was introduced in [13]. Here the dote denotes differentiation with respect to the time t , and $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m respectively.

For this subclass of polynomial vector fields we have a simplified version of Hilbert's problem, see [14] and [22].

Many results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single generate singular point, that are so called small amplitude limit cycles, see [17]. We denote by $\hat{H}(m, n)$ the maximum number of small amplitude limit cycles for systems of the form (1). The values of $\hat{H}(m, n)$ give a lower bound for the maximum number $H(m, n)$ (i. e. the Hilbert number) of limit cycles that the differential equation (1) can have with n and m fixed. For more information about the Hilbert's 16th problem see [10] and [11].

Now we shall describe briefly the main results about the limit cycles on Liénard differential systems. Let $[x]$ denotes the integer part function.

In 1928 Liénard [13] proved if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = 0$ and is monotone increasing for $x \geq 0$, then equation (1) has a unique limit cycle. In 1973 Rychkov [21] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is an odd polynomial of degree five, then equation (1) has at most two limit cycles. In 1977 Lins, de Melo and Pugh [14] proved that $H(1, 1) = 0$ and $H(1, 2) = 1$. In 1998 Coppel [4] proved that $H(2, 1) = 1$. Dumortier, Li and Rousseau in [7] and [5] proved that $H(3, 1) = 1$. In 1997 Dumortier and Chengzhi [6] proved that $H(2, 2) = 1$. Blows, Lloyd and Lynch [2, 19 and 20] proved by using inductive argument the following results : if g is odd then $\hat{H}(m, n) = [n/2]$; if f is even then $\hat{H}(m, n) = n$ whatever g is; if f is odd then $\hat{H}(m, n+1) = \left\lceil \frac{m-2}{2} \right\rceil + n$ and if $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m, 2) = m$. Christopher and Lynch [3, 20 and 21] have developped a new algebraic method for determining the Liapunov quantities of systems (1) and proved the following results: $\hat{H}(m, 2) = \left\lceil \frac{2m+1}{3} \right\rceil$; $\hat{H}(2, n) = \left\lceil \frac{2n+1}{3} \right\rceil$; $\hat{H}(m, 3) = 2 \left\lceil \frac{3m+2}{8} \right\rceil$ for all $1 \leq m \leq 50$; $\hat{H}(3, n) = 2 \left\lceil \frac{3n+2}{8} \right\rceil$ for all $1 \leq m \leq 50$; $\hat{H}(4, k) = \hat{H}(k, 4)$ for $k = 6, 7, 8, 9$ and $\hat{H}(5, 6) = \hat{H}(6, 5)$. In 1998 Gasull and Torregrosa [8] obtained upper bonds for $\hat{H}(6, 7)$, $\hat{H}(7, 6)$, $\hat{H}(7, 7)$ and $\hat{H}(4, 20)$. In 2006 Yu and Han in [25] proved that $\hat{H}(m, n) = \hat{H}(n, m)$ for $n = 4, m = 10, 11, 12, 13$; $n = 5, m = 6, 7, 8, 9$; $n = 6, m = 5, 6$. In 2009 Llibre, Mereu and Teixeira [16] by using the averaging theory studied the maximum number of limit cycles $\tilde{H}(m, n)$ which can bifurcate from periodic solutions of a linear center perturbed inside the class of all generalized polynomial Liénard differential equations of degrees m and n as follows

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -x - \sum_{k \geq 1} \varepsilon^k (f_n^k(x)y + g_m^k(x))\end{aligned}$$

where for every k the polynomials $f_n^k(x)$ and $g_m^k(x)$ have degrees n and m respectively, and ε is a small parameter and prove the following results: $\tilde{H}_1(m, n) = \left\lfloor \frac{n}{2} \right\rfloor$ (using the first order averaging theory), $\tilde{H}_2(m, n) = \max \left\{ \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{m}{2} \right\rfloor, \left\lfloor \frac{n}{2} \right\rfloor \right\}$ (using the second order averaging theory), $\tilde{H}_3(m, n) = \left\lfloor \frac{n+m-1}{2} \right\rfloor$ (using the third order averaging theory). In 2014, Llibre and Makhlof [15] proved that the generalized Liénard polynomial differential system

