The Total Restrained Monophonic Number of a Graph

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Abstract
For a connected graph $G = (V,E)$ of order at least two, a total restrained monophonic set $S$ of a graph $G$ is a restrained monophonic set $S$ such that the subgraph induced by $S$ has no isolated vertices. The minimum cardinality of a total restrained monophonic set of $G$ is the total restrained monophonic number of $G$ and is denoted by $m_{tr}(G)$. A total restrained monophonic set of cardinality $m_{tr}(G)$ is called a $m_{tr}$-set of $G$. We determine bounds for it and characterize graphs which
realize these bounds. It is shown that if $p,d$ and $k$ are positive integers such that $2 \leq d \leq p - 2$, $3 \leq k \leq p$ and $p - d - k + 2 \geq 0$, there exists a connected graph $G$ of order $p$, monophonic diameter $d$ and $m_r(G) = k$.

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1. **Introduction**

By a graph $G = (V,E)$ we mean a finite undirected connected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$, respectively. For basic graph theoretic terminology we refer to Harary [1, 2]. The neighborhood of a vertex $v$ is the set $N(v)$ consisting of all vertices $u$ which are adjacent with $v$. A vertex $v$ is an extreme vertex if the subgraph induced by its neighbors is complete. A vertex $v$ of a connected graph $G$ is called a support vertex of $G$ if it is adjacent to an endvertex of $G$.

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called a monophonic path if it is a chordless path. A set $S$ of vertices of $G$ is a monophonic set of $G$ if each vertex $v$ of $G$ lies on an $x-y$ monophonic path for some $x$ and $y$ in $S$. The minimum cardinality of a monophonic set of $G$ is the monophonic number of $G$ and is denoted by $m(G)$ [6]. A restrained monophonic set $S$ of a graph $G$ is a monophonic set such that either $S = V$ or the subgraph induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained monophonic set of $G$ is the restrained monophonic number of $G$ and is denoted by $m_r(G)$. The restrained monophonic number of a graph was introduced and studied in [7]. A set $S$ of vertices of $G$ is a restrained edge geodetic set of $G$ if $S$ is an edge geodetic set, and if either $S = V$ or the subgraph $G[V - S]$ induced by $V - S$ has no isolated vertices. The minimum cardinality of a restrained edge geodetic set of $G$ is the restrained edge geodetic number, denoted by $eg_r(G)$. The restrained edge geodetic number of a graph was introduced and studied in [3].

A connected restrained monophonic set of $G$ is a restrained monophonic set $S$ such that the subgraph $G[S]$ induced by $S$ is connected. The minimum cardinality of a connected restrained monophonic set of $G$ is the connected restrained monophonic number of $G$ and is denoted by $m_{cr}(G)$. The connected restrained monophonic number of a graph was introduced and studied in [8].

For any two vertices $u$ and $v$ in a connected graph $G$, the monophonic distance $d_m(u,v)$ from $u$ to $v$ is defined as the length of a longest $u-v$ monophonic path in $G$. The monophonic eccentricity $e_m(v)$ of a vertex $v$ in $G$ is $e_m(v) = \max \{d_m(v,u) : u \in V(G)\}$. The monophonic radius, $rad_m(G)$ of $G$ is $rad_m(G) = \min \{e_m(v) : v \in V(G)\}$ and the monophonic diameter, $diam_m(G)$ of $G$ is $diam_m(G) = \max \{e_m(v) : v \in V(G)\}$. A vertex $u$ in $G$ is monophonic eccentric vertex of a vertex $v$ in $G$ if $e_m(u) = d_m(u,v)$. The monophonic distance was introduced and studied in [4, 5].
The following theorems will be used in the sequel.

**Theorem 1.1.** [7] Each extreme vertex of a connected graph $G$ belongs to every restrained monophonic set of $G$.

**Theorem 1.2.** [7] Let $G$ be a connected graph with cutvertices and let $S$ be a restrained monophonic set of $G$. If $v$ is a cutvertex of $G$, then every component of $G - v$ contains an element of $S$.

**Theorem 1.3.** [8] Every cutvertex of a connected graph $G$ belongs to every connected restrained monophonic set of $G$.

