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Kannan - Type Fixed Point Theorem for Four Maps in Cone Pentagonal Metric Spaces

Abba Auwalu

Department of Mathematics,
Near East University,
Nicosia-TRNC, Mersin 10, Turkey.

Evren Hınçal

Department of Mathematics,
Near East University, Nicosia-TRNC,
Mersin 10, Turkey.

Abstract

In this paper, we prove Kannan - type fixed point theorem for four self mappings in non-normal cone pentagonal metric spaces. Our results extend and improve the recent results announced by many authors.

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1. Introduction

Let (X, d) be a metric space and $S : X \to X$ be a mapping. Then S is called Kannan contraction if there exists $\alpha \in [0, 1/2)$ such that

$$d(Sx, Sy) \leq \alpha\big[d(x, Sx) + d(y, Sy)\big], \quad \text{for all } x, y \in X. \tag{1}$$

Kannan [10] proved that if X is complete, then every Kannan contraction has a fixed point.

The study of existence and uniqueness of fixed points of a mapping and common fixed points of two or more mappings has become a subject of great interest. Many authors proved the Kannan contraction principle in various generalized metric spaces (e.g., see [2, 7, 8, 11]).

Long-Guang and Xian [7] introduced the concept of a cone metric space and proved some fixed point theorems for contractive type conditions in cone metric spaces. Later on many authors have (for e.g., [1, 3, 6, 13]) proved some fixed point theorems for different contractive types conditions in cone metric spaces.

Recently, Garg and Agarwal [6] introduced the notion of cone pentagonal metric space and proved Banach contraction mapping principle in a normal cone pentagonal metric space setting.

Motivated and inspired by the results of [6, 12], it is our purpose in this paper to continue the study of common fixed points for four self mappings in non-normal cone pentagonal metric space setting. Our results extend and improve the results of [2, 8, 11, 12], and many others.

2. Preliminaries

The following definitions and lemmas are needed in the sequel.

Definition 2.1. [7] Let E be a real Banach space and P subset of E. P is called a cone if and only if:

(1) P is closed, nonempty, and $P \neq \{0\}$;

(2) $a, b \in \mathbb{R}, \;\; a, b \geq 0$ and $x, y \in P \Longrightarrow ax + by \in P$;

(3) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Given a cone $P \subseteq E$, we defined a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in int(P)$, where $int(P)$ denotes the interior of P.

In this paper, we always suppose that E is a real Banach space and P is a cone in E with $int(P) \neq \emptyset$ and $\leq$ is a partial ordering with respect to P.

Definition 2.2. [7] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all $x, y, z \in X$.

Then ρ is called a cone metric on X, and (X, ρ) is called a cone metric space.

The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = [0, \infty)$ (e.g., see [7]).

Definition 2.3. [3] Let X be a nonempty set. Suppose the mapping $\rho : X \times X \to E$ satisfies:

(1) $0 < \rho(x, y)$ for all $x, y \in X$ and $\rho(x, y) = 0$ if and only if $x = y$;

(2) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;

(3) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y)$ for all $x, y, z \in X$ and for all distinct points $w, z \in X - \{x, y\}$ [Rectangular property].

Then ρ is called a cone rectangular metric on X, and (X, ρ) is called a cone rectangular metric space.

Remark 2.4. Every cone metric space is cone rectangular metric space. The converse is not necessarily true (e.g., see [3]).

Definition 2.5. [6] Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for $x, y \in X$;

(3) $d(x, y) \leq d(x, z) + d(z, w) + d(w, u) + d(u, y)$ for all $x, y, z, w, u \in X$ and for all distinct points $z, w, u, \in X - \{x, y\}$ [Pentagonal property].

Then d is called a cone pentagonal metric on X, and (X, d) is called a cone pentagonal metric space.

Remark 2.6. Every cone rectangular metric space and so cone metric space is cone pentagonal metric space. The converse is not necessarily true (e.g., see [6]).

Let (X, d) be a cone pentagonal metric space. Let $\{x_n\}$ be a sequence in (X, d) and $x \in X$. If for every $c \in E$ with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ and that for all $n > n_0$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$. If for every $c \in E$, with $0 \ll c$ there exist $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called Cauchy sequence in (X, d). If every Cauchy sequence is convergent in (X, d), then X is called a complete cone pentagonal metric space.

Let T and S be self maps of a nonempty set X. If $w = Tx = Sx$ for some $x \in X$, then x is called a coincidence point of T and S and w is called a point of coincidence of T and S. Also, T and S are said to be weakly compatible if they commute at their coincidence points, that is, $Tx = Sx$ implies that $TSx = STx$.