$$\begin{aligned}\dot{x} &= y^{2p-1} \\ \dot{y} &= -x^{2q-1} - \varepsilon f(x)y^{2n-1},\end{aligned}\tag{2}$$

where p, q and n are positive integers, ε is a small parameter and $f(x)$ is a polynomial of degree m can have $\left\lfloor \frac{m}{2} \right\rfloor$ limit cycles. System (2) with $p = q = n = 1$ was studied by Lins et al. [14] in 1977, and for $p = n = 1$ and q arbitrary has been studied by Urbina et al. [23] in 1993.

In this paper we want to study the maximum number of limit cycles of the following class of generalized Liénard polynomial differential system

$$\begin{aligned}\dot{x} &= y^{2p-1} \\ \dot{y} &= -x^{2q-1} - \varepsilon f(x, y)\end{aligned}\tag{3}$$

where p, q are positive integers, ε is a small parameter, $f(x, y)$ is a polynomial of degree m

$$f(x, y) = \sum_{i+j=0}^m a_{ij} x^i y^j.$$

Note that system (3) is more general than system (2).

System (3) with $\varepsilon = 0$ is an Hamiltonian system with Hamiltonian

$$H(x, y) = \frac{1}{2q} x^{2q} + \frac{1}{2p} y^{2p}.$$

This system has a global center at the origin of coordinates, i.e., the periodic orbits surrounding the origin filled the whole plane \mathbb{R}^2 , and we want to study how many periodic orbits persist after perturbing the periodic orbits of this center as in the system (3) for $\varepsilon = 0$ sufficiently small.

Let $[x]$ denotes the integer part function of $x \in \mathbb{R}$. Our main result is the following one.

Theorem 1.1. Let

$$l = \begin{cases} m & \text{if } m \text{ is odd} \\ m - 1 & \text{if } m \text{ is even.} \end{cases}$$

For $\varepsilon \neq 0$ sufficiently small, the maximum number of limit cycles of the polynomial differential system (3) is bounded by $\tilde{H}(p, q, l) = \left\lceil \frac{l \cdot \max(p, q) - q}{2} \right\rceil$.

Corollary 1.2. Let $p = q = 1$ and $f(x, y)$ is a polynomial of degree $m = 5$

$$\begin{aligned} f(x, y) &= \sum_{i+j=0}^5 a_{ij} x^i y^j \\ &= a_{00} + a_{10}x + a_{01}y + a_{21}x^2y + a_{12}xy^2 \\ &\quad + a_{03}y^3 + a_{40}x^4 + a_{23}x^2y^3 + a_{41}x^4y + a_{05}y^5. \end{aligned}$$

where

$$\begin{aligned} a_{00} &= 2, \quad a_{10} = -0.25, \quad a_{01} = 1.9, \quad a_{12} = 1.2, \quad a_{21} = -2.5, \\ a_{03} &= -1.2, \quad a_{23} = 0.8, \quad a_{40} = 3.2, \quad a_{41} = 0.6, \quad a_{05} = 0.2. \end{aligned}$$

For $\varepsilon \neq 0$ sufficiently small, system (3) has two limit cycles bifurcating from periodic solutions of the unperturbed system (for $\varepsilon = 0$). The bound is reached.

Corollary 1.3. Consider system (3) with $p = 1, q = 2, m = 6$

$$\begin{aligned} f(x, y) &= \sum_{i+j=0}^6 a_{ij} x^i y^j \\ &= a_{01}y + a_{21}x^2y + a_{03}y^3 + a_{23}x^2y^3 + a_{41}x^4y \\ &\quad + a_{05}y^5 + a_{33}x^3y^3 + a_{60}x^6 + a_{06}y^6. \end{aligned}$$

where

$$\begin{aligned} a_{01} &= 9.7, \quad a_{21} = -73.7, \quad a_{23} = -44.26, \quad a_{03} = 28.3, \\ a_{33} &= 2, \quad a_{41} = 14.1, \quad a_{05} = 2.07, \quad a_{60} = 0.1, \quad a_{06} = 4. \end{aligned}$$

For $\varepsilon \neq 0$ sufficiently small, system (3) has four limit cycles bifurcating from the periodic solutions of the unperturbed system. The bound is reached.

