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# Convergence results of implicit iterative scheme for two asymptotically quasi-I-nonexpansive mappings in Banach spaces

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**Abstract**

In this article, we consider an implicit iterative scheme for two asymptotically quasi-I-nonexpansive mappings $\mathcal{S}_1,\ \mathcal{S}_2$ and two asymptotically quasi-nonexpansive mapping $I_1,\ I_2$ in Banach space. We obtain convergence results for considered iteration to common fixed point of two asymptotically quasi-I-nonexpansive mappings, asymptotically quasi-nonexpansive mapping and equilibrium problem in frame work of real Banach spaces. A comparison table is prepared using a numeric example which shows that the proposed iterative algorithm is faster than known iterative algorithm by mathematica software. Our main results improve and compliment some known results.

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## 1. Introduction

Let $D$ be a nonempty subset of a real normed linear space $E$ and let $\mathcal{T} : D \to D$ be a mapping. Throughout this article, we assume that $\mathbb{N}$ is the set of natural numbers, we consider that $E$ is real Banach space and $F(\mathcal{T})$ is nonempty. Now, let us recall some known definitions.

**Definition 1.1.** Let $D$ be a nonempty closed convex subset of real Banach space $E$. A mapping $\mathcal{T} : D \to D$ is said to be:

(i) *nonexpansive* [4] if for all $x,\ y \in D$ and $F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,
$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|x - y\|,$$

(ii) *quasi-nonexpansive* [18] if for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,
$$\|\mathcal{T}x - q\| \leq \|x - q\|,$$

(iii) *uniformly $L$-Lipschitzian* if there exists a constant $L > 0$ such that, for any $x,\ y \in D$,
$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq L\|x - y\|, \quad \forall\, n \in \mathbb{N},$$

(iv) *asymptotically nonexpansive* [6] with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that, for all $x,\ y \in D$,
$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq k_n\|x - y\|, \quad \forall\, n \in \mathbb{N},$$

(v) *asymptotically quasi-nonexpansive* [10] with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,
$$\|\mathcal{T}^n x - q\| \leq k_n\|x - q\| \quad \forall\, n \in \mathbb{N}.$$

In 1916, Tricomi [18] introduced quasi-nonexpansive for real functions and later studied by Diaz and Metcalf [2] for mappings in Banach spaces. Ghosh and Debnath [5] established a necessary and sufficient condition for convergence of Ishikawa iterates of a quasi-nonexpansive mapping on a closed convex subset of a Banach space. In 1972, the class of asymptotically nonexpansive mappings was introduced as a generalization of the class of nonexpansive mappings by Goebel and Kirk [6]. In 2001, the class of asymptotically quasi-nonexpansive mapping was introduced as a generalization of the class of asymptotically nonexpansive mappings by Qihou [10]. Furthermore, it is easy to observe that, if $F(T) \neq \emptyset$, then a nonexpansive mapping must be quasi-nonexpansive and an asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping. But the converse implications need not be true.

There are many methods for approximating fixed point of a nonexpansive mapping. Xu and ori [19] introduced implicit iteration process to approximate a common fixed point of a finite family of nonexpansive mappings in a Hilbert space. After two years later, Sun [15] has extended an implicit iteration process for a finite of nonexpansive mappings, due to Xu and ori [19], to the case of asymptotically quasi-nonexpansive mappings in a setting of Banach spaces.

In 2006, Rhodes and Temir [12] are proved strong convergence result of Mann iteration for I-nonexpansive mapping. Temir and Gul [17] are proved a weakly convergence result for asymptotically I-nonexpansive mapping in Hilbert space. In [8] weak and strong convergence of an implicit iteration process for asymptotically quasi I- nonexpansive mapping in Banach space has been proved. Recently, in [20] implicit iteration process for approximating the common fixed points of two asymptotically quasi I- nonexpansive mappings were studied.

There are many concepts which generalize a notion of nonexpansive mapping. One of such is $I$-nonexpansivity of a mapping $\mathcal{T}$ [14]. Let us recall some notions.

**Definition 1.2.** Let $D$ be a nonempty closed convex subset of real Banach space $E$. A mapping $\mathcal{T},\ I : D \to D$ be two mappings of nonempty subset $D$ of a real normed linear space $E$. Then $\mathcal{T}$ is said to be:

(i) *I-nonexpansive* if for all $x, y \in D$ and $F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,
$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|Ix - Iy\|,$$

(ii) *asymptotically-I-nonexpansive* with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ such that, for all $x,\ y \in D$,
$$\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq k_n \|I^n x - I^n y\|, \quad \forall\, n \in \mathbb{N},$$

(iii) *asymptotically quasi-I-nonexpansive* with a sequence $\{k_n\} \subset [1, \infty)$ and $\lim_{n\to\infty} k_n = 1$ if, for all $x \in D$ and $q \in F(\mathcal{T}) = \{x \in D : \mathcal{T}x = x\}$, the set of fixed points of $\mathcal{T}$,
$$\|\mathcal{T}^n x - q\| \leq k_n \|I^n x - q\| \quad \forall\, n \in \mathbb{N}.$$

**Remark 1.3.** If $F(\mathcal{T}) \cap F(I)$ is nonempty then an asymptotically I-nonexpansive mapping is a asymptotically quasi-I-nonexpansive. But, there exists a nonlinear continuous asymptotically quasi I-nonexpansive mappings which is asymptotically I-nonexpansive.

Let $\phi$ be a bifunction of $D \times D$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $\phi : D \times D \to \mathbb{R}$ is to find $x \in D$ such that
$$\phi(x,\ y) \geq 0, \quad \forall\, y \in D. \tag{1}$$

The set of solutions of (1) is denoted by $EP(\phi)$. Given a mapping $\mathcal{T} : D \to D$, let $\phi(x,\ y) = (\mathcal{T}x,\ y - x)$ for all $x,\ y \in D$. For solving the equilibrium problem for a bifunction $\phi : D \times D \to \mathbb{R}$, let us assume that $\phi$ satisfies the following conditions:

(C1) $\phi(x, x) = 0$ for all $x \in D$,
(C2) $\phi$ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in D$,
(C3) for each $x, y, z \in D$,

$$\lim_{t \downarrow 0} \phi(tz + (1-t)x, y) \leq \phi(x, y),$$

(C4) for each $x \in D$, $y \mapsto \phi(x, y)$ is convex and lower semicontinuous.