Lemma 2.7. **[1]** Let T and S be weakly compatible self mappings of nonempty set X. If T and S have a unique point of coincidence $w = Tx = Sx$, then w is the unique common fixed point of T and S.

Lemma 2.8. **[9]** Let (X, d) be a cone metric space with cone P not necessary to be normal. Then for $a, c, u, v, w \in E$, we have

(1) If $a \preceq ha$ and $h \in [0, 1)$, then $a = 0$.

(2) If $0 \preceq u \ll c$ for each $0 \ll c$, then $u = 0$.

(3) If $u \preceq v$ and $v \ll w$, then $u \ll w$.

(4) If $c \in int(P)$ and $a_n \to 0$, then $\exists n_0 \in \mathbb{N} : \forall n > n_0$, $a_n \ll c$.

Lemma 2.9. Let (X, d) be a complete cone pentagonal metric space. Let $\{x_n\}$ be a Cauchy sequence in X and suppose that there is natural number N such that:

1. $x_n \neq x_m$ for all $n, m > N$;
2. x_n, x are distinct points in X for all $n > N$;
3. x_n, y are distinct points in X for all $n > N$;
4. $x_n \to x$ and $x_n \to y$ as $n \to \infty$.

Then $x = y$.

3. Main Results

In this section, we prove Kannan - type theorem for four self mappings in cone pentagonal metric spaces. We give an example to illustrate the result.

Theorem 3.1. Let (X, d) be a cone pentagonal metric space. Suppose the mappings $f, g, U, V : X \to X$ satisfy the following contractive conditions:

(C1) $d(fx, gy) \preceq \lambda\big(d(fx, Ux) + d(gy, Vy)\big)$;

(C2) $d(fx, fy) \preceq \lambda\big(d(fx, Ux) + d(fy, Uy)\big)$;

(C3) $d(gx, gy) \preceq \lambda\big(d(gx, Vx) + d(gy, Vy)\big)$;

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$ and one of $f(X)$, $g(X)$, $U(X)$ or $V(X)$ is a complete subspace of X, then the pairs (f, U) and (g, V) have a unique point of coincidence in X. Moreover, if (f, U) and (g, V) are weakly compatible pairs then f, g, U and V have a unique common fixed point in X.

Proof. Let $x_0 \in X$. Since $f(X) \subseteq V(X)$ and $g(X) \subseteq U(X)$, starting with x_0, we define a sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n} = Vx_{2n+1} \text{ and } y_{2n+1} = gx_{2n+1} = Ux_{2n+2}, \text{ for all } n = 0, 1, 2, \ldots.$$

Suppose that $y_k = y_{k+1}$ for some $k \in \mathbb{N}$. If $k = 2m$, then $y_{2m} = y_{2m+1}$ for some $m \in \mathbb{N}$, then from (C1), we obtain

$$\begin{aligned} d(y_{2m+2}, y_{2m+1}) &= d(fx_{2m+2}, gx_{2m+1}) \\ &\leq \lambda\big(d(fx_{2m+2}, Ux_{2m+2}) + d(gx_{2m+1}, Vx_{2m+1})\big) \\ &\leq \lambda\big(d(y_{2m+2}, y_{2m+1}) + d(y_{2m+1}, y_{2m})\big) \\ &= \lambda d(y_{2m+2}, y_{2m+1}), \end{aligned}$$

which implies that $d(y_{2m+2}, y_{2m+1}) = 0$. That is, $y_{2m+2} = y_{2m+1}$.
In similar way, we can deduce that $y_{2m+2} = y_{2m+3} = y_{2m+4} = \cdots$.
Hence $y_n = y_k$, for all $n \geq k$. Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d). Now, assume that $y_n \neq y_{n+1}$, for all $n \in \mathbb{N}$. Then from (C1), we have