**Theorem 1.4.** [8] Let $G$ be a connected graph of order $p \geq 2$. Then $G = K_2$ if and only if $m_{cr}(G) = 2$.

**Theorem 1.5.** [8] For the complete graph $K_p (p \geq 2)$, $m_{cr}(K_p) = p$.

**Theorem 1.6.** [8] For the complete bipartite graph

$$G = K_{m,n} (2 \leq m \leq n), m_{cr}(G) = \begin{cases} n + 2 & \text{if } 2 = m \leq n \\ 4 & \text{if } 3 \leq m \leq n \end{cases}$$

**Theorem 1.7.** [8] If $G = K_1 + \bigcup m_j K_j$, where $j \geq 2, \sum m_j \geq 2$, then $m_{cr}(G) = p$.

Throughout this paper $G$ denotes a connected graph with at least two vertices.

### 2. Total restrained monophonic number

**Definition 2.1.** A total restrained monophonic set $S$ of a graph $G$ is a restrained monophonic set such that the subgraph $G[S]$ induced by $S$ has no isolated vertices. The minimum cardinality of a total restrained monophonic set of $G$ is the total restrained monophonic number of $G$ and is denoted by $m_{tr}(G)$. A total restrained monophonic set of cardinality $m_{tr}(G)$ is called a $m_{tr}$-set of $G$.

![Figure 2.1: $G$](image)

**Example 2.2.** For the graph $G$ in Figure 2.1, every vertex of $G$ is either a cutvertex or an extreme vertex. By Theorems 1.1 and 1.3, we have $m_{cr}(G) = 7$. Let $S =$
\{v_1, v_3, v_6, v_7, v_2\} be the set of all extreme vertices and support vertex of $G$. It is easily verified that the set $S - \{v_2\}$ is a minimum restrained monophonic set of $G$ and so $m_r(G) = 4$. The subgraph induced by $S - \{v_2\}$ has the isolated vertices $v_1, v_3$ so that $S - \{v_2\}$ is not a total restrained monophonic set of $G$. It is clear that $S$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = 5$. Thus the restrained monophonic number, total restrained monophonic number and connected restrained monophonic number of a graph are all different.

It is easily observed that every connected restrained monophonic set of $G$ is a total restrained monophonic set of $G$. The next theorem follows from Theorems 1.1 and 1.3.

**Theorem 2.3.** Each extreme vertex and each support vertex of a connected graph $G$ belongs to every total restrained monophonic set of $G$. If the set $S$ of all extreme vertices and support vertices form a total restrained monophonic set, then it is the unique minimum total restrained monophonic set of $G$.

**Corollary 2.4.** For the complete graph $K_p (p \geq 2)$, $m_{tr}(K_p) = p$.

**Theorem 2.5.** Let $G$ be a connected graph with cutvertices and let $S$ be a total restrained monophonic set of $G$. If $v$ is a cutvertex of $G$, then every component of $G - v$ contains an element of $S$.

**Proof.** Since every total restrained monophonic set of $G$ is a restrained monophonic set of $G$, the result follows from Theorem 1.2. □

**Theorem 2.6.** For a connected graph $G$ of order $p$, $2 \leq m_r(G) \leq m_{tr}(G) \leq m_{cr}(G) \leq p, m_r(G) = m_{cr}(G) = m_{tr}(G) \neq p - 1$.

**Proof.** Any restrained monophonic set of $G$ needs at least two vertices and so $m_r(G) \geq 2$. Since every total restrained monophonic set of $G$ is also a restrained monophonic set of $G$, it follows that $m_r(G) \leq m_{tr}(G)$. Also, since every connected restrained monophonic set of $G$ is a total restrained monophonic set of $G$ we have $m_{tr}(G) \leq dm_{cr}(G)$. Since $V(G)$ is a connected restrained monophonic set of $G$, it is clear that $m_{tr}(G) \leq p$. Hence $2 \leq m_r(G) \leq m_{tr}(G) \leq m_{cr}(G) \leq p$. From the definitions of restrained, connected restrained and total restrained monophonic number, we have $m_r(G) = m_{cr}(G) = m_{tr}(G) \neq p - 1$. □