In 2010, Farrukh and Saburov [8] used the following implicit iterative scheme to prove weak and strong convergence results for asymptotically quasi I-nonexpansive in Banach space,

$$\begin{cases} x_0 \in D, \\ x_n = (1-b_n)x_{n-1} + b_n \mathcal{S}^n y_n, \\ y_n = (1-c_n)x_n + c_n I^n x_n. \end{cases} \tag{2}$$

Motivated by above works, in this paper, we proposed a new implicit iteration scheme for approximating the common fixed points of asymptotically quasi I-nonexpansive mappings $\mathcal{S}_1, \mathcal{S}_2$, asymptotically quasi-nonexpansive mapping $I_1, I_2$ and equilibrium problem:

$$\begin{cases} x_0 \in D, \\ x_n = a_n x_{n-1} + b_n \mathcal{S}_1^n y_n + c_n \mathcal{S}_2^n x_n, \\ y_n = \widehat{a_n} x_n + \widehat{b_n} I_2^n x_n + \widehat{c_n} I_1^n x_n. \end{cases} \tag{3}$$

where

$$\{a_n\}, \{b_n\}, \{c_n\}, \{\widehat{a_n}\}, \{\widehat{b_n}\}, \{\widehat{c_n}\}$$

are six real sequences in $(0, 1)$ satisfying

$$a_n + b_n + c_n = 1 = \widehat{a_n} + \widehat{b_n} + \widehat{c_n}.$$

## 2. Preliminaries

Recall that a Banach space $E$ is said to satisfy Opial condition [9] if for each sequence $\{x_n\}$ in $E$ such that $\{x_n\}$ converges weakly to $x$ implies that

$$\lim_{n\to\infty} \inf \|x_n - x\| < \lim_{n\to\infty} \inf \|x_n - y\| \tag{4}$$

for all $y \in E$ with $y \neq x$. It is well know that [3] inequality (4) is equivalent to

$$\lim_{n\to\infty} \sup \|x_n - x\| < \lim_{n\to\infty} \sup \|x_n - y\|. \tag{5}$$

**Definition 2.1.** Let $E$ be a closed subset of a real Banach space $E$ and let $\mathcal{T} : D \to D$ be a mapping.

(i) A mapping $\mathcal{T}$ is said to be semi-closed(demi-closed) at zero, if for each bounded sequence $\{x_n\}$ in $D$, the conditions $x_n$ converges weakly to $x \in D$ and $\mathcal{T}x_n$ converges strongly to zero imply $\mathcal{T}x = 0$.

(ii) A mapping $\mathcal{T}$ is said to be semicompact, if for any bounded sequence $\{x_n\}$ in $D$ such that $\|x_n - \mathcal{T}x_n\| \to 0,\ n \to \infty$, then there exists a subsequence $\{x_{n_p}\} \subset \{x_n\}$ such that $x_{n_p} \to x^* \in D$ strongly. ∎

We restate the following lemmas which play key roles in our proofs.

**Lemma 2.2. [13]** Let $D$ be a uniformly convex Banach space and let $0 < \beta < \gamma < 1$. suppose that $\{t_n\}$ is a sequence in $[\beta,\ \gamma]$ and $\{x_n\},\ \{y_n\}$ are two sequence in $D$ such that

$$\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = d, \quad \lim_{n\to\infty} \sup \|x_n\| \le d, \quad \lim_{n\to\infty} \sup \|y_n\| \le d, \tag{6}$$

holds some $d \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

**Lemma 2.3. [16]** Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$. If one of the following conditions is satisfied:

(1) $\alpha_{n+1} \le \alpha_n + \beta_n,\ \ n \ge 1,$

(2) $\alpha_{n+1} \le (1 + \beta_n)\alpha_n,\ \ n \ge 1,$ then the limit $\lim_{n\to\infty} \alpha_n$ exists.

**Lemma 2.4. [1]** Let $D$ be a closed convex subset of a smooth, strictly convex and reflexive Banach space $E$ and $\phi$ be a bifunction of $D \times D$ into $\mathbb{R}$ satisfying (C1) - (C4), let $r > 0$ and $x \in E$. Then, there exists $z \in D$ such that

$$\phi(z,\ y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle \ge 0 \ \ \forall\, y \in D.$$

**Lemma 2.5. [11]** Let $D$ be a closed convex subset of a uniformly smooth, strictly convex and reflexive Banach space $E$ and $\phi$ be a bifunction of $D \times D$ into $\mathbb{R}$ satisfying (C1) - (C4). For $r > 0$ and $x \in E$, define a mapping $T_r : E \to D$ as follows:

$$T_r(x) = \left\{ z \in D : \phi(z,\ y) + \frac{1}{r}\langle y - z,\ Jz - Jx\rangle, \quad \forall\, y \in D \right\}$$

for all $x \in E$. Then, the following hold:

(1) $T_r$ is single-valued,

(2) $T_r$ is firmly nonexpansive-type mapping that is, for all $x,\ y \in E$,

$$\langle T_r x - T_r y,\ JT_r x - JT_r y\rangle \le \langle T_r x - T_r y,\ Jx - Jy\rangle,$$

(3) $\phi(T_r) = EP(\phi)$,

(4) $EP(\phi)$ is closed and convex.

**Lemma 2.6. [7]** Let $E$ be a uniformly convex Banach space satisfying the Opial's condition, $D$ be a nonempty closed subset of $E$ and $\mathcal{T} : D \to D$ an asymptotically nonexpansive mapping. If the sequence $\{x_n\} \subset D$ is a weakly convergent sequence with the limit $p$ and if $\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$, then $\mathcal{T}p = p$.

## 3. Main Results

**Lemma 3.1.** Let $E$ be a real Banach space and let $D$ be a nonempty closed convex subset of $E$. Let $\mathcal{S}_1,\ \mathcal{S}_2 : D \to D$ be two asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\},\ \{h_n\} \subset [1,\ \infty)$ and $I_1,\ I_2$ be two asymptotically quasi-nonexpansive self mapping of $D$ with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}} \{k_n,\ h_n,\ g_n,\ t_n\}$ and assume that $R = \sup_n(1 - a_n),\ \mathcal{M} = \sup_n \mu_n^2 \geq 1$ such that $\mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$ is nonempty and $q^* \in \mathcal{F}$. And $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ which satisfy the following conditions:

(i) $R(\mathcal{M}^2 + \mathcal{M}) < 1$,

(ii) $\sum_{n=1}^{\infty}(1 - a_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$.

If $\{x_n\}$ is the implicit iterative sequence defined by (3), then

(1) $\lim_{n\to\infty} \|x_n - q^*\|$ exists for each $q^* \in \mathcal{F}$.

(2) The sequence $\{x_n\}$ generated by (3) converges strongly to common fixed point in $\mathcal{F}$ if and only if $\lim \inf_{n\to\infty} d(x_n,\ \mathcal{F}) = 0$.

