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Regular cubeco graphs

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Abstract

We study regular cube-complementary graphs, that is, regular graphs whose complement and cube are isomorphic. We prove several necessary conditions for a graph to be regular cube-complementary, and characterize all cube-complementary circulant graphs with number of vertices is $9k$ where k is an integer.

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1. Introduction

Given a graph G and a positive integer d, a new graph G^d, called the d^{th} power of G, is defined as vertex set $V(G^d) = V(G)$ and two distinct vertices x and y are adjacent in G^d if the distance between x and y, $d(x, y)$, is at most d. Recall that a graph G is called kth complementary if the graph G^k, called the $kth - co$ of G, is isomorphic to the complement of G, $\overline{G}$. That is, $G^k \cong \overline{G}$. Cube-complementary graphs were studied by [1].

A graph G is caled $r-$regular if every vertex has degree r. Motivated by the study of cube-complementary graphs (cubeco for short), we study the cube-complementary regular graphs. These graphs are defined as graphs G for which G is regular graph and

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G^3 is isomorphic to the complement of G, i.e. $G^3 \cong \overline{G}$. Of course, also we will have $G \cong \overline{G^3}$.

After introducing the necessary basic terms and definitions, we provide basic examples of regular cubeco graphs. In Section 3, we give an upper-bound for n and show that there exist no regular cubeco circulant graphs of certain jumps for n larger than this upper-bound. This upper-bound improves computations significantly. We also characterize an infinite family of regular cubeco graphs. Basic properties in terms of girth, cute-vertex, radius, and diameter are also studied in Section 3. We finally end up with some possible open problems.

Unless stated otherwise, all graphs considered in this paper will be finite, simple and undirected. Let G be a graph. A $k - vertex$ of G is a vertex of degree k in G. An $n - vertex$ graph is a graph of order n, that is, a graph on exactly n vertices. We denote by $n(G)$ the number of vertices of G and by $m(G)$ the number of its edges. Given a vertex v in a graph G, we denote by $\deg(v, G)$ its degree, that is, the size of its neighborhood $N_G(v) := \{u \in V(G) : uv \in E(G)\}$. The closed neighborhood of v is the set $N_G(v) := N_G(v) \cup \{v\}$. Vertices that are further away from v by more than k distance is the set $N_{>k}(G, v) := \{u \in G : d(u, v) > k\}$. Vertices that are further away from v by exactly k distance is the set $N_k(G, v) := \{u \in G : d(u, v) = k\}$. The ball $B_k(v, G := \{u \in G : d(u, v) \leq k\}$.

By $\Delta(G)$ and $\delta(G)$ we denote the maximum and the minimum degrees of a vertex in G respectively. For two vertices u, v in a graph G, we denote by $d_G(u, v)$ the distance between u and v, that is, the number of edges on a shortest path connecting u and v; if there is no path connecting the two vertices, then the distance is defined to be infinite.

The eccentricity $ecc_G(u)$ of a vertex u in a graph G is maximum of the numbers $d_G(u, v)$ where $v \in V(G)$. The radius of a graph G, denoted $radius(G)$, is the minimum of the eccentricities of the vertices of G. The diameter of a graph G, denoted $diam(G)$, is the maximum of the eccentricities of the vertices of G, or, equivalently, the maximum distance between any two vertices in G. The girth of a graph G, denoted $girth(G)$, is the length of a shortest cycle in G (or infinity, if G has no cycles).

Given two graphs G and H, an isomorphism between G and H is a bijective mapping $\phi : V(G) \to V(H)$ such that for every two vertices $u, v \in V(G)$, we have $uv \in E(G)$ if and only if $\phi(u)\phi(v) \in E(H)$. If there exists an isomorphism between graphs G and H, we say that G and H are isomorphic, and denote this relation by $G \cong H$. An automorphism of a graph G is an isomorphism between G and itself. The complement of a graph G is the graph $\overline{G}$ with $V(\overline{G}) = V(G)$, in which two distinct vertices are adjacent if and only if they are not adjacent in G.

2. Preview

In this section, we give some results that were proved in [1].

Lemma 2.1. [1] If the cycle, C_n, is cubeco graph, then $n = 9$.

