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The Total Restrained Monophonic Number of a Graph

A.P. Santhakumaran

Department of Mathematics,
Hindustan University,
Hindustan Institute of Technology and Science,
Chennai - 603 103, India.

T. Venkata Raghu

Department of Applied Sciences and Humanities,
Sasi Institute of Technology and Engineering,
Tadepalligudem - 534 101, India.

P. Titus

Department of Mathematics,
University College of Engineering Nagercoil,
Anna University, Tirunelveli Region,
Nagercoil - 629 004, India.

K. Ganesamoorthy

Department of Mathematics,
Coimbatore Institute of Technology,
(Government Aided Autonomous Institution),
Coimbatore - 641 014, India.

Abstract

For a connected graph $G = (V, E)$ of order at least two, a *total restrained monophonic set* S of a graph G is a restrained monophonic set S such that the subgraph induced by S has no isolated vertices. The minimum cardinality of a total restrained monophonic set of G is the *total restrained monophonic number* of G and is denoted by $m_{tr}(G)$. A total restrained monophonic set of cardinality $m_{tr}(G)$ is called a m_{tr}*-set* of G. We determine bounds for it and characterize graphs which realize these bounds. It is shown that if p, d and k are positive integers such that $2 \leq d \leq p-2$, $3 \leq k \leq p$ and $p-d-k+2 \geq 0$, there exists a connected graph G of order p, monophonic diameter d and $m_{tr}(G) = k$.

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1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q, respectively. For basic graph theoretic terminology we refer to Harary [1, 2]. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete. A vertex v of a connected graph G is called a *support vertex* of G if it is adjacent to an endvertex of G.

A *chord* of a path P is an edge joining two non-adjacent vertices of P. A path P is called a *monophonic path* if it is a chordless path. A set S of vertices of G is a *monophonic set* of G if each vertex v of G lies on an $x-y$ monophonic path for some x and y in S. The minimum cardinality of a monophonic set of G is the *monophonic number* of G and is denoted by $m(G)$ [6]. A *restrained monophonic set* S of a graph G is a *monophonic set* such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of G is the *restrained monophonic number* of G and is denoted by $m_r(G)$. The restrained monophonic number of a graph wast introduced and studied in [7]. A set S of vertices of G is a *restrained edge geodetic set* of G if S is an edge geodetic set, and if either $S = V$ or the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained edge geodetic set of G is the *restrained edge geodetic number*, denoted by $eg_r(G)$. The restrained edge geodetic number of a graph was introduced and studied in [3].

A *connected restrained monophonic set* of G is a restrained monophonic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected restrained monophonic set of G is the *connected restrained monophonic number* of G and is denoted by $m_{cr}(G)$. The connected restrained monophonic number of a graph was introduced and studied in [8].

For any two vertices u and v in a connected graph G, the *monophonic distance* $d_m(u, v)$ from u to v is defined as the length of a longest $u-v$ monophonic path in G. The *monophonic eccentricity* $e_m(v)$ of a vertex v in G is $e_m(v) = \max\{d_m(v, u) : u \in V(G)\}$. The *monophonic radius*, $rad_m(G)$ of G is $rad_m(G) = \min\{e_m(v) : v \in V(G)\}$ and the *monophonic diameter*, $diam_m(G)$ of G is $diam_m(G) = \max\{e_m(v) : v \in V(G)\}$. A vertex u in G is *monophonic eccentric vertex* of a vertex v in G if $e_m(u) = d_m(u, v)$. The monophonic distance was introduced and studied in [4, 5].

The following theorems will be used in the sequel.

Theorem 1.1. [7] Each extreme vertex of a connected graph G belongs to every restrained monophonic set of G.

Theorem 1.2. [7] Let G be a connected graph with cutvertices and let S be a restrained monophonic set of G. If v is a cutvertex of G, then every component of $G-v$ contains an element of S.

Theorem 1.3. [8] Every cutvertex of a connected graph G belongs to every connected restrained monophonic set of G.

Theorem 1.4. [8] Let G be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $m_{cr}(G) = 2$.

Theorem 1.5. [8] For the complete graph $K_p(p \geq 2)$, $m_{cr}(K_p) = p$.