*Proof.* As $q^* \in \mathcal{F}$, it follows from (3) that

$$\begin{aligned}
\|x_n - q^*\| &= \|a_n x_{n-1} + b_n \mathcal{S}_1^n y_n + c_n \mathcal{S}_2^n x_n - q^*\| \\
&\leq a_n\|x_{n-1} - q^*\| + b_n\|\mathcal{S}_1^n y_n - q^*\| + c_n\|\mathcal{S}_2^n x_n - q^*\| \\
&\leq a_n\|x_{n-1} - q^*\| + b_n k_n\|I_1^n y_n - q^*\| + c_n h_n\|I_2^n x_n - q^*\| \\
&\leq a_n\|x_{n-1} - q^*\| + (1 - a_n - c_n)k_n g_n\|y_n - q^*\| \\
&\quad + (1 - a_n - b_n)h_n t_n\|x_n - q^*\| \\
&\leq a_n\|x_{n-1} - q^*\| + (1 - a_n)\mu_n^2\|y_n - q^*\| + (1 - a_n)\mu_n^2\|x_n - q^*\| \qquad (7)
\end{aligned}$$

Again from (3), we obtain

$$\begin{aligned}\|y_n - q^*\| &= \|\widehat{a_n}x_n + \widehat{b_n}I_2^n x_n + \widehat{c_n}I_1^n x_n - q^*\|\\ &\le \widehat{a_n}\|x_n - q^*\| + \widehat{b_n}t_n\|x_n - q^*\| + \widehat{c_n}g_n\|x_n - q^*\|\\ &\le \widehat{a_n}\|x_n - q^*\| + \widehat{b_n}\mu_n\|x_n - q^*\| + \widehat{c_n}\mu_n\|x_n - q^*\|\\ &\le \widehat{a_n}\mu_n^2\|x_n - q^*\| + \widehat{b_n}\mu_n^2\|x_n - q^*\| + \widehat{c_n}\mu_n^2\|x_n - q^*\|\\ &\le \mu_n^2\|x_n - q^*\| \end{aligned} \tag{8}$$

Then from (7), we get

$$\begin{aligned}\|x_n - q^*\| &\le a_n\|x_{n-1} - q^*\| + (1-a_n)\mu_n^4\|x_n - q^*\| + (1-a_n)\mu_n^2\|x_n - q^*\|\\ &\le a_n\|x_{n-1} - q^*\| + (1-a_n)(\mu_n^4 + \mu_n^2)\|x_n - q^*\|\end{aligned}$$

By transposing, we have

$$[1 - (1-a_n)(\mu_n^4 + \mu_n^2)]\|x_n - q^*\| \le a_n\|x_{n-1} - q^*\| \tag{9}$$

By condition (i) we obtain $(1-a_n)(\mu_n^4+\mu_n^2) \le \sup(1-a_n)\sup(\mu_n^4+\mu_n^2) = R(\mathcal{M}^2 + \mathcal{M}) < 1$,
and therefore

$$1 - (1-a_n)(\mu_n^4 + \mu_n^2) \ge 1 - R(\mathcal{M}^2 + \mathcal{M}) > 0.$$

Therefore (9) we take

$$\begin{aligned}\|x_n - q^*\| &\le \frac{a_n}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\|x_{n-1} - q^*\|\\ &\le \left[1 + \frac{a_n + (1-a_n)(\mu_n^4+\mu_n^2) - 1}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\right]\|x_{n-1} - q^*\|\\ &\le \left[1 + \frac{(1-a_n)(\mu_n^4+\mu_n^2 - 1)}{1-(1-a_n)(\mu_n^4+\mu_n^2)}\right]\|x_{n-1} - q^*\|\\ &\le \left[1 + \frac{(1-a_n)(\mu_n^4+\mu_n^2 - 1)}{1-R(\mathcal{M}^2+\mathcal{M})}\right]\|x_{n-1} - q^*\|\end{aligned}$$

Let

$$\beta_n = \frac{(1-a_n)(\mu_n^4+\mu_n^2-1)}{1-R(\mathcal{M}^2+\mathcal{M})}.$$

Then the last inequality can be written as follows:

$$\|x_n - q^*\| \le (1+\beta_n)\|x_{n-1} - q^*\|. \tag{10}$$

From condition (ii) we find

$$\sum_{n=1}^{\infty}\beta_n = \frac{1}{1-R(\mathcal{M}^2+\mathcal{M})}\sum_{n=1}^{\infty}(1-a_n)(\mu_n^4+\mu_n^2-1) < \infty.$$

Now taking $\alpha_n = \|x_{n-1} - q^*\|$ in (10) we obtain

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n$$

and according to Lemma 2.3 the limit $\lim_{n\to\infty} \alpha_n$ exists. This means the limit

$$\lim_{n\to\infty} \|x_n - q^*\| = d \tag{11}$$

exists, where $d \geq 0$ is a constant.

It follows from (10) that

$$d(x_n, \ F) \leq (1 + \beta_n) d(x_{n-1}, \ F).$$

So from Lemma 2.3, we obtain $\lim_{n\to\infty} d(x_n, \ F)$ exists. Furthermore, since $\liminf_{n\to\infty} d(x_n, \ F) = 0$, then $\lim_{n\to\infty} d(x_n, \ F) = 0$. Next we show that $\{x_n\}$ is a Cauchy sequence in $D$. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, \ F) = 0$, there exists a positive integer $m_0$ such that

$$d(x_n, \ F) < \frac{\epsilon}{2}, \quad \forall \ \ n \geq m_0.$$

In particular, $\inf\{\|x_{m_0} - q\| : \ q \in F\} < \frac{\epsilon}{2}$. Thus there must exists $q \in F$ such that

$$\|x_{m_0} - q\| < \frac{\epsilon}{2}.$$

Now, for all $m, \ n \geq m_0$, we have

$$\begin{aligned} \|x_{m+n} - x_n\| &\leq \|x_{m+n} - q\| + \|x_n - q\| \\ &\leq 2\|x_{m_0} - q\| \\ &\leq 2\Big(\frac{\epsilon}{2}\Big) = \epsilon. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in a closed subset $D$ of a Banach space $E$ and so it must converge to a point $q^*$ in $D$. And $\lim_{n\to\infty} d(x_n, \ F) = 0$ gives that $d(q^*, \ F) = 0$. By the routine proof, we know $F$ is a closed subset of $D$. Thus $q^* \in F$. This completes the proof. ■

**Theorem 3.2.** Let $E$ be a real uniformly convex Banach space and let $D$ be a nonempty closed convex subset of $E$. Let $\phi : D \times D \to \mathbb{R}$ be a bifunction which satisfy the conditions (C1)-(C4). Let $\mathcal{S}_1, \ \mathcal{S}_2 : D \to D$ be two uniformly $L_1$ and $L_2$- Lipschitizian asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\}, \ \{h_n\} \subset [1, \ \infty)$ and $I_1, \ I_2$ be two uniformly $L_3$ and $L_4$- Lipschitizian asymptotically quasi-nonexpansive self mapping of $D$ with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}} \{k_n,\ h_n,\ g_n,\ t_n\}$ and we assume that $R = \sup_n(1-a_n),\ \mathcal{M} = \sup_n \mu_n^2 \geq 1$ such that $\mathcal{F} = F(\mathcal{S}_1)\cap F(\mathcal{S}_2)\cap F(I_1)\cap F(I_2)\cap EP(\phi)$ is nonempty and $q^* \in \mathcal{F}$. For an initial point $x_0 \in D$, generate a sequence $\{x_n\}$ by $v_n \in D$ such that

$$\begin{cases} \phi(v_n,\ y) + \dfrac{1}{r_n}\langle y - v_n,\ Jv_n - Jx_n\rangle \geq 0, \quad \forall\ y \in D, \\ x_n = a_n x_{n-1} + b_n \mathcal{S}_1^n y_n + c_n \mathcal{S}_2^n x_n, \\ y_n = \widehat{a_n} x_n + \widehat{b_n} I_2^n x_n + \widehat{c_n} I_1^n x_n, \quad \forall\ n \geq 1, \end{cases} \tag{12}$$

where $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ satisfying $a_n + b_n + c_n = 1 = \widehat{a_n} + \widehat{b_n} + \widehat{c_n}$ and $\{r_n\} \subset [\rho,\ \infty)$ for $\rho > 0$, which satisfy the following conditions:

(i) $R(\mathcal{M}^2 + \mathcal{M}) < 1$,

(ii) $\sum_{n=1}^{\infty}(1 - a_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$,

(iii) $\lim\inf_{n\to\infty} r_n > 0$.