Theorems 1.1, Corollaries 1.2 and 1.3 are proved in section 2 by using the first order averaging theory. See the appendix for a summary of the results on averaging theory used here. Note that the maximum number of limit cycles obtained by using the averaging theory of the first order in [15] only depends on m the degree of $f(x)$. But our results obtained for the polynomial differential system (3) depend on p, q and the degree m .

2. Proof of Theorem 1.1

In [12], Liapunov introduced the (p, q) -trigonometric functions, $z(\theta) = Cs(\theta)$, $w(\theta) = Sn(\theta)$ as the solution of the following initial value problem

$$\begin{aligned}\dot{z} &= -w^{2p-1} \\ \dot{w} &= z^{2q-1} \\ z(0) &= p^{-\frac{1}{q}}, \quad w(0) = 0.\end{aligned}$$

It is easy to check that the functions $Cs(\theta)$ and $Sn(\theta)$ satisfy the equality

$$pCs^{2q}(\theta) + qSn^{2p}(\theta) = 1.$$

For $p = q = 1$ the (p, q) -trigonometric functions are the classical ones

$$Cs(\theta) = \cos \theta \text{ and } Sn(\theta) = \sin \theta.$$

It is known that $Cs(\theta)$ and $Sn(\theta)$ are T -periodic functions with

$$T = 2p^{-\frac{1}{2q}}q^{-\frac{1}{2p}} \frac{\Gamma\left(\frac{1}{2p}\right)\Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p} + \frac{1}{2q}\right)},$$

where Γ is the Gamma function. We consider the (p, q) -polar coordinates (r, θ) defined by

$$x = r^p Cs(\theta) \text{ and } y = r^q Sn(\theta).$$

System (3) in the coordinates (r, θ) can be written as

$$\begin{aligned}\dot{r} &= -\varepsilon r^{1-q} Sn^{2p-1}(\theta) f(r^p Cs(\theta), r^q Sn(\theta)) \\ \dot{\theta} &= -r^{2pq-p-q} - \varepsilon p r^{-q} Cs(\theta) f(r^p Cs(\theta), r^q Sn(\theta)).\end{aligned}\tag{4}$$

Taking the angular variable θ as the independent variable, system (4) becomes

$$\begin{aligned}\frac{dr}{d\theta} &= \varepsilon r^{-2pq+p+1} Sn^{2p-1}(\theta) f(r^p Cs(\theta), r^q Sn(\theta)) + O(\varepsilon^2) \\ &= \varepsilon F_1(\theta, R) + O(\varepsilon^2).\end{aligned}$$

We apply Theorem 4.1 (see appendix) with

$$\mathbf{x} = \mathbf{y} = r, \quad t = \theta, \quad F_1(t, \mathbf{x}) = F_1(\theta, r).$$

$$F_1(\theta, r) = r^{-2pq+p+1} Sn^{2p-1}(\theta) \sum_{i+j=0}^m a_{ij} r^{pi+qj} Cs^i(\theta) Sn^j(\theta).\tag{5}$$

Then according to (5) we obtain

$$\mathcal{F}(r) = r^{-2pq+p+1} \sum_{i+j=0}^m a_{ij} r^{pi+qj} \int_0^T C s^i(\theta) S n^{j+2p-1}(\theta) d\theta$$

then

$$\mathcal{F}(r) = r^{-2pq+p+1} \sum_{i+j=0}^m a_{ij} r^{pi+qj} I_{i,j+2p-1}$$

where

$$I_{i,j} = \int_0^T C s^i(\theta) S n^j(\theta) d\theta.$$

It is known that

$$\begin{aligned} I_{i,j} &= 0 \text{ if } i \text{ or } j \text{ is odd} \\ I_{i,j} &> 0 \text{ if } i \text{ and } j \text{ are even.} \end{aligned}$$