Theorem 1.6. [8] For the complete bipartite graph

$$G = K_{m,n}(2 \leq m \leq n), m_{cr}(G) = \begin{cases} n+2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n \end{cases}.$$

Theorem 1.7. [8] If $G = K_1 + \bigcup m_j K_j$, where $j \geq 2, \sum m_j \geq 2$, then $m_{cr}(G) = p$.

Throughout this paper G denotes a connected graph with at least two vertices.

2. Total restrained monophonic number

Definition 2.1. A *total restrained monophonic set* S of a graph G is a restrained monophonic set such that the subgraph $G[S]$ induced by S has no isolated vertices. The minimum cardinality of a total restrained monophonic set of G is the *total restrained monophonic number* of G and is denoted by $m_{tr}(G)$. A total restrained monophonic set of cardinality $m_{tr}(G)$ is called a m_{tr}*-set* of G.

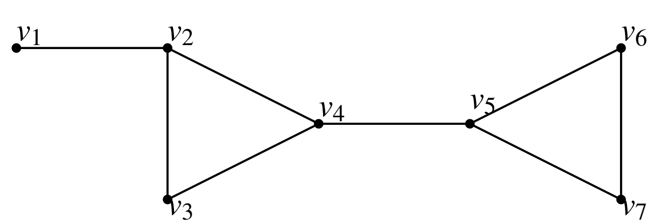

Figure 2.1: G

Example 2.2. For the graph G in Figure 2.1, every vertex of G is either a cutvertex or an extreme vertex. By Theorems 1.1 and 1.3, we have $m_{cr}(G) = 7$. Let $S =$

$\{v_1, v_3, v_6, v_7, v_2\}$ be the set of all extreme vertices and support vertex of G. It is easily verified that the set $S - \{v_2\}$ is a minimum restrained monophonic set of G and so $m_r(G) = 4$. The subgraph induced by $S - \{v_2\}$ has the isolated vertices v_1, v_3 so that $S - \{v_2\}$ is not a total restrained monophonic set of G. It is clear that S is a minimum total restrained monophonic set of G and so $m_{tr}(G) = 5$. Thus the restrained monophonic number, total restrained monophonic number and connected restrained monophonic number of a graph are all different.

It is easily observed that every connected restrained monophonic set of G is a total restrained monophonic set of G. The next theorem follows from Theorems 1.1 and 1.3.

Theorem 2.3. Each extreme vertex and each support vertex of a connected graph G belongs to every total restrained monophonic set of G. If the set S of all extreme vertices and support vertices form a total restrained monophonic set, then it is the unique minimum total restrained monophonic set of G.

Corollary 2.4. For the complete graph $K_p (p \geq 2)$, $m_{tr}(K_p) = p$.

Theorem 2.5. Let G be a connected graph with cutvertices and let S be a total restrained monophonic set of G. If v is a cutvertex of G, then every component of $G - v$ contains an element of S.

Proof. Since every total restrained monophonic set of G is a restrained monophonic set of G, the result follows from Theorem 1.2. ∎

Theorem 2.6. For a connected graph G of order p, $2 \leq m_r(G) \leq m_{tr}(G) \leq m_{cr}(G) \leq p, m_r(G) = m_{cr}(G) = m_{tr}(G) \neq p - 1$.

Proof. Any restrained monophonic set of G needs at least two vertices and so $m_r(G) \geq 2$. Since every total restrained monophonic set of G is also a restrained monophonic set of G, it follows that $m_r(G) \leq m_{tr}(G)$. Also, since every connected restrained monophonic set of G is a total restrained monophonic set of G we have $m_{tr}(G) \leq dm_{cr}(G)$. Since $V(G)$ is a connected restrained monophonic set of G, it is clear that $m_{tr}(G) \leq p$. Hence $2 \leq m_r(G) \leq m_{tr}(G) \leq m_{cr}(G) \leq p$. From the definitions of restrained, connected restrained and total restrained monophonic number, we have $m_r(G) = m_{cr}(G) = m_{tr}(G) \neq p - 1$. ∎

Corollary 2.7. Let G be a connected graph. If $m_{tr}(G) = 2$, then $m_r(G) = 2$.

For any non-trivial path of order at least 4, the restrained monophonic number is 2 and the total restrained monophonic number is 4. This shows that the converse of the Corollary 2.7 need not be true.