Then the implicit iterative sequence $\{x_n\}$ defined by (12), satisfies the following :

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1 x_n\| = \lim_{n\to\infty} \|x_n - \mathcal{S}_2 x_n\| = \lim_{n\to\infty} \|x_n - I_1 x_n\| = \lim_{n\to\infty} \|x_n - I_2 x_n\| = 0.$$

*Proof.* We divide the proof into two steps.

*Step 1.* First, we will prove that

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1^n x_n\| = \lim_{n\to\infty} \|x_n - \mathcal{S}_2^n x_n\| = \lim_{n\to\infty} \|x_n - I_1^n x_n\| = \lim_{n\to\infty} \|x_n - I_2^n x_n\| = 0.$$

According to Lemma 3.1, we know that $\lim_{n\to\infty} \|x_n - q^*\|$ exists for any $q^* \in F$. We have suppose that $\lim_{n\to\infty} \|x_n - q^*\| = d$. It follows from (12) that

$$\begin{aligned} \lim_{n\to\infty} \|x_n - q^*\| &= \lim_{n\to\infty} \|a_n x_{n-1} + b_n \mathcal{S}_1^n y_n + c_n \mathcal{S}_2^n x_n - q^*\| \\ &= \lim_{n\to\infty} \|a_n(x_{n-1} - q^*) + b_n(\mathcal{S}_1^n y_n - q^*) + c_n(\mathcal{S}_2^n x_n - q^*)\| \\ &= \lim_{n\to\infty} \Big\| a_n(x_{n-1} - q^*) + (1 - a_n)\Big[\frac{b_n}{1 - a_n}(\mathcal{S}_1^n y_n - q^*) \\ &\qquad + \frac{c_n}{1 - a_n}(\mathcal{S}_2^n x_n - q^*)\Big]\Big\| = d. \end{aligned} \tag{13}$$

It follows from $\lim_{n\to\infty} \|x_n - q^*\| = d$ that $\lim_{n\to\infty} \|x_{n-1} - q^*\| = d$. Taking lim sup on both sides, we have

$$\lim_{n\to\infty} \sup \|x_{n-1} - q^*\| = 0. \tag{14}$$

In addition, from (13), we have

$$\begin{aligned}
&\lim_{n\to\infty} \sup \left\| \frac{b_n}{1-a_n}(\mathcal{S}_1^n y_n - q^*) + \frac{c_n}{1-a_n}(\mathcal{S}_2^n x_n - q^*) \right\| \\
&\leq \lim_{n\to\infty} \sup \left[ \frac{b_n}{1-a_n} \|\mathcal{S}_1^n y_n - q^*\| + \frac{c_n}{1-a_n} \|\mathcal{S}_2^n x_n - q^*\| \right] \\
&\leq \lim_{n\to\infty} \sup \left[ \frac{b_n}{1-a_n} \mu_n^2 \|y_n - q^*\| + \frac{c_n}{1-a_n} \mu_n^2 \|x_n - q^*\| \right] \\
&\leq \lim_{n\to\infty} \sup \left[ \frac{b_n}{1-a_n} \mu_n^4 \|x_n - q^*\| + \frac{c_n}{1-a_n} \mu_n^4 \|x_n - q^*\| \right] \\
&\leq \lim_{n\to\infty} \sup \left[ \frac{\mu_n^4}{1-a_n} (b_n + c_n) \|x_n - q^*\| \right] \\
&= \lim_{n\to\infty} \sup \mu_n^4 \|x_n - q^*\| = d.
\end{aligned} \tag{15}$$

From (13), (14), (15) and Lemma 2.2, we get

$$\begin{aligned}
&\lim_{n\to\infty} \left\| (x_{n-1} - q^*) - \left[ \frac{b_n}{1-a_n}(\mathcal{S}_1^n y_n - q^*) + \frac{c_n}{1-a_n}(\mathcal{S}_2^n x_n - q^*) \right] \right\| \\
&= \lim_{n\to\infty} \left( \frac{1}{1-a_n} \right) \|(1-a_n)(x_{n-1} - q^*) - b_n(\mathcal{S}_1^n y_n - q^*) - c_n(\mathcal{S}_2^n x_n - q^*)\| \\
&= \lim_{n\to\infty} \left( \frac{1}{1-a_n} \right) \|x_n - x_{n-1}\| = 0.
\end{aligned}$$

Since the sequence $\{a_n\}$ in $(0,\ 1)$, there are some constants $a,\ b \in (0,\ 1)$ such that $0 < a \leq a_n < b < 1$. Therefore, we have

$$\lim_{n\to\infty} \|x_n - x_{n-1}\| = 0. \tag{16}$$

On the other hand, from (13), we have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q^*\| = \lim_{n\to\infty} \Big\| b_n(\mathcal{S}_1^n y_n - q^*) + (1-b_n)\Big[ &\frac{a_n}{1-b_n}(x_{n-1} - q^*) \\
&+ \frac{c_n}{1-b_n}(\mathcal{S}_2^n x_n - q^*) \Big] \Big\| = d.
\end{aligned} \tag{17}$$

Since $\mathcal{S}_1$ is uniformly $L_1$-Lipschitizian asymptotically quasi-nonexpansive mapping, we have

$$\lim_{n\to\infty} \|\mathcal{S}_1^n y_n - q^*\| \leq \mu_n^2 \|y_n - q^*\| \leq \mu_n^4 \|x_n - q^*\|.$$

Taking lim sup on both sides, we take

$$\lim_{n\to\infty}\sup \|\mathcal{S}_1^n y_n - q^*\| \leq d, \tag{18}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\sup \left\| \frac{a_n}{1-b_n}(x_n - q^*) + \frac{c_n}{1-b_n}(\mathcal{S}_2^n x_n - q^*) \right\| \\
&\leq \lim_{n\to\infty}\sup \left[ \frac{a_n}{1-b_n}(x_n - q^*) + \frac{c_n}{1-b_n}\mu_n^2\|x_n - q^*\| \right] \\
&\leq \lim_{n\to\infty}\sup \left[ \frac{a_n}{1-b_n}\mu_n^2\|x_n - q^*\| + \frac{c_n}{1-b_n}\mu_n^2\|x_n - q^*\| \right] \\
&\leq \lim_{n\to\infty}\sup \left[ \frac{\mu_n^2}{1-b_n}(a_n + c_n)\|x_n - q^*\| \right] \\
&= \lim_{n\to\infty}\sup \mu_n^2\|x_n - q^*\| = d.
\end{aligned} \tag{19}$$