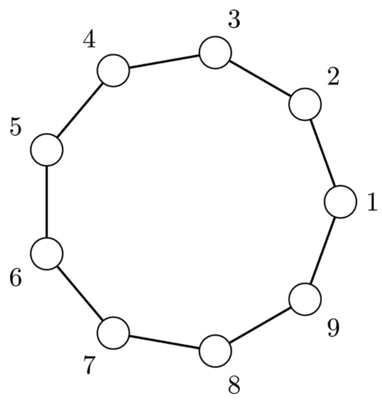

Figure 1: C_9

Theorem 2.2. [1] Let G be a cubeco graph. For every nonempty proper subset of S of $V(G)$ there exists a $u \in S$ and $v \in V(G) \backslash S$ such that $d_G(u, v) \geq 4$.

Theorem 2.3. [1] If G is a nontrivial cubeco graph, then $4 \leq radius(G) \leq diam(G) \leq 6$.

Recall that the graph G is called a circulant graph if it is a Cayley graph over the cyclic graph of order n denoted by $C_n(D)$, where $D \subseteq [\lfloor \frac{n}{2} \rfloor] := \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. In fact, the circulant $C_n(D)$ is the graph with vertex set $\{0, 1, \ldots, n-1\}$ and two distinct vertices $i, j \in [0, 1, \ldots, n-1]$ are adjacent if $|i - j| \in D$. The cycle C_n is the circulant graph $C_n\{1\}$. By Theorem 2.1, C_9 is a cubeco graph. Other circulant graphs are also cubeco graphs. In fact, it is known that the two circulant graphs $C_n(D)$ and $C_n(D')$ are isomorphic if there is a unit u in the ring Z_n with $uD = D'$.

The following are examples of non-isomorphic circulant cubeco graphs that were obtained using a computer:

$$C_{18}\{1, 8\},$$
$$C_{27}\{1, 8, 10\},$$
$$C_{29}\{1, 12\},$$
$$C_{27}\{1, 5\},$$
$$C_{36}\{1, 8, 10, 17\},$$
$$C_{43}\{1, 6, 7\},$$
$$C_{45}\{4, 5, 13, 14, 22\},$$
$$C_{61}\{1, 5, 24\},$$

and

$$C_{63}\{1, 5, 25\}.$$

We also showed that for any positive integer k, if G is cubeco graph, then $G[k]$ is also cubeco graph. $G[k]$ is defined as follows: Given an n-vertex graph G with vertices labeled $v_1, \dots, v_n$ and positive integers $k_1, \dots, k_n$, we denote by $G[k_1, k_2, \dots, k_n]$ the graph obtained from G by replacing each vertex v_i of G with a set U_i of nonadjacent k_i (new) vertices and joining vertices $u_i \in U_i$ and $u_j \in U_j$ with an edge if and only if v_i and v_j are adjacent in G. If $k_1 = \dots = k_n = k$, then we write $G[k]$ instead of $G[k_1, \dots, k_n]$,

3. Properties of regular cubeco graphs

In this section we find out several necessary conditions that every regular cube-complementary graph must satisfy. First we introduce the following theorem which shows that we have infinitely many circulant cubeco graphs. It should be mentioned that one can use the same technique to show that we can construct infinitely many cubeco based on any cubeco circulant graph.

Theorem 3.1. For any positive integer k, the circulant graph $G = C_{9k}(i, i \equiv 1 \mod 9)$ is cubeco.

Proof. For each $0 \leq j \leq 8$, define $U_j = \{m : m \equiv j \mod 9\}$, then for each $x \in U_j$ and $y \in U_{j+1}$, we have $x - y = 1 + 9k$ or $x - y = 1$. This means that x and y are adjacent, therefore, $G \cong C_9[k]$. ■

In the general case (non-regular), an open problem is that whether a cubeco graph with diameter 5 or 6 exists. The following theorem addresses this open problem for regular cubeco graphs.

Theorem 3.2. If G is regular cubeco graph, then, $diam(G) = 4$.

Proof. Let G be a regular cubeco graph with $diam(G) > 4$. Let u, v be two vertices with $d(u, v) = 5$ or 6. If $deg_G(u) = deg_G(v) = 5$, then, because G is cubeco graph, we have, $|N_{>3}(v, G)| > \delta + 1$. This is a contradiction, therefore, $diam(G) = 4$. ■

Based on our previous work [1] and above result, we can conclude that, if G is a regular cubeco graph, then $diam(G) = radius(G) = 4$.

It is still an open problem to show that whether there exists a cubeco graph of $girth > 9$. This is related to the existence of cut vertex in cube-complementary graphs, that is, if such graph exists, then it must contain a cut vertex, which is another open problem. The above discussion is true for non-regular cubeco graphs, the following theorem solves this problem for regular cubeco graphs.

Theorem 3.3. If G is regular cubeco graph, then, G can not have a cut vertex.