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &= d(fx_{2m}, gx_{2m+1}) \\ &\leq \lambda\big(d(fx_{2m}, Ux_{2m}) + d(gx_{2m+1}, Vx_{2m+1})\big) \\ &= \lambda\big(d(y_{2m}, y_{2m-1}) + d(y_{2m+1}, y_{2m})\big), \end{aligned}$$

which implies that

$$d(y_{2m}, y_{2m+1}) \leq \frac{\lambda}{1-\lambda} d(y_{2m-1}, y_{2m}) = \alpha d(y_{2m-1}, y_{2m}), \tag{2}$$

where $\alpha = \dfrac{\lambda}{1-\lambda} \in [0, 1)$. Also

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &= d(fx_{2m+1}, gx_{2m+2}) \\ &\leq \lambda\big(d(fx_{2m+2}, Ux_{2m+2}) + d(gx_{2m+1}, Vx_{2m+1})\big) \\ &= \lambda\big(d(y_{2m+2}, y_{2m+1}) + d(y_{2m+1}, y_{2m})\big), \end{aligned}$$

which implies that

$$d(y_{2m+1}, y_{2m+2}) \leq \frac{\lambda}{1-\lambda} d(y_{2m}, y_{2m+1}) = \alpha d(y_{2m}, y_{2m+1}). \tag{3}$$

From (2) and (3), it follows that

$$\begin{aligned} d(y_{2m}, y_{2m+1}) &\leq \alpha d(y_{2m-1}, y_{2m}) \\ &\leq \alpha^2 d(y_{2m-2}, y_{2m-1}) \\ &\vdots \\ &\leq \alpha^{2m} d(y_0, y_1), \ \forall m \geq 1, \end{aligned} \tag{4}$$

and

$$\begin{aligned} d(y_{2m+1}, y_{2m+2}) &\le \alpha d(y_{2m}, y_{2m+1}) \\ &\le \alpha^2 d(y_{2m-1}, y_{2m}) \\ &\vdots \\ &\le \alpha^{2m+1} d(y_0, y_1), \ \forall m \ge 1. \end{aligned} \tag{5}$$

Hence, from (4) and (5), we deduce that

$$d(y_n, y_{n+1}) \le \alpha^n d(y_0, y_1), \ \forall n \ge 1. \tag{6}$$

From (C2), (C3), (6) and the fact that $0 \le \lambda \le \alpha < 1$, we obtain

$$\begin{aligned} d(y_{2m}, y_{2m+2}) &= d(fx_{2m}, fx_{2m+2}) \\ &\le \lambda\big(d(fx_{2m}, Ux_{2m}) + d(fx_{2m+2}, Ux_{2m+2})\big) \\ &= \lambda\big(d(y_{2m}, y_{2m-1}) + d(y_{2m+2}, y_{2m+1})\big) \\ &\le \lambda\big(\alpha^{2m-1} d(y_0, y_1) + \alpha^{2m+1} d(y_0, y_1)\big) \\ &\le \alpha^{2m} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1) \\ &= (1 + \alpha^2)\alpha^{2m} d(y_0, y_1) \\ &\le (1 + \alpha)\alpha^{2m} d(y_0, y_1), \ \forall m \ge 1, \end{aligned} \tag{7}$$

and

$$\begin{aligned} d(y_{2m+1}, y_{2m+3}) &= d(gx_{2m+1}, gx_{2m+3}) \\ &\le \lambda\big(d(gx_{2m+1}, Vx_{2m+1}) + d(gx_{2m+3}, Vx_{2m+3})\big) \\ &\le \lambda\big(d(y_{2m+1}, y_{2m}) + d(y_{2m+3}, y_{2m+2})\big) \\ &\le \lambda\big(\alpha^{2m} d(y_0, y_1) + \alpha^{2m+2} d(y_0, y_1)\big) \\ &\le \alpha^{2m+1} d(y_0, y_1) + \alpha^{2m+3} d(y_0, y_1) \\ &= (1 + \alpha^2)\alpha^{2m+1} d(y_0, y_1) \\ &\le (1 + \alpha)\alpha^{2m} d(y_0, y_1), \ \forall m \ge 1. \end{aligned} \tag{8}$$

Hence, from (7) and (8), we have

$$d(y_n, y_{n+2}) \le (1 + \alpha)\alpha^n d(y_0, y_1), \ \forall n \ge 1. \tag{9}$$

For the sequence $\{y_n\}$, we consider $d(y_n, y_{n+p})$ in two cases as follows:

If p is odd say $p = 2k + 1$, where $k \geq 1$, then by pentagonal property and (6), we have

$$\begin{aligned}
d(y_n, y_{n+2k+1}) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+2k+1}) \\
&\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \cdots \\
&\quad + d(y_{n+2k-1}, y_{n+2k}) + d(y_{n+2k}, y_{n+2k+1}) \\
&\leq \alpha^n d(y_0, y_1) + \alpha^{n+1} d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \cdots \\
&\quad + \alpha^{n+2k-1} d(y_0, y_1) + \alpha^{n+2k} d(y_0, y_1) \\
&\leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n \geq 1.
\end{aligned}$$