From (17), (18), (19) and Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1^n y_n\| = 0. \tag{20}$$

Similarly, we obtain

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_2^n x_n\| = 0. \tag{21}$$

From (16) and (20), we take

$$\lim_{n\to\infty} \|x_{n-1} - \mathcal{S}_1^n y_n\| \leq \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - \mathcal{S}_1^n y_n\| = 0. \tag{22}$$

Consider

$$\begin{aligned}
\|x_{n-1} - q^*\| &\leq \|x_{n-1} - \mathcal{S}_1^n y_n\| + \|\mathcal{S}_1^n y_n - q^*\| \\
&\leq \|x_{n-1} - \mathcal{S}_1^n y_n\| + k_n g_n\|y_n - q^*\| \\
&\leq \|x_{n-1} - \mathcal{S}_1^n y_n\| + \mu_n^2\|y_n - q^*\|,
\end{aligned}$$

by transporting, we take

$$\|x_{n-1} - q^*\| - \|x_{n-1} - \mathcal{S}_1^n y_n\| \leq \mu_n^4\|x_n - q^*\|.$$

From (13) and (22) with squeeze theorem yield

$$\lim_{n\to\infty} \|y_n - q^*\| = d. \tag{23}$$

Again from (12), we can see that

$$\begin{aligned}\lim_{n\to\infty}\|y_n-q^*\| &= \lim_{n\to\infty}\|\widehat{a_n}x_n+\widehat{b_n}I_2^nx_n+\widehat{c_n}I_1^nx_n-q^*\|\\ &= \lim_{n\to\infty}\|\widehat{a_n}(x_n-q^*)+\widehat{b_n}(I_2^nx_n-q^*)+\widehat{c_n}(I_1^nx_n-q^*)\|\\ &= \lim_{n\to\infty}\Big\|\widehat{a_n}(x_n-q^*)+(1-\widehat{a_n})\Big[\frac{\widehat{b_n}}{1-\widehat{a_n}}(I_2^nx_n-q^*)\\ &\qquad+\frac{\widehat{c_n}}{1-\widehat{a_n}}(I_1^nx_n-q^*)\Big]\Big\| = d. \end{aligned} \tag{24}$$

In addition, from (24), we get

$$\begin{aligned}&\lim_{n\to\infty}\sup\Big\|\frac{\widehat{b_n}}{1-\widehat{a_n}}(I_2^nx_n-q^*)+\frac{\widehat{c_n}}{1-\widehat{a_n}}(I_1^nx_n-q^*)\Big\|\\ &\le \lim_{n\to\infty}\sup\Big[\frac{\widehat{b_n}}{1-\widehat{a_n}}\|I_2^nx_n-q^*\|+\frac{\widehat{c_n}}{1-\widehat{a_n}}\|I_1^nx_n-q^*\|\Big]\\ &\le \lim_{n\to\infty}\sup\Big[\frac{\widehat{b_n}}{1-\widehat{a_n}}\mu_n^2\|x_n-q^*\|+\frac{\widehat{c_n}}{1-\widehat{a_n}}\mu_n^2\|x_n-q^*\|\Big]\\ &\le \lim_{n\to\infty}\sup\Big[\frac{\mu_n^2}{1-a_n}(\widehat{b_n}+\widehat{c_n})\|x_n-q^*\|\Big]\\ &= \lim_{n\to\infty}\sup\mu_n^2\|x_n-q^*\| = d.\end{aligned} \tag{25}$$

From (11), (24), (25) and Lemma 2.2, we get

$$\begin{aligned}&\lim_{n\to\infty}\Big\|(x_n-q^*)-\Big[\frac{\widehat{b_n}}{1-\widehat{a_n}}(I_2^nx_n-q^*)+\frac{\widehat{c_n}}{1-\widehat{a_n}}(I_1^nx_n-q^*)\Big]\Big\|\\ &= \lim_{n\to\infty}\Big(\frac{1}{1-\widehat{a_n}}\Big)\|(1-\widehat{a_n})(x_n-q^*)-\widehat{b_n}(I_2^nx_n-q^*)-\widehat{c_n}(I_1^nx_n-q^*)\|\\ &= \lim_{n\to\infty}\Big(\frac{1}{1-\widehat{a_n}}\Big)\|x_n-y_n\| = 0.\end{aligned}$$

Since the sequence $\{\widehat{a_n}\}$ in $(0,\ 1)$, there are some constants $a,\ b \in (0,\ 1)$ such that $0 < a \le \widehat{a_n} < b < 1$. Therefore, we have

$$\lim_{n\to\infty}\|x_n-y_n\| = 0. \tag{26}$$

In a similar way, we obtain

$$\lim_{n\to\infty}\|x_n-I_1^nx_n\| = 0. \tag{27}$$

and

$$\lim_{n\to\infty}\|x_n-I_2^nx_n\| = 0. \tag{28}$$

From (16), (22) and (26), we take

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1^n x_n\| \le \lim_{n\to\infty} [\|x_n - x_{n-1}\| + \|x_{n-1} - \mathcal{S}_1^n y_n\| + \|\mathcal{S}_1^n y_n - \mathcal{S}_1^n x_n\|]$$
$$\le \lim_{n\to\infty} [\|x_n - x_{n-1}\| + \|x_{n-1} - \mathcal{S}_1^n y_n\| + L_1\|y_n - x_n\|] = 0. \tag{29}$$

Consider, from (16) and (27), we take

$$\lim_{n\to\infty} \|x_{n-1} - \mathcal{S}_2^n x_n\| \le \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - \mathcal{S}_2^n x_n\| = 0. \tag{30}$$

From (16) and (21), we take

$$\lim_{n\to\infty} \|x_{n-1} - I_1^n x_n\| \le \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - I_1^n x_n\| = 0. \tag{31}$$

and by (16) and (28), we take

$$\lim_{n\to\infty} \|x_{n-1} - I_2^n x_n\| \le \lim_{n\to\infty} \|x_{n-1} - x_n\| + \lim_{n\to\infty} \|x_n - I_2^n x_n\| = 0. \tag{32}$$