Proof. Let G be a regular cubeco graph with a cur vertex x. Let A, B be the connected components of $G - \{x\}$.

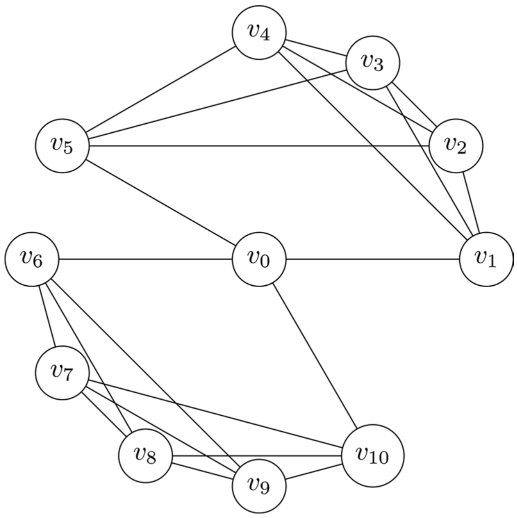

Figure 2: regular graph with v_0 cut-vertex

Using Theorem 2.2, then, there exists a $y \in G$ such that $d(y, x) > 3$. Since $diam(G) = 4$, we have $d(x, y) = 4$, this means either A or B is empty set which is a contradiction. ■

Theorem 3.4. If G is regular cubeco graph of $girth \geq 9$, then, $G \cong C_9$.

Proof. Let G be a regular cubeco graph with $girth(G) \geq 9$. Let v be any vertex in G, then, $\delta = |N(v, G)| = |N_{>3}(v, G)|$.

Since $diam(G) = 4$ and $girth(G) \geq 9$, one can conclude that $girth(G) = 9$. Now, suppose that $\delta > 3$, then, since $girth(G) = 9$, we have $|N_3(v, G)| = \delta(\delta - 1)^2$ vertex. Moreover, since $girth(G) = 9$ and $diam(G) = 4$, we have each vertex in $N_{>3}(v, G)$ is adjacent to exactly one vertex in $N_3(v, G)$. This means $|N_{>3}(v, G)| = \delta(\delta - 1)^2$ or $\delta = \delta(\delta - 1)^2$, hence, $\delta = 2$, and therefore, $G \cong C_9$. ■

An example of a regular cubeco graph of $girth = 3$ is $C_{43}\{1, 6, 7\}$, and of $girth = 4$ is $C_{18}\{1, 8\}$, and of $girth = 9$ is C_9. In general, one can easily show that if $|D| > 2$, then $girth(C_n(D)) < 4)$.

Theorem 3.5. A regular cubeco graph can not have $girth$ 8.

Proof. Let G be a regular cubeco graph of degree k. Suppose that $girth(G) = 8$ and let $v \in V(G)$, then, $|N(v, G)| = k$ and since $girth(G) = 8$, we have, $|N_2(v, G)| = k(k-1)$ and $|N_3(v, G)| = k^(k - 1)^2$, moreover, each vertex in $N_3(v, G)$ is adjacent to exactly one vertex in $B_3(v, G)$, so, the number of edges from $N_3(v, G)$ to $N_4(v, G)$ is at most $k(k - 1)^3$.

On the other hand, since $diam(G) = 4$ and G is cubeco, we have $|N_{>3}(v, G)| = k$, so, the number of edges from $N_{>3}(v, G)$ to $N_3(v, G)$ is at most k^2, therefore, $k \leq 2$, so, G is regular graph of degree 2 and therefore it must be a cycle or $G \cong C_8$ which is impossible. ■

It should be mentioned that we used computers to search for regular cubeco graphs, the search is time consuming, the following theorem puts a good upper bound on the number of vertices of the graph and the degree of each vertex, which reduces the computer search significantly especially for large n.

Theorem 3.6. If G is δ-regular cubeco graph, then, $n \leq \delta(\delta^2 - 2\delta + 2) + 1$.

Proof. Let G be a regular cubeco graph. Let v be any vertex in G, then, G contains at most $1 + \delta + \delta(\delta - 1) + \delta(\delta - 1)^2$ vertices in $B_3(v, G)$.

On the other hand, $N_{>3}(v, G) = N(v, \overline{G^3}) = \delta$. That is,

$$\delta \geq n - (\delta(\delta - 1)^2 + \delta(\delta - 1) + \delta + 1),$$

therefore, $n \leq \delta(\delta^2 - 2\delta + 2) + 1$. ■

4. Summary

In this paper we have studied cube-complementary regular