If p is even say $p = 2k$, where $k \geq 1$, then by pentagonal property, (6) and (9), we have

$$\begin{aligned}
d(y_n, y_{n+2k}) &\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + d(y_{n+4}, y_{n+2k}) \\
&\leq d(y_n, y_{n+2}) + d(y_{n+2}, y_{n+3}) + d(y_{n+3}, y_{n+4}) + \cdots \\
&\quad + d(y_{n+2k-2}, y_{n+2k-1}) + d(y_{n+2k-1}, y_{n+2k}) \\
&\leq (1+\alpha)\alpha^n d(y_0, y_1) + \alpha^{n+2} d(y_0, y_1) + \alpha^{n+3} d(y_0, y_1) + \cdots \\
&\quad + \alpha^{n+2k-2} d(y_0, y_1) + \alpha^{n+2k-1} d(y_0, y_1) \\
&\leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n \geq 1.
\end{aligned}$$

Therefore, combining the above two cases, we get

$$d(y_n, y_{n+p}) \leq \frac{\alpha^n}{1-\alpha} d(y_0, y_1), \ \forall n, p \in \mathbb{N}. \tag{10}$$

Since $\alpha \in [0, 1)$, we get, as $n \to \infty$, $\frac{\alpha^n}{1-\alpha} \to 0$. Hence, for every $c \in E$ with $c \gg 0$, $\exists n_0 \in \mathbb{N}$ such that

$$d(y_n, y_{n+p}) \ll c, \ \text{for all } n \geq n_0.$$

Therefore, $\{y_n\}$ is a Cauchy sequence in (X, d). Suppose $U(X)$ is a complete subspace of X, there exists a points $p, q \in U(X)$ such that $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} U_{2n+2} = q = Up$.

Now, we show that $Up = fp$. Given $c \gg 0$, we choose a natural numbers M_1, M_2, M_3 such that $d(y_{2n+2}, q) \ll \frac{c(1-\lambda)}{4}, \ \forall n \geq M_1$, $d(y_{2n-1}, y_{2n}) \ll \frac{c(1-\lambda)}{4\lambda}, \ \forall n \geq M_2$ and $d(y_{2n}, y_{2n+1}) \ll \frac{c(1-\lambda)}{4}, \ \forall n \geq M_3$. Since $y_n \neq y_m$ for $n \neq m$, by pentagonal property and (C2), we have

$$\begin{aligned} d(fp,q) &\le d(fp,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &= d(fp,fx_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &\le \lambda\big(d(fp,Up) + d(fx_{2n},Ux_{2n})\big) + d(y_{2n},y_{2n+1}) \\ &\quad + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &= \lambda d(fp,q) + \lambda d(y_{2n},y_{2n-1}) + d(y_{2n},y_{2n+1}) \\ &\quad + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q), \end{aligned}$$

which implies that,

$$\begin{aligned} d(fp,q) &\le \frac{1}{1-\lambda}\big(\lambda d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1}) \\ &\quad + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q)\big) \\ &\ll \frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4} = c, \text{ for all } n \ge K_1, \end{aligned}$$

where $K_1 := \max\{M_1, M_2, M_3\}$. Since c is arbitrary, we have $d(fp,q) \ll \frac{c}{m}$, $\forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(fp,q) \to -d(fp,q)$ as $m \to \infty$. Since P is closed, $-d(fp,q) \in P$. Hence $d(fp,q) \in P \cap -P$. By definition of cone we get that $d(fp,q) = 0$, and so $Up = fp = q$. Hence, q is a point of coincidence of f and U.

Since $q = fp \in f(X)$ and $f(X) \subseteq V(X)$, there exists $r \in X$ such that $q = Vr$. Now, we show that $Vr = gr$. Given $c \gg 0$, we choose a natural numbers M_4, M_5, M_6 such that $d(y_{2n+2},q) \ll \frac{c(1-\lambda)}{4}$, $\forall n \ge M_4$, $d(y_{2n-1},y_{2n}) \ll \frac{c(1-\lambda)}{4\lambda}$, $\forall n \ge M_5$ and $d(y_{2n},y_{2n+1}) \ll \frac{c(1-\lambda)}{4}$, $\forall n \ge M_6$. Since $y_n \ne y_m$ for $n \ne m$, by pentagonal property and (C1), we have that