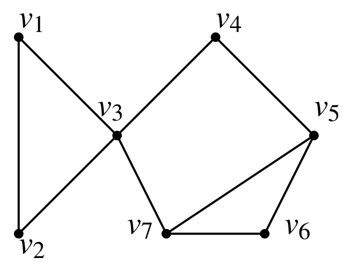

Figure 2.2: G

Remark 2.8. The bounds in Theorem 2.6 are sharp. For the complete graph $G = K_p$, then $m_r(G) = m_{tr}(G) = p$ and $m_{cr}(K_p) = p$. Also, all the inequalities in Theorem 2.6 are strict. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_2, v_6\}$ is the unique minimum restrained monophonic set of G so that $m_r(G) = 3$. The subgraph induced by S_1 is not connected and it has an isolated vertex v_6. It is clear that $S_2 = S_1 \cup \{v_5\}$ and $S_3 = S_1 \cup \{v_7\}$ are the two minimum total restrained monophonic sets of G and so $m_{tr}(G) = 4$. The subgraph induced by S_i, $i = 2, 3$ are not connected. Clearly $S_3 \cup \{v_3\}$ is a minimum connected restrained monophonic set of G, it follows that $m_{cr}(G) = 5$. Thus, we have $2 < m_r(G) < m_{tr}(G) < m_{cr}(G) < p$.

Theorem 2.9. For any non-trivial tree T, the set of all endvertices and support vertices of T is the unique minimum total restrained monophonic set of G.

Proof. Since the set of all endvertices and support vertices of T forms a total restrained monophonic set, the result follows from Theorem 2.3. ■

Theorem 2.10. For any connected graph G, $m_{tr}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $m_{tr}(G) = 2$. Conversely, let $m_{tr}(G) = 2$. Let $S = \{u, v\}$ be a minimum total restrained monophonic set of G. Then uv is an edge. It is clear that a vertex different from u and v cannot lie on a $u - v$ monophonic path and so $G = K_2$. ■

Theorem 2.11. For any connected graph G, $m_{tr}(G) = 3$ if and only if $m_{cr}(G) = 3$.

Proof. Suppose $m_{cr}(G) = 3$. Let $S = \{x, y, z\}$ is a minimum connected restrained monophonic set of G. Therefore, S is a total restrained monophonic set of G. It follows from Theorem 2.10 that S is a minimum total restrained monophonic set of G and so $m_{tr}(G) = 3$. Conversely, let $m_{tr}(G) = 3$. By Theorem 1.4 and the argument similar to the first part, we have $m_{cr}(G) = 3$. ■

Theorem 2.12. For the cycle $G = C_3$ or $G = C_n (n \geq 5)$ or $G = \overline{K}_2 + H$ $(p \geq 5)$, where H is a 2-connected graph of order $p - 2$, then $m_{tr}(G) = 3$.

Proof. First, suppose that $G = C_3$, it is a complete graph, by Corollary 2.4, we have $m_{tr}(G) = 3$. For any cycle $C_n (n \geq 5)$, it is easily verified that any three consecutive vertices of C_n is a minimum total restrained monophonic set of C_n and so $m_{tr}(C_n) = 3$.

Next, suppose that $G = \overline{K}_2 + H$, where H is a connected graph of order $p-2$. Let $V(\overline{K}_2) = \{u_1, u_2\}$. Then for any vertex v of H, the set $S = \{v, u_1, u_2\}$ is a minimum total restrained monophonic set of G and so $m_{tr}(G) = 3$. ■

Problem 2.13. Characterize graphs G for which $m_{tr}(G) = 3$.

The next two observations follow from Theorems 1.6 and 1.7.

Observation 2.14. For the complete bipartite graph

$$G = K_{m,n}(2 \le m \le n), m_{tr}(G) = \begin{cases} n+2 & \text{if } 2 = m \le n \\ 4 & \text{if } 3 \le m \le n \end{cases}.$$

Observation 2.15. If $G = K_1 + \bigcup m_j K_j$, where $j \ge 1, \sum m_j \ge 2$, then $m_{tr}(G) = p$.

Problem 2.16. Characterize the class of graphs G of order p for which $m_{tr}(G) = p$.