Finally, we get

$$\begin{aligned}
\|x_n - \mathcal{S}_1 x_n\| &\le \|x_n - \mathcal{S}_1^n x_n\| + \|\mathcal{S}_1^n x_n - \mathcal{S}_1 x_n\| \\
&\le \|x_n - \mathcal{S}_1^n x_n\| + L_1\|\mathcal{S}_1^{n-1} x_n - x_n\| \\
&\le \|x_n - \mathcal{S}_1^n x_n\| + L_1\big[\|\mathcal{S}_1^{n-1} x_n - \mathcal{S}_1^{n-1} x_{n-1}\| \\
&\quad + \|\mathcal{S}_1^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\big] \\
&\le \|x_n - \mathcal{S}_1^n x_n\| + L_1\big[L_1\|x_n - x_{n-1}\| \\
&\quad + \|\mathcal{S}_1^{n-1} x_{n-1} - x_{n-1}\| + \|x_{n-1} - x_n\|\big] \\
&\le \|x_n - \mathcal{S}_1^n x_n\| + L_1(L_1+1)\|x_n - x_{n-1}\| + L_1\|\mathcal{S}_1^{n-1} x_{n-1} - x_{n-1}\|
\end{aligned}$$

with (16) and (29), we obtain

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1 x_n\| = 0. \tag{33}$$

Analogously, one has

$$\|x_n - \mathcal{S}_2 x_n\| \le \|x_n - \mathcal{S}_2^n x_n\| + L_2(L_2+1)\|x_n - x_{n-1}\| + L_2\|\mathcal{S}_2^{n-1} x_{n-1} - x_{n-1}\|,$$

$$\|x_n - I_1 x_n\| \le \|x_n - I_1^n x_n\| + L_3(L_3+1)\|x_n - x_{n-1}\| + L_3\|I_1^{n-1} x_{n-1} - x_{n-1}\|,$$

and

$$\|x_n - I_2 x_n\| \le \|x_n - I_2^n x_n\| + L_4(L_2+1)\|x_n - x_{n-1}\| + L_4\|I_2^{n-1} x_{n-1} - x_{n-1}\|,$$

which with (16), (21), (27) and (28) imply

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_2 x_n\| = 0 \tag{34}$$

$$\lim_{n\to\infty} \|x_n - I_1 x_n\| = 0 \tag{35}$$

and

$$\lim_{n\to\infty} \|x_n - I_2 x_n\| = 0. \tag{36}$$

*Step 2.* Assume $z^* \in \mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2) \cap EP(\phi)$. From the definition of notation $T_r$ in Lemma 2.5, we know that $v_n = T_{r_n} x_n$. So, it follows that

$$\|v_n - z^*\| = \|T_{r_n} x_n - T_{r_n} z^*\| \leq \|x_n - z^*\|.$$

Since $\{v_n\}$ is bounded, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $\{v_{n_k}\}$ converges weakly to $z^* \in D$ when $z^* = J^{-1} w^*$ for some $w^* \in J(D)$. By (10), we have that $\{x_{n_k}\}$ converges weakly to $z^* \in D$ and from (26), we also have that $\{y_{n_k}\}$ converges weakly to $z^* \in D$. Also, from (21), (27), (28),(29) and Lemma 2.6, we obtain $z^* \in F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$.

Next, we show that $z^* \in EP(\phi)$, that is $Jz^* = w^* \in J(EP(\phi))$. By $v_n = T_{r_n} x_n$, since $J$ is uniformly norm-to-norm continuous on bounded subset of $E$, it follows that

$$\lim_{n\to\infty} \|Jx_n - Jv_n\| = 0. \tag{37}$$

From the assumption $r_n \in [\rho, \infty)$, one sees

$$\lim_{n\to\infty} \frac{\|Jx_n - Jv_n\|}{r_n} = 0. \tag{38}$$

Since $\{x_n\}$ is bounded and so is $\{Jx_n\}$, there exists a subsequence $\{Jx_{n_k}\}$ of $\{Jx_n\}$ such that $\{Jx_n \rightharpoonup w^*\}$. Since $\{v_n\}$ is bounded, by (38), we also obtain $\{Jv_n \rightharpoonup w^*\}$. Noticing that $v_n = T_{r_n} x_n$, we obtain

$$\begin{aligned} &\phi(v_n, y) + \frac{1}{r_n}\langle y - v_n, Jv_n - Jx_n\rangle \geq 0, \quad y \in D, \\ &\phi(v_{n_k}, y) + \langle y - v_{n_k}, \frac{Jv_{n_k} - Jx_{n_k}}{r_{n_k}}\rangle \geq 0, \quad y \in D, \end{aligned} \tag{39}$$

According to (38), we obtain $\lim_{k\to\infty}\left[\frac{Jv_{n_k} - Jx_{n_k}}{r_{n_k}}\right] = 0$. Then, from (C2), we find

$$\begin{aligned} &\frac{1}{r_n}\langle y - v_n, Jv_n - Jx_n\rangle \geq -\phi(v_n, y), \\ &\quad \langle y - v_n, \frac{Jv_n - Jx_n}{r_n}\rangle \geq \phi(y, v_n). \end{aligned} \tag{40}$$

Since $\dfrac{\|Jx_n - Jv_n\|}{r_n} \to 0$ and $\{Jv_n \rightharpoonup w^*\}$, we obtain

$$\phi(y,\ w^*) \leq 0 \quad y \in D. \tag{41}$$

For $t$ with $0 \leq t \leq 1$ and $y \in D$, let $y_t = ty + (1-t)w^*$. Since $y \in D$ and $w^* \in D$, we have $y_t \in D$ and hence $\phi(y_t,\ w^*) \leq 0$. So from condition (C1) and (C3), we have

$$0 = \phi(y_t,\ y_t) \leq t\phi(y_t,\ y) + (1-t)\phi(y_t,\ w^*) \leq t\phi(y_t,\ y), \tag{42}$$

and hence $0 \leq \phi(y_t,\ y)$. From (C3), we have

$$0 \leq \phi(w^*,\ y), \quad \forall\, y \in D,$$

and hence $w^* \in EP(\phi)$. This completes the proof. ■

**Theorem 3.3.** Let $E$ be a real uniformly convex Banach space satisfying Opial condition and let $D$ be a nonempty closed convex subset of $E$. Suppose $X : E \to E$ is an identity mapping, let $\mathcal{S}_1,\ \mathcal{S}_2 : D \to D$ be two uniformly $L_1$ and $L_2$- Lipschitizian asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\},\ \{h_n\} \subset [1,\ \infty)$ and $I_1,\ I_2$ be two uniformly $L_3$ and $L_4$- Lipschitizian asymptotically quasi-nonexpansive self mapping of $D$ with a sequence $\{g_n\},\ \{t_n\} \subset [1,\ \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}} \{k_n,\ h_n,\ g_n,\ t_n\}$ and we assume that $R = \sup_n(1-a_n),\ \mathcal{M} = \sup_n \mu_n^2 \geq 1$ such that $\mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$ is nonempty and $q^* \in \mathcal{F}$. And $\{a_n\},\ \{b_n\},\ \{c_n\},\ \{\widehat{a_n}\},\ \{\widehat{b_n}\},\ \{\widehat{c_n}\}$ are six real sequences in $(0,\ 1)$ which satisfy the following conditions:

(i) $R(\mathcal{M}^2 + \mathcal{M}) < 1$,

(ii) $\sum_{n=1}^{\infty}(1-a_n)(\mu_n^4 + \mu_n^2 - 1) < \infty$.

If the mappings $X - \mathcal{S}_1,\ X - \mathcal{S}_2,\ X - I_1,\ X - I_2$ are semiclosed at zero, then the implicitly iterative sequence $\{x_n\}$ defined by (3) converges weakly to common fixed point of $\mathcal{F}$.