$$\begin{aligned} d(gr,q) &\le d(gr,y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &= d(gr,fx_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &\le \lambda\big(d(fx_{2n},Ux_{2n}) + d(gr,Vr)\big) + d(y_{2n},y_{2n+1}) \\ &\quad + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q) \\ &= \lambda d(y_{2n},y_{2n-1}) + \lambda d(gr,q) + d(y_{2n},y_{2n+1}) \\ &\quad + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q), \end{aligned}$$

which implies that,

$$\begin{aligned} d(gr,q) &\le \frac{1}{1-\lambda}\big(\lambda d(y_{2n-1},y_{2n}) + d(y_{2n},y_{2n+1}) + d(y_{2n+1},y_{2n+2}) + d(y_{2n+2},q)\big) \\ &\ll \frac{c}{4}+\frac{c}{4}+\frac{c}{4}+\frac{c}{4} = c, \text{ for all } n \ge K_2, \end{aligned}$$

where $K_2 := \max\{M_4, M_5, M_6\}$. Since c is arbitrary, we have $d(gr, q) \ll \frac{c}{m}, \ \forall m \in \mathbb{N}$. Since $\frac{c}{m} \to 0$ as $m \to \infty$, we conclude $\frac{c}{m} - d(gr, q) \to -d(gr, q)$ as $m \to \infty$. Since P is closed, $-d(gr, q) \in P$. Hence $d(gr, q) \in P \cap -P$. By definition of cone we get that $d(gr, q) = 0$, and so $Vr = gr = q$. Hence, q is a point of coincidence of g and V.

Thus, the pairs (f, U) and (g, V) have common point of coincidence q in X. Now, suppose the pairs (f, U) and (g, V) are weakly compatible mappings. Then

$$fq = fUp = Ufp = Uq = q_1, \text{ for some } q_1 \in X,$$

and

$$gq = gVr = Vgr = Vq = q_2, \text{ for some } q_2 \in X.$$

Hence, from (C1), we have

$$\begin{aligned} d(q_1, q_2) &= d(fq, gq) \\ &\leq \lambda\big(d(fq, Uq) + d(gq, Vq)\big) \\ &= \lambda\big(d(q_1, q_1) + d(q_2, q_2)\big) = 0. \end{aligned}$$

That is, $q_1 = q_2$. Therefore,

$$fq = gq = Uq = Vq.$$

Also,

$$\begin{aligned} d(q, gq) &= d(fp, gq) \\ &\leq \lambda\big(d(fp, Uq) + d(gq, Vq)\big) \\ &= \lambda\big(d(q, gq) + d(gq, gq)\big) \\ &\leq \lambda d(q, gq), \end{aligned}$$

which implies that

$$d(q, gq) = 0.$$

Hence, $gq = q$, or $fq = gq = Uq = Vq = q$. Thus, q is the common fixed point of f, g, U, and V. Next, we show that q is unique. For suppose q' be another common fixed point of f, g, U, and V. That is,

$$fq' = gq' = Uq' = Vq' = q',$$

for some $q' \in X$. Then from (C1), we have

$$\begin{aligned} d(q, q') &= d(fq, gq') \\ &\leq \lambda\big(d(fq, Uq) + d(gq', Vq')\big) \\ &= \lambda\big(d(fq, fq) + d(gq', gq')\big) = 0. \end{aligned}$$

Hence $q = q'$. Therefore, the mappings f, g, U and V have a unique common fixed point in X. Similarly, if $f(X)$, $g(X)$ or $V(X)$ is a complete subspace of X, then we can easily prove that f, g, U and V have unique common fixed point in X. This completes the proof of the theorem. ■

Remark 3.2. If P is a normal cone, and (X, d) a cone rectangular metric space in the above Theorem 3.1, then we get the Theorem 2.1 in [12].

The following example illustrates the result of Theorem 3.1.

Example 3.3. Let $X = \{1, 2, 3, 4, 5\}$, $E = \mathbb{R}^2$ and $P = \{(x, y) : x, y \geq 0\}$ is a cone in E. Define $d : X \times X \to E$ as follows:

$$\begin{aligned} d(x,x) &= 0, \forall x \in X; \\ d(1,2) &= d(2,1) = (4,16); \\ d(1,3) = d(3,1) = d(3,4) = d(4,3) &= d(2,3) = d(3,2) = d(2,4) \\ &= d(4,2) = d(1,4) = d(4,1) = (1,4); \\ d(1,5) = d(5,1) = d(2,5) &= d(5,2) = d(3,5) = d(5,3) = d(4,5) \\ &= d(5,4) = (5,20). \end{aligned}$$