3. Some realization results on the total restrained monophonic number

Theorem 3.1. If p, d and k are positive integers such that $2 \le d \le p-2$, $3 \le k \le p$ and $p - d - k + 2 \ge 0$, then there exists a connected graph G of order p, monophonic diameter d and $m_{tr}(G) = k$.

Proof. We prove this theorem by considering two cases.
Case 1. Let $d = 2$. First, let $k = 3$. Let $P_3 : v_1, v_2, v_3$ be the path of order 3. Now, add $p-3$ new vertices $w_1, w_2, ..., w_{p-3}$ to P_3. Let G be the graph obtained from P_3 by joining each $w_i (1 \le i \le p-3)$ to v_1 and v_3, and joining each $w_j (1 \le j \le p-4)$ to $w_k (j+1 \le k \le p-3)$. The graph G is shown in Figure 3.1. Then G has order p and monophonic diameter $d = 2$. Clearly $S = \{v_1, v_2, v_3\}$ is a minimum total restrained monophonic set of G so that $m_{tr}(G) = k = 3$.

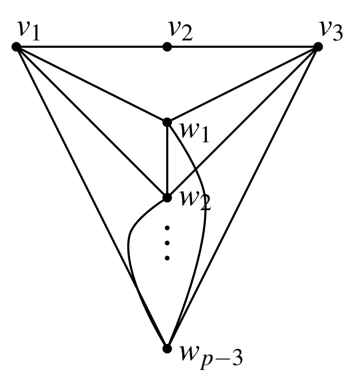

Figure 3.1: G

Now, let $4 \leq k \leq p$. Let K_{p-2} be the complete graph of order $p-2$ with the vertex set $\{w_1, w_2, \ldots, w_{p-k}, v_1, v_2, \ldots, v_{k-2}\}$. Now, add two new vertices x and y to K_{p-2} and let G be the graph obtained from K_{p-2} by joining x and y with each vertex $w_i (1 \leq i \leq p-k)$, and joining the vertices x and y. The graph G is shown in Figure 3.2. Then G has order p and monophonic diameter $d = 2$. Let $S = \{v_1, v_2, \ldots, v_{k-2}, x, y\}$ be the set of all extreme vertices of G. By Theorem 2.3, every total restrained monophonic set of G contains S. It is easily verified that S is a minimum total restrained monophonic set of G and so $m_{tr}(G) = k$.

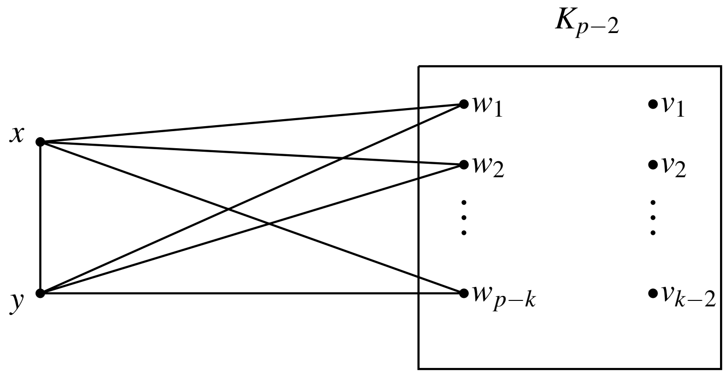

Figure 3.2: G

Case 2. $d \geq 3$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be the cycle of order $d+2$. Add $p-d-2$ new vertices $w_1, w_2, \ldots, w_{p-d-2}$ to C and join each vertex $w_i (1 \leq i \leq p-d-2)$ to both v_1 and v_3, thereby producing the graph G of Figure 3.3. Then G has order p and monophonic diameter d. It is clear that $S = \{v_3, v_4, v_5\}$ is a minimum total restrained monophonic set of G and so $m_{tr}(G) = 3 = k$.