**Corollary 2.7.** Let $G$ be a connected graph. If $m_{tr}(G) = 2$, then $m_r(G) = 2$.

For any non-trivial path of order at least 4, the restrained monophonic number is 2 and the total restrained monophonic number is 4. This shows that the converse of the Corollary 2.7 need not be true.
Remark 2.8. The bounds in Theorem 2.6 are sharp. For the complete graph $G = K_p$, then $m_r(G) = m_{tr}(G) = p$ and $m_{cr}(K_p) = p$. Also, all the inequalities in Theorem 2.6 are strict. For the graph $G$ given in Figure 2.2, $S_1 = \{v_1, v_2, v_6\}$ is the unique minimum restrained monophonic set of $G$ so that $m_r(G) = 3$. The subgraph induced by $S_1$ is not connected and it has an isolated vertex $v_6$. It is clear that $S_2 = S_1 \cup \{v_3\}$ and $S_3 = S_1 \cup \{v_7\}$ are the two minimum total restrained monophonic sets of $G$ and so $m_{tr}(G) = 4$. The subgraph induced by $S_i$, $i = 2, 3$ are not connected. Clearly $S_3 \cup \{v_3\}$ is a minimum connected restrained monophonic set of $G$, it follows that $m_{cr}(G) = 5$. Thus, we have $2 < m_r(G) < m_{tr}(G) < m_{cr}(G) < p$.

Theorem 2.9. For any non-trivial tree $T$, the set of all endvertices and support vertices of $T$ is the unique minimum total restrained monophonic set of $G$.

Proof. Since the set of all endvertices and support vertices of $T$ forms a total restrained monophonic set, the result follows from Theorem 2.3.

Theorem 2.10. For any connected graph $G$, $m_{tr}(G) = 2$ if and only if $G = K_2$.

Proof. If $G = K_2$, then $m_{tr}(G) = 2$. Conversely, let $m_{tr}(G) = 2$. Let $S = \{u, v\}$ be a minimum total restrained monophonic set of $G$. Then $uv$ is an edge. It is clear that a vertex different from $u$ and $v$ cannot lie on a $u-v$ monophonic path and so $G = K_2$.

Theorem 2.11. For any connected graph $G$, $m_{tr}(G) = 3$ if and only if $m_{cr}(G) = 3$.

Proof. Suppose $m_{cr}(G) = 3$. Let $S = \{x, y, z\}$ is a minimum connected restrained monophonic set of $G$. Therefore, $S$ is a total restrained monophonic set of $G$. It follows from Theorem 2.10 that $S$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = 3$. Conversely, let $m_{tr}(G) = 3$. By Theorem 1.4 and the argument similar to the first part, we have $m_{cr}(G) = 3$.

Theorem 2.12. For the cycle $G = C_3$ or $G = C_n (n \geq 5)$ or $G = \overline{K}_2 + H (p \geq 5)$, where $H$ is a 2-connected graph of order $p - 2$, then $m_{tr}(G) = 3$.

Proof. First, suppose that $G = C_3$, it is a complete graph, by Corollary 2.4, we have $m_{tr}(G) = 3$. For any cycle $C_n (n \geq 5)$, it is easily verified that any three consecutive vertices of $C_n$ is a minimum total restrained monophonic set of $C_n$ and so $m_{tr}(C_n) = 3$.
Next, suppose that $G = \overline{K}_2 + H$, where $H$ is a connected graph of order $p - 2$. Let $V(\overline{K}_2) = \{u_1, u_2\}$. Then for any vertex $v$ of $H$, the set $S = \{v, u_1, u_2\}$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = 3$. □

**Problem 2.13.** Characterize graphs $G$ for which $m_{tr}(G) = 3$.

The next two observations follow from Theorems 1.6 and 1.7.