*Proof.* Let $q^* \in \mathcal{F}$, then according to Lemma 3.1 the sequence $\{\|x_n - q^*\|\}$ converges. This provides that $\{x_n\}$ is bounded sequence. Since $E$ be a uniformly convex, then every bounded subset of $E$ is weakly compact. Since $\{x_n\}$ is bounded sequence in $D$, then there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $q \in D$. Therefore, from (33), (34), (35) and (36) it follows that

$$\begin{aligned}\lim_{n_r\to\infty} \|x_{n_r} - \mathcal{S}_1 x_{n_r}\| &= \lim_{n_r\to\infty} \|x_{n_r} - \mathcal{S}_2 x_{n_r}\| = \lim_{n_r\to\infty} \|x_{n_r} - I_1 x_{n_r}\| \\ &= \lim_{n_r\to\infty} \|x_{n_r} - I_2 x_{n_r}\| = 0. \end{aligned} \tag{43}$$

Since the mapping $X - \mathcal{S}_1,\ X - \mathcal{S}_2,\ X - I_1$ and $X - I_2$ are semiclosed at zero, hence, we find $\mathcal{S}_1 q = q,\ \mathcal{S}_2 q = q,\ I_1 q = q$ and $I_2 q = q$ which means $q \in \mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$.

Finally, let us prove that $\{x_n\}$ converges weakly to $q$. In fact, suppose the contrary, that is, there exists some subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $\{x_{n_r}\}$ converges weakly to $q_1 \in D$ and $q_1 \neq q$. Then by the same method as given above, we can also prove that $q_1 \in \mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$.

From Lemma 3.1, we can prove that the $\lim_{n\to\infty} \|x_n - q\|$ and $\lim_{n\to\infty} \|x_n - q_1\|$ exist, and we have

$$\lim_{n\to\infty} \|x_n - q\| = d, \ \ \lim_{n\to\infty} \|x_n - q_1\| = d_1, \tag{44}$$

where $d$ and $d_1$ are two nonnegative numbers. By asset of the Opial condition of $E$, we take

$$\begin{aligned} d &= \lim_{n_r\to\infty} \sup \|x_{n_r} - q\| < \lim_{n_r\to\infty} \sup \|x_{n_r} - q_1\| = d_1 \\ &= \lim_{n_k\to\infty} \sup \|x_{n_k} - q_1\| < \lim_{n_k\to\infty} \sup \|x_{n_k} - q\|. \end{aligned} \tag{45}$$

This is a contradiction. Therefore $q_1 = q$. This implies that $\{x_n\}$ converges weakly to $q$. This completes the proof. ∎

Now we formulate next results concerning strong convergence of the sequence $\{x_n\}$.

**Theorem 3.4.** Let $E$ be a real uniformly convex Banach space and let $D,\ \mathcal{S}_1,\ \mathcal{S}_2,\ I_1,\ I_2$, $\{x_n\}$ be same as in Theorem 3.3. Suppose that the conditions in Theorem 3.3 is satisfied. If at least one mapping of the mappings $\mathcal{S}_1,\ \mathcal{S}_2,\ I_1$ and $I_2$ are semicompact, then an explicitly iterative sequence $\{x_n\}$ defined by (3) converges strongly to a common fixed point in $\mathcal{F}$.

*Proof.* Without any loss of generality, we may assume the $\mathcal{S}_1,\ \mathcal{S}_2,\ I_1$ and $I_2$ are semicompact. Then from (33), (34), (35) and (36), we have

$$\lim_{n\to\infty} \|x_n - \mathcal{S}_1 x_n\| = \lim_{n\to\infty} \|x_n - \mathcal{S}_2 x_n\| = \lim_{n\to\infty} \|x_n - I_1 x_n\| = \lim_{n\to\infty} \|x_n - I_2 x_n\| = 0.$$

From the semicompactness $\mathcal{S}_1,\ \mathcal{S}_2,\ I_1$ and $I_2$ there exists a subsequence $\{x_{n_r}\} \subset \{x_n\}$ such that $x_{n_r} \to q$ converges strongly to a $q \in D$. Again, using (33), (34), (35) and (36), we obtain

$$\lim_{n_r\to\infty} \|x_{n_r} - \mathcal{S}_1 x_{n_r}\| = \|q - \mathcal{S}_1 q\| = 0, \quad \lim_{n_r\to\infty} \|x_{n_r} - \mathcal{S}_2 x_{n_r}\| = \|q - \mathcal{S}_2 q\| = 0,$$
$$\lim_{n_r\to\infty} \|x_{n_r} - I_1 x_{n_r}\| = \|q - I_1 q\| = 0, \quad \lim_{n_r\to\infty} \|x_{n_r} - I_2 x_{n_r}\| = \|q - I_2 q\| = 0.$$

This shows that $q \in \mathcal{F} = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap F(I_2)$. According to Lemma 3.1 the limit $\lim_{n\to\infty} \|x_n - q\|$ exists. Then

$$\lim_{n\to\infty} \|x_n - q\| = \lim_{n_r\to\infty} \|x_{n_r} - q\| = 0,$$

which means that $\{x_n\}$ converges to $q \in \mathcal{F}$. This completes the proof. ■

If $\mathcal{S}_1 = \mathcal{S}$, $I_2 = I$ and relaxing conditions (i) and (ii) in Theorem 3.2, then we obtain following result:

**Theorem 3.5.** Let $E$ be a real Banach space and let $D$ be a nonempty closed convex subset of $E$. Let $\mathcal{S} : D \to D$ be a uniformly $L_1$-Lipschitizian asymptotically quasi-I-nonexpansive mapping with a sequences $\{k_n\} \subset [1, \infty)$ and $I$ be a uniformly $L_2$-Lipschitizian asymptotically quasi-nonexpansive self mapping of $D$ with a sequence $\{g_n\} \subset [1, \infty)$. Let $\mu_n = \max_{n\in\mathbb{N}} \{k_n, g_n\}$ and we assume that $R = \sup_n b_n$, $\mathcal{M} = \sup_n \mu_n^2 \geq 1$ such that $\mathcal{F} = F(\mathcal{S}) \cap F(I)$ is nonempty and $q^* \in \mathcal{F}$. For an initial point $x_0 \in D$, let sequence $\{x_n\}$ define as follows:

$$\begin{cases} x_n = (1 - b_n)x_{n-1} + b_n\mathcal{S}^n y_n, \quad n \geq 1, \\ y_n = (1 - \widehat{b_n})x_n + \widehat{b_n} I^n x_n, \end{cases} \tag{46}$$

where $\{b_n\}$, $\{\widehat{b_n}\}$ are two real sequences in $(0, 1)$ which satisfy the following conditions:

(i) $R\mathcal{M}^2 < 1$,

(ii) $\sum_{n=1}^{\infty}(\mu_n^4 - 1)b_n < \infty$.

Then iterative sequence $\{x_n\}$ satisfies the following :

$$\lim_{n\to\infty} \|x_n - \mathcal{S}x_n\| == \lim_{n\to\infty} \|x_n - Ix_n\| = 0.$$

*Proof.* Proof by condition of proposed iteration (3) put $c_n = 0 = \widehat{c_n}$ then $a_n = (1 - b_n)$ and $\widehat{a_n} = (1 - \widehat{b_n})$. Then (3) becomes (46), the rest of the proof follows on the proof technique of our Theorem 3.2 for step 1. ■

## 4. A comparison table of iterative algorithms

In this section, using Example 4.1 below, we numerically demonstrate the convergence of the algorithm defined in this paper and compare its behavior with the iterative algorithms of Farrukh and Saburov (2).