We put

$$l = \begin{cases} m & \text{if } m \text{ is odd} \\ m - 1 & \text{if } m \text{ is even,} \end{cases}$$

then

$$\begin{aligned} \mathcal{F}(r) &= r^{-2pq+p+1} \sum_{\substack{i+j=0 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+qj} \\ &= r^{-2pq+p+2} \sum_{\substack{i+j=1 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+qj-1} \end{aligned}$$

with

$$\tilde{a}_{ij} = a_{ij} I_{i,j+2p-1}.$$

Then

$$\begin{aligned}
 \mathcal{F}(r) &= r^{-2pq+p+2} \sum_{\substack{i+j=1 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+qj-1} \\
 &= r^{-2pq+p+2} (\tilde{a}_{01} r^{q-1} + \sum_{\substack{i+j=3 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+qj-1}) \\
 &= r^{-2pq+p+q+1} (\tilde{a}_{01} + \sum_{\substack{i+j=3 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+q(j-1)}) \\
 &= r^{-2pq+p+q+1} \sum_{\substack{i+j=1 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+q(j-1)}.
 \end{aligned}$$

In order to answer our problem, we should know how many positive solutions which can have the following algebraic equation

$$\mathcal{F}(r) = \sum_{\substack{i+j=1 \\ i: \text{even} \\ j: \text{odd}}}^l \tilde{a}_{ij} r^{pi+q(j-1)} = 0. \quad (6)$$

The degree of $\mathcal{F}(r)$ is the maximum of $pi + q(j - 1)$ with $i + j \leq l$, i is even and j is odd. We have

$$\begin{aligned}
 pi + qj - q &\leq \max(p, q).i + \max(p, q).j - q \\
 &\leq \max(p, q)(i + j) - q \\
 &\leq l. \max(p, q) - q.
 \end{aligned}$$

Then, the degree of $\mathcal{F}(r)$ is bounded by $l. \max(p, q) - q$. Since $r = 0$ is not a solution which can provide limit cycles we omit it. The variable r appears in the equation (6) through r^2 because $pi + q(j - 1)$ is even. So if r^* with $r^* \neq 0$ is a solution of (6) then

$-r^*$ is a solution too. We omit this last solution because r must be positive. If we take in account that we only are interested in solutions of the form $r^* > 0$, then the number of solutions of the equation (6) is bounded by $\left\lceil \frac{l \cdot \max(p, q) - q}{2} \right\rceil$ where l is given by

$$l = \begin{cases} m & \text{if } m \text{ is odd} \\ m - 1 & \text{if } m \text{ is even.} \end{cases}$$

Equivalently system (3) can have at most $\tilde{H}(p, q, l)$ limit cycles. This completes the proof of Theorem 1.2. ■

3. Proof of corollaries

3.1. Proof of corollary 1.2

Consider the polynomial differential system (3) with $p = q = 1$ and $m = 5$. Applying Theorem 1.1, we prove that system (3) can have at most $\left\lceil \frac{5 * 1 - 1}{2} \right\rceil = 2$ limit cycles.

We have

$$\mathcal{F}(r) = \tilde{a}_{01} + (\tilde{a}_{21} + \tilde{a}_{03})r^2 + (\tilde{a}_{23} + \tilde{a}_{41} + \tilde{a}_{05})r^4$$

where

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij} I_{i,j+1} \\ &= a_{ij} \int_0^T C s^i(\theta) S n^{j+1}(\theta) d\theta. \end{aligned}$$

For $p = q = 1$ the $(1, 1)$ -trigonometric functions are the classical ones

$$Cs(\theta) = \cos(\theta) \text{ and } Sn(\theta) = \sin(\theta) \text{ with } T = 2\pi.$$