**Observation 2.14.** For the complete bipartite graph

$$G = K_{m,n}(2 \leq m \leq n), m_{tr}(G) = \begin{cases} n+2 & \text{if } 2 \leq m \leq n \\ 4 & \text{if } 3 \leq m \leq n \end{cases}$$

**Observation 2.15.** If $G = K_1 + \bigcup m_jK_j$, where $j \geq 1, \sum m_j \geq 2$, then $m_{tr}(G) = p$.

**Problem 2.16.** Characterize the class of graphs $G$ of order $p$ for which $m_{tr}(G) = p$.

3. Some realization results on the total restrained monophonic number

**Theorem 3.1.** If $p, d$ and $k$ are positive integers such that $2 \leq d \leq p - 2$, $3 \leq k \leq p$ and $p - d - k + 2 \geq 0$, then there exists a connected graph $G$ of order $p$, monophonic diameter $d$ and $m_{tr}(G) = k$.

**Proof.** We prove this theorem by considering two cases.

**Case 1.** Let $d = 2$. First, let $k = 3$. Let $P_3 : v_1, v_2, v_3$ be the path of order 3. Now, add $p - 3$ new vertices $w_1, w_2, \ldots, w_{p-3}$ to $P_3$. Let $G$ be the graph obtained from $P_3$ by joining each $w_i (1 \leq i \leq p - 3)$ to $v_1$ and $v_3$, and joining each $w_j (1 \leq j \leq p - 4)$ to $w_k (j + 1 \leq k \leq p - 3)$. The graph $G$ is shown in Figure 3.1. Then $G$ has order $p$ and monophonic diameter $d = 2$. Clearly $S = \{v_1, v_2, v_3\}$ is a minimum total restrained monophonic set of $G$ so that $m_{tr}(G) = k = 3$.

![Figure 3.1: $G$](image-url)
Now, let $4 \leq k \leq p$. Let $K_{p-2}$ be the complete graph of order $p - 2$ with the vertex set \{\(w_1, w_2, \ldots, w_{p-k}, v_1, v_2, \ldots, v_{k-2}\). Now, add two new vertices $x$ and $y$ to $K_{p-2}$ and let $G$ be the graph obtained from $K_{p-2}$ by joining $x$ and $y$ with each vertex $w_i(1 \leq i \leq p-k)$, and joining the vertices $x$ and $y$. The graph $G$ is shown in Figure 3.2. Then $G$ has order $p$ and monophonic diameter $d = 2$. Let $S = \{v_1, v_2, \ldots, v_{k-2}, x, y\}$ be the set of all extreme vertices of $G$. By Theorem 2.3, every total restrained monophonic set of $G$ contains $S$. It is easily verified that $S$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = k$.

**Case 2.** $d \geq 3$. First, let $k = 3$. Let $C_{d+2} : v_1, v_2, \ldots, v_{d+2}, v_1$ be the cycle of order $d + 2$. Add $p - d - 2$ new vertices $w_1, w_2, \ldots, w_{p-d-2}$ to $C$ and join each vertex $w_i(1 \leq i \leq p - d - 2)$ to both $v_1$ and $v_3$, thereby producing the graph $G$ of Figure 3.3. Then $G$ has order $p$ and monophonic diameter $d$. It is clear that $S = \{v_3, v_4, v_5\}$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = 3 = k$.

Now, let $k \geq 4$. Let $P_{d+1} : v_0, v_1, \ldots, v_d$ be a path of length $d$. Add $p - d - 1$ new vertices $w_1, w_2, \ldots, w_{p-d-k+2}, u_1, u_2, \ldots, u_{k-3}$ to $P_{d+1}$ and join $w_1, w_2, \ldots, w_{p-d-k+2}$ to both $v_0$
and $v_2$ and also join $u_1, u_2, \ldots, u_{k-3}$ to $v_d$; and join each $w_j(1 \leq j \leq p - d + k + 1)$ to $w_k(j + 1 \leq k \leq p - d + k + 2)$, thereby producing the graph $G$ of Figure 3.4.

Then $G$ has order $p$ and monophonic diameter $d$. Let $S = \{u_1, u_2, \ldots, u_{k-3}, v_d\}$ be the set of all endvertices and support vertex of $G$. By Theorem 2.3, every total restrained monophonic set of $G$ contains $S$. It is clear that $S$ is not a total restrained monophonic set of $G$. Also, for any $x \notin S$, $S \cup \{x\}$ is not a total restrained monophonic set of $G$. It is easily seen that $S \cup \{v_0, v_1\}$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = k$.