Then (X, d) is a complete cone pentagonal metric space, but (X, d) is not a complete cone rectangular metric space because it lacks the rectangular property:

$$\begin{aligned} (4,16) = d(1,2) &> d(1,3) + d(3,4) + d(4,2) \\ &= (1,4) + (1,4) + (1,4) \\ &= (3,12), \text{ as } (4,16) - (3,12) = (1,4) \in P. \end{aligned}$$

Define a mapping $f, g, U, V : X \to X$ as follows:

$$f(x) = 4, \ \forall x \in X.$$

$$g(x) = \begin{cases} 4, & \text{if } x \neq 5; \\ 2, & \text{if } x = 5. \end{cases}$$

$$U(x) = \begin{cases} 3, & \text{if } x = 1; \\ 1, & \text{if } x = 2; \\ 2, & \text{if } x = 3; \\ 4, & \text{if } x = 4; \\ 5, & \text{if } x = 5. \end{cases}$$

$$V(x) = x, \ \forall x \in X.$$

Clearly $f(X) \subseteq V(X)$, $g(X) \subseteq U(X)$, and the pairs (f, U) and (g, V) are weakly compatible mappings. The conditions of Theorem 3.1 holds for all $x, y \in X$, where $\lambda = \frac{1}{5}$, and 4 is the unique common fixed point of the mappings f, g, U and V.

Now as corollaries, we recover, extend and generalize the recent results of [2, 11, 8], and many others in the literature, to a more general cone pentagonal metric space.

Corollary 3.4. Let (X, d) be a cone pentagonal metric space. Suppose the mappings $f, g, U : X \to X$ satisfies the contractive conditions:

(C1) $d(fx, gy) \leq \lambda\big(d(fx, Ux) + d(gy, Uy)\big)$;

(C2) $d(fx, fy) \leq \lambda\big(d(fx, Ux) + d(fy, Uy)\big)$;

(C3) $d(gx, gy) \leq \lambda\big(d(gx, Ux) + d(gy, Uy)\big)$;

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \cup g(X) \subseteq U(X)$, and if $U(X)$, or $f(X) \cup g(X)$ is a complete subspace of X, then the pairs (f, U) and (g, U) have a unique point of coincidence in X. Moreover, if (f, U) and (g, U) are weakly compatible pairs then f, g and U have a unique common fixed point in X.

Proof. Putting $V = U$ in Theorem 3.1. This completes the proof. ■

Corollary 3.5. Let (X, d) be a cone pentagonal metric space. Suppose the mappings $f, U : X \to X$ satisfies the contractive conditions:

$$d(fx, fy) \leq \lambda\big(d(fx, Ux) + d(fy, Uy)\big), \tag{11}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \subseteq U(X)$, and if $U(X)$, or $f(X)$ is a complete subspace of X, then the pair (f, U) have a unique point of coincidence in X. Moreover, if f and U is weakly compatible pairs then f and U have a unique common fixed point in X.

Proof. Putting $g = f$ and $V = U$ in Theorem 3.1. This completes the proof. ■

Corollary 3.6. (see [2]) Let (X, d) be a complete cone pentagonal metric space and P be a normal cone with normal constant k. Suppose the mapping $f : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \leq \lambda\big(d(x, fx) + d(y, fy)\big), \tag{12}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. f has a unique fixed point in X.

2. For any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

Proof. Putting $g = f$, $V = U = I$, and P is a normal cone in Theorem 3.1. This completes the proof. ■

Corollary 3.7. (see [11]) Let (X, d) be a cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mappings $f, g : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \le \lambda\big(d(gx, fx) + d(gy, fy)\big),$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Suppose that $f(X) \subseteq g(X)$, and $f(X)$ or $g(X)$ is a complete subspace of X, then the mappings f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible then f and g have a unique common fixed point in X.

Proof. This follows from the Remark 2.6, putting $g = f$, $V = U$, and P is a normal cone in Theorem 3.1. ■

Corollary 3.8. (see [8]) Let (X, d) be a complete cone rectangular metric space and P be a normal cone with normal constant k. Suppose the mapping $f : X \to X$ satisfies the contractive condition:

$$d(fx, fy) \le \lambda\big[d(x, fx) + d(y, fy)\big], \tag{13}$$

for all $x, y \in X$, where $\lambda \in [0, 1/2)$. Then

1. f has a unique fixed point in X.
2. For any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

Proof. This follows from the Remark 2.6, putting $g = f$, $V = U = I$, and P is a normal cone in Theorem 3.1. ■

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