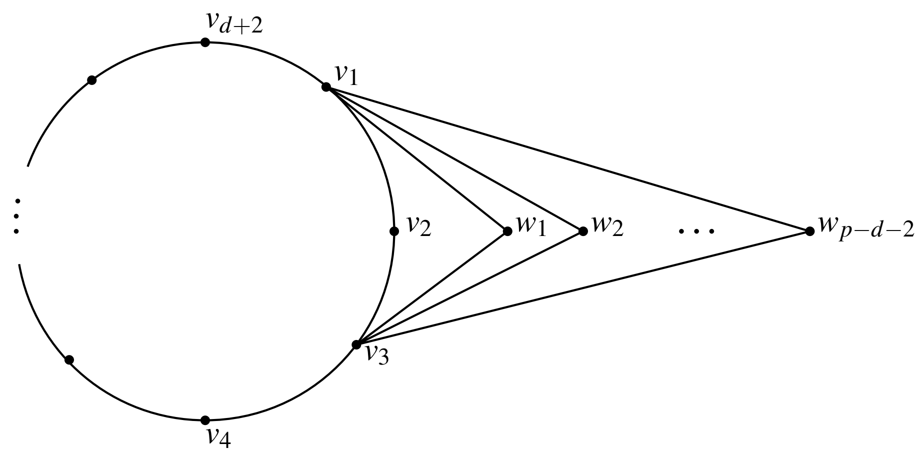

Figure 3.3: G

Now, let $k \geq 4$. Let $P_{d+1} : v_0, v_1, \ldots, v_d$ be a path of length d. Add $p-d-1$ new vertices $w_1, w_2, \ldots, w_{p-d-k+2}, u_1, u_2, \ldots, u_{k-3}$ to P_{d+1} and join $w_1, w_2, \ldots, w_{p-d-k+2}$ to both v_0

and v_2 and also join $u_1, u_2, \ldots, u_{k-3}$ to v_d; and join each $w_j (1 \leq j \leq p-d+k+1)$ to $w_k (j+1 \leq k \leq p-d+k+2)$, thereby producing the graph G of Figure 3.4.

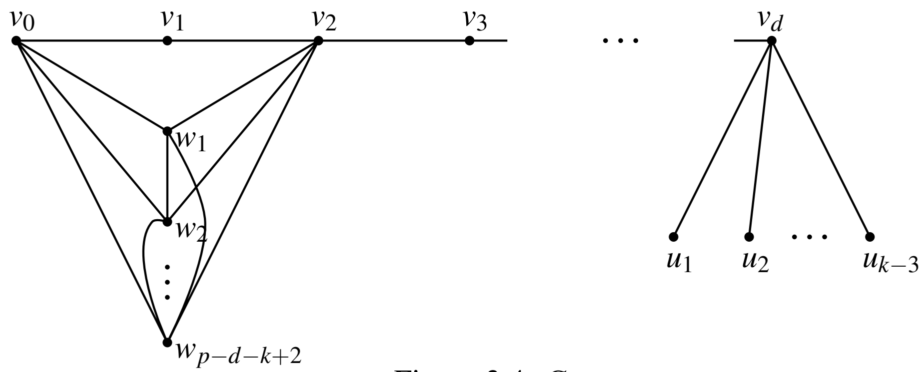

Figure 3.4: G

Then G has order p and monophonic diameter d. Let $S = \{u_1, u_2, \ldots, u_{k-3}, v_d\}$ be the set of all endvertices and support vertex of G. By Theorem 2.3, every total restrained monophonic set of G contains S. It is clear that S is not a total restrained monophonic set of G. Also, for any $x \notin S$, $S \cup \{x\}$ is not a total restrained monophonic set of G. It is easily seen that $S \cup \{v_0, v_1\}$ is a minimum total restrained monophonic set of G and so $m_{tr}(G) = k$. ■

Theorem 3.2. If a, b are two positive integers such that $3 \leq a \leq b$, then there exists a connected graph G of order p with $m_{tr}(G) = a$ and $m_{cr}(G) = b$.

Proof. We prove this theorem by considering two cases.
Case 1. $a = b$. Let G be the complete graph of order b. Then by Corollary 2.4 and Theorem 1.5, we have $m_{tr}(G) = m_{cr}(G) = b$.