**Theorem 3.2.** If $a, b$ are two positive integers such that $3 \leq a \leq b$, then there exists a connected graph $G$ of order $p$ with $m_{tr}(G) = a$ and $m_{cr}(G) = b$.

**Proof.** We prove this theorem by considering two cases.

**Case 1.** $a = b$. Let $G$ be the complete graph of order $b$. Then by Corollary 2.4 and Theorem 1.5, we have $m_{tr}(G) = m_{cr}(G) = b$.

**Case 2.** $3 \leq a < b$. Let $P_{b-a} : u_1, u_2, \ldots, u_{b-a}$ be a path of order $b - a$. Let $H$ be the graph obtained from $P_{b-a}$ by adding $a$ new vertices $v_1, v_2, \ldots, v_{a-2}$, $u, v$ to $P_{b-a}$ and joining the vertices $u, v$ to $u_{b-a}$; and joining the vertices $v_1, v_2, \ldots, v_{a-2}$ to the vertices $u, v$; and joining the vertices $v_j(1 \leq j \leq a - 3)$ to $v_k(j + 1 \leq k \leq a - 2)$. The graph $G$ is obtained from $H$ and the complete graph $K_2$ with the vertex set $V(K_2) = \{x, y\}$, by joining the vertices $x, y$ to $u_1$; and joining the vertices $u$ and $v$, thereby
producing the graph $G$ and is shown in Figure 3.5. Let $S = \{v_1, v_2, \ldots, v_{a-2}, x, y\}$ be the set of all extreme vertices of $G$. By Theorem 2.3, every total restrained monophonic set of $G$ contains $S$. It is clear that, $S$ is a minimum total restrained monophonic set of $G$ and so $m_{tr}(G) = a$.

Let $S_1 = S \cup \{u_1, u_2, \ldots, u_{b-a}\}$ be the set of all extreme vertices and cutvertices of $G$. By Theorems 1.1 and 1.3, every connected restrained monophonic set of $G$ contains $S_1$. It is easily verified that $S_1$ is a minimum connected restrained monophonic set of $G$ and so $m_{cr}(G) = b$. ■

**Theorem 3.3.** For positive integers $a, b$ such that $3 \leq a \leq b$ with $b \leq 2a$, there exists a connected graph $G$ such that $m_r(G) = a$ and $m_{tr}(G) = b$.

**Proof.** Case 1. For $a = b$, the complete graph $K_a$ has the desired properties.

Case 2. $a < b$. Let $b = a + k$ where $1 \leq k \leq a$. Let $C_i : x_i, y_i, z_i, u_i, v_i, x_i (1 \leq i \leq k)$ be “$k$” copies of $C_5$. Let $H$ be the graph obtained from $C_i$ by identifying the vertices $x_i (1 \leq i \leq k)$, say $x$ be the identified vertices and joining the vertices $y_i$ and $u_i (1 \leq i \leq k)$. Let $G$ be the graph obtained from $H$ and the complete graph $K_{a-k}$ with the vertex set $V(K_{a-k}) = \{w_1, w_2, \ldots, w_{a-k}\}$ by joining each vertex $w_j (1 \leq j \leq a-k)$ to the vertex $x$ of $H$. The graph $G$ is shown in Figure 2.8. Let $S = \{w_1, w_2, \ldots, w_{a-k}, z_1, z_2, \ldots, z_k\}$ be the set of all extreme vertices of $G$. By Theorem 1.1, every restrained monophonic set of $G$ contains $S$. It is easily seen that $S$ is a minimum restrained monophonic set of $G$ and so $m_r(G) = a$.

By Theorem 2.3, every total restrained monophonic set of $G$ contains $S$. We observe that every minimum total restrained monophonic set of $G$ contains exactly one vertex from $\{y_i, u_i\}$ for every $i (1 \leq i \leq k)$. Thus $m_{tr}(G) \geq b$. Since $S_1 = S \cup \{u_1, u_2, \ldots, u_k\}$ is a total restrained monophonic set of $G$, it follows that $m_{tr}(G) = a + k = b$. ■
References