Computing the integrals, we get

$$I_{0,2} = \pi, \quad I_{2,2} = \frac{1}{4}\pi, \quad I_{0,4} = \frac{3}{4}\pi, \quad I_{2,4} = I_{4,2} = \frac{1}{8}\pi, \quad I_{0,6} = \frac{5}{8}\pi$$

and

$$\begin{aligned} \tilde{a}_{01} &= 5.969026043, \quad \tilde{a}_{21} = -1.963495409, \quad \tilde{a}_{03} = -2.827433388, \\ \tilde{a}_{23} &= 0.3141592654, \quad \tilde{a}_{41} = 0.2356194490, \quad \tilde{a}_{05} = 0.3926990818. \end{aligned}$$

Then, according to Theorem 1.1, the algebraic equation $\mathcal{F}(r) = 0$ has two positive zeros

$$r_1 = 1.478399984 \text{ and } r_2 = 1.702253454$$

1839 which satisfy

$$\begin{aligned} \frac{d\mathcal{F}(r)}{dr} \Big|_{r=r_1} &= -1.98414430 \neq 0 \\ \frac{d\mathcal{F}(r)}{dr} \Big|_{r=r_2} &= 2.28457558 \neq 0. \end{aligned}$$

Equivalently, system (3) can have at most two limit cycles (see Figure 1). This completes the proof of Corollary 1.2. ■

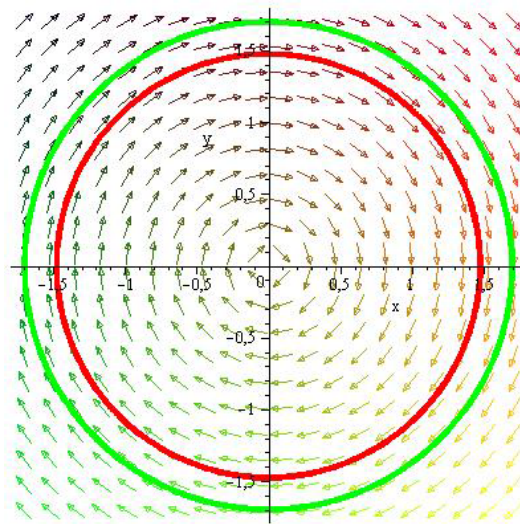


Figure 1: Two limit cycles for $\varepsilon = 0.001$.

3.2. Proof of corollary 1.3

Now, we have to apply Theorem 1.1 with $p = 1, q = 2, m = 6$. Then system (3) can have at most $\left\lfloor \frac{(6-1) * 2 - 2}{2} \right\rfloor = 4$ limit cycles. We have

$$\mathcal{F}(r) = \tilde{a}_{01} + \tilde{a}_{21}r^2 + (\tilde{a}_{03} + \tilde{a}_{41})r^4 + \tilde{a}_{23}r^6 + \tilde{a}_{05}r^8$$

where

$$\tilde{a}_{ij} = a_{ij}I_{i,j+1}.$$

In this case and according to Liapunov [12], $Cs(\theta)$ and $Sn(\theta)$ are the elliptic functions

$$Cs(\theta) = cn(\theta) \text{ and } Sn(\theta) = sn(\theta)dn(\theta) \text{ of modulus } \frac{1}{\sqrt{2}}$$

with the period

$$T = 2(p)^{-\frac{1}{2q}}(q)^{-\frac{1}{2p}} \frac{\Gamma\left(\frac{1}{2p}\right) \Gamma\left(\frac{1}{2q}\right)}{\Gamma\left(\frac{1}{2p} + \frac{1}{2q}\right)} = 7.416298712.$$

Computing the following integrals using an algebraic manipulation as Maple or Mathematica

$$I_{i,j+1} = \int_0^T cn^i(\theta)(sn(\theta)dn(\theta))^{j+1}d\theta.$$

we find

$$\begin{aligned} I_{0,2} &= 2.472099570, \quad I_{2,2} = .6777704678, \quad I_{0,4} = 1.059471244, \\ I_{2,4} &= 0.2259234893, \quad I_{4,2} = 0.3531570814, \quad I_{0,6} = 0.4815778383. \end{aligned}$$