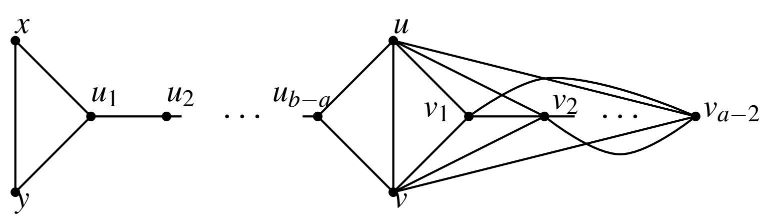

Figure 3.5: G

Case 2. $3 \leq a < b$. Let $P_{b-a} : u_1, u_2, \ldots, u_{b-a}$ be a path of order $b-a$. Let H be the graph obtained from P_{b-a} by adding a new vertices $v_1, v_2, \ldots, v_{a-2}, u, v$ to P_{b-a} and joining the vertices u, v to u_{b-a}; and joining the vertices $v_1, v_2, \ldots, v_{a-2}$ to the vertices u, v; and joining the vertices $v_j (1 \leq j \leq a-3)$ to $v_k (j+1 \leq k \leq a-2)$. The graph G is obtained from H and the complete graph K_2 with the vertex set $V(K_2) = \{x, y\}$, by joining the vertices x, y to u_1; and joining the vertices u and v, thereby producing the graph G and is shown in Figure 3.5. Let $S = \{v_1, v_2, \ldots, v_{a-2}, x, y\}$ be the set of all extreme vertices of G. By Theorem 2.3, every total restrained monophonic set of G contains S. It is clear that, S is a minimum total restrained monophonic set of G and so $m_{tr}(G) = a$.

Let $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$ be the set of all extreme vertices and cutvertices of G. By Theorems 1.1 and 1.3, every connected restrained monophonic set of G contains S_1. It is easily verified that S_1 is a minimum connected restrained monophonic set of G and so $m_{cr}(G) = b$. ■

Theorem 3.3. For positive integers a, b such that $3 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph G such that $m_r(G) = a$ and $m_{tr}(G) = b$.

Proof. **Case 1.** For $a = b$, the complete graph K_a has the desired properties.
Case 2. $a < b$. Let $b = a + k$ where $1 \leq k \leq a$. Let $C_i : x_i, y_i, z_i, u_i, v_i, x_i \, (1 \leq i \leq k)$ be "k"copies of C_5. Let H be the graph obtained from C_i by identifying the vertices $x_i \, (1 \leq i \leq k)$, say x be the identified vertices and joining the vertices y_i and $u_i \, (1 \leq i \leq k)$. Let G be the graph obtained from H and the complete graph K_{a-k} with the vertex set $V(K_{a-k}) = \{w_1, w_2, \cdots, w_{a-k}\}$ by joining each vertex $w_j (1 \leq j \leq a-k)$ to the vertex x of H. The graph G is shown in Figure 2.8. Let $S = \{w_1, w_2, \cdots, w_{a-k}, z_1, z_2, \ldots, z_k\}$ be the set of all extreme vertices of G. By Theorem 1.1, every restrained monophonic set of G contains S. It is easily seen that S is a minimum restrained monophonic set of G and so $m_r(G) = a$.

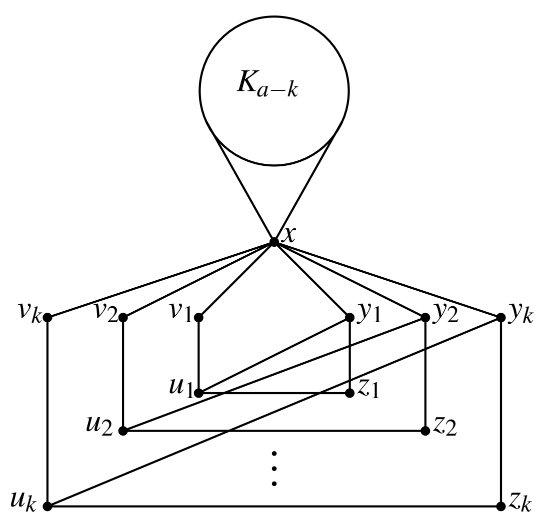

Figure: 3.6 G

By Theorem 2.3, every total restrained monophonic set of G contains S. We observe that every minimum total restrained monophonic set of G contains exactly one vertex from $\{y_i, u_i\}$ for every $i (1 \leq i \leq k)$. Thus $m_{tr}(G) \geq b$. Since $S_1 = S \cup \{u_1, u_2, \ldots, u_k\}$ is a total restrained monophonic set of G, it follows that $m_{tr}(G) = a + k = b$. ■

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