**Example 4.1.** Let $E = \mathbb{R}$ be the set of real numbers equipped with the norm $\|\cdot\| = |\cdot|$, $D = (0, 1)$, and let $a_n, b_n, c_n, \widehat{a_n}, \widehat{b_n}, \widehat{c_n}$ are six sequences in $(0, 1)$ define as follows:

$$a_n = \Big(1 - \frac{1}{2n} - \frac{1}{3n^2}\Big), \quad b_n = \frac{1}{2n}, \quad c_n = \frac{1}{3n^2},$$

$$\widehat{a_n} = \Big(1 - \frac{1}{n^2+2} - \frac{1}{n^2+1}\Big), \quad \widehat{b_n} = \frac{1}{n^2+2} \quad \text{and} \quad \widehat{c_n} = \frac{1}{n^2+1} \quad \text{for all } n \geq 1.$$

Let $\mathcal{S}_1,\ \mathcal{S}_2,\ I_1,\ I_2 : D \to D$ are four operators defined, respectively, by

$$\mathcal{S}_1 x = \frac{x}{1+x},\quad \mathcal{S}_2 x = \frac{x}{1+2x},\ \text{and}$$
$$I_1 x = \frac{3x}{3+x}\quad \text{and}\quad I_2 x = \frac{2x}{2+x}\quad \text{for all}\ \ x \in D.$$

It can easily verified that $\mathcal{S}_1,\ \mathcal{S}_2$ are asymptotically quasi-I-nonexpansive and $I_1,\ I_2$ are asymptotically quasi-nonexpansive mapping.

To check Farrukh and Saburov iteration, first we assume that $\mathcal{S}x = \mathcal{S}_1 x$ and $Ix = I_1 x$ for all $x \in D$.

Notice that $F = F(\mathcal{S}_1) \cap F(\mathcal{S}_2) \cap F(I_1) \cap (I_2) = \{0\}$. Suppose initial value $x_0 = 0.5$. The comparison given in the following table illustrate the proposed iteration scheme (3) and compared it with the well known iteration schemes proposed by Farrukh and Saburov (2) [scheme up to the accuracy of six decimal places by Mathematica software.]

Table 1: A comparison table of proposed iterative algorithms with known iterative algorithms

| Steps | items | proposed iteration | Farrukh and Saburov |
|---|---|---|---|
| 1 | $x_0$ | 0.500000 | 0.500000 |
| 2 | $x_1$ | 0.218889 | 0.385191 |
| 3 | $x_2$ | 0.210187 | 0.338928 |
| 4 | $x_3$ | 0.190387 | 0.309046 |
| 5 | $x_4$ | 0.176899 | 0.287074 |
| 6 | $x_5$ | 0.168160 | 0.269833 |
| 7 | $x_6$ | 0.159843 | 0.255745 |
| 8 | $x_7$ | 0.152914 | 0.243907 |
| 9 | $x_8$ | 0.146978 | 0.233750 |
| 10 | $x_9$ | 0.140597 | 0.224893 |
| 11 | $x_{10}$ | 0.136095 | 0.217070 |
| 12 | $x_{11}$ | 0.132068 | 0.210086 |

## 5. Conclusion

In the foregoing discussion, a new implicit iterative scheme is proposed which enable us to prove exists-ness, some weak and strong convergence results, related to two asymptotically quasi I-nonexpansive and two asymptotically quasi-nonexpansive mappings in real Banach space. We compute comparison of proposed implicit iterative scheme (3) to known iterative scheme (2) in fact that the iterative sequence generated by the proposed iterative scheme converges faster than to iterative sequence generated by known iterative scheme as shown in Example 4.1 above.

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## References

[1] E. Blum, W. Oettli, From optimization and variational ineqalities to equilibrium problems, *Math. Student* **63** (1994), 123–145.

[2] J. B. Diaz, F. B. Metcalf, On the structure of the set of subsequential limit points of succeessive approximation, *Bull. Amer. Math. Soc.* **73** (1967), 516–519.

[3] E. Lami Dozo, Multivalued nonexpansive mappings and Opial's condition, *Proc. Amer. Math. Soc.* **38** (1973), 286–292.

[4] M. Edelstein, On nonexpansive mappings, *Proc. Amer. Math. Soc.* **15** (1964), 689–695.

[5] M. K. Ghosh, L. Debnath, Convergence of Ishikawa iterates of quasi-nonexpansive mappings, *J. Math. Anal. Appl.* **207** (1997), 96–103.

[6] K. Goebel, W. A. Kirk, A fixed points theorem for asymptotically nonexpansive mappings, *Proc. Amer. Math. Soc.* **35** (1972), 171–174.

[7] J. Gornicki, Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces, *Comment. Math. Univ. Carolin.* **30 (2)** (1989), 249–252.

[8] F. Mukhamedov and M. Saburov, Weak and strong convergence of an implicit iteration process for an asymptotically quasi I-nonexpansive mapping in Banach space, *Fixed Point Theory and Appl.* **2010** (2010), Article ID 719631, 13 pages.

[9] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, *Bull. Amer. Math. Soc.* **73** (1967), 591–597.

[10] L. Qihou, Iterative sequences for asymptotically quasi-nonexpansive mappings, *J. Math. Anal. and Appl.* **259** (2001), 1–7.

[11] X. Qin, Y. J. Cho, S. M. Kang, Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces, *J. Comput. Appl. Math.* **225** (2009), 20–30.

[12] B. H. Rhodes and S. Temir, Convergence theorems for I-nonexpansive mapping, *Int. J. Mathematics and Mathematical Sciences* **2006** (2006), Article ID 63435, 4 pages.

[13] J. Schu, Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, *Bull. Australian Math. Soc.* **43(1)** (1991), 153–159.

[14] N. Shahzad, Generalized I-nonexpansive maps and best approximations in Banach spaces, *Demonstration Mathemaica* **37(3)** (2004), 597–600.

[15] Z. H. Sun, Strong convergence of an implicit iteration process for a finite family of

asymptotically quasi-nonexpansive mappings *J. Math. Anal. Appl.* **286(1)** (2003), 351–358.

[16] K. K. Tan and H. K. Xu, Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process, *J. Math. Anal. Appl.* **178(2)** (1993), 301–308.

[17] S. Temir and O. Gul, Convergence theorems for I- asymptotically quasi-nonexpansive mapping in Hilbert space, *J. Math. Anal. Appl.* **329(2)** (2007), 759–765.

[18] F. Tricomi, Un teorema sull convergenza delle successioni formate delle successive iterate di una funzione di una variable reale, *Giorn. Mat. Battaglini* **54** (1916), 1–9.

[19] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, *Numerical Functional Analysis and Optimization* **22(5-6)** (2001), 767–773.

[20] I. Yildirin, On the convergence theorems of an implicit iteration process for asymptotically quasi I-nonexpansive mappings, *Hacettepe J. of Mathematics and Statistics* **42(6)** (2013), 617–626.