Then

$$\begin{aligned} \tilde{a}_{01} &= 23.97936583, \quad \tilde{a}_{21} = -49.95168348, \quad \tilde{a}_{03} = 29.98303621, \\ \tilde{a}_{23} &= -9.999373636, \quad \tilde{a}_{41} = 4.979514848, \quad \tilde{a}_{05} = 0.9968661253. \end{aligned}$$

The equation $\mathcal{F}(r) = 0$ has four positive real roots given by

$$\begin{aligned} r_1 &= 0.9989866774, \quad r_2 = 1.438456003 \\ r_4 &= 1.67128889, \quad r_4 = 2.042173432. \end{aligned}$$

The derivatives of $\mathcal{F}(r)$ for these roots are

$$\begin{aligned} \left. \frac{d\mathcal{F}(r)}{dr} \right|_{r=r_1} &= -12.15098498 \\ \left. \frac{d\mathcal{F}(r)}{dr} \right|_{r=r_2} &= 4.6739814 \\ \left. \frac{d\mathcal{F}(r)}{dr} \right|_{r=r_3} &= -5.9652117 \\ \left. \frac{d\mathcal{F}(r)}{dr} \right|_{r=r_4} &= 37.382648. \end{aligned}$$

Since they are different from zero, we conclude that the differential system (3) has four limit cycles (see Figure 2). This completes the proof of Corollary 1.3. ■

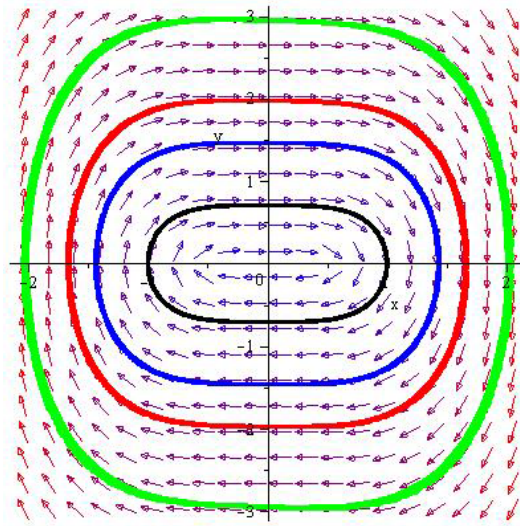
4. Appendix: Averaging theory of first order

We consider the initial value problems

$$\dot{\mathbf{x}} = \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (7)$$

and

$$\dot{\mathbf{y}} = \varepsilon \mathcal{F}(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \quad (8)$$

Figure 2: Four limit cycles for $\varepsilon = 0.0001$.

with x, y and x_0 in some open Ω of \mathbb{R}^n , $t \in [0, \infty)$, $\varepsilon \in (0, \varepsilon_0]$. We assume that F_1 and F_2 are periodic of period T in the variable t , and we set

$$\mathcal{F}(y) = \frac{1}{T} \int_0^T F_1(t, y) dt.$$

Theorem 4.1. Assume that F_1 , $D_x F_1$, $D_{xx} F_1$ and F_2 are continuous and bounded by a constant independent of ε in $[0, \infty) \times \Omega \times (0, \varepsilon_0]$, and that $y(t) \in \Omega$ for $t \in [0, 1/\varepsilon]$. Then the following statements holds:

1. For $t \in [0, 1/\varepsilon]$ we have $x(t) - y(t) = O(\varepsilon)$ as $\varepsilon \rightarrow 0$.
2. If $p \neq 0$ is a singular point of (8), then there exists a solution $\phi(t, \varepsilon)$ of period T for system (7) which is closed to p and such that $\phi(t, \varepsilon) - p = O(\varepsilon)$ as $\varepsilon \rightarrow 0$.
3. the stability of the periodic solution $\phi(t, \varepsilon)$ is given by the stability of the singular point.

We have used the notation $D_x \mathcal{F}$ for all the first derivatives of \mathcal{F} , and $D_{xx} \mathcal{F}$ for all the second derivatives of \mathcal{F} . For a proof of Theorem 2 see [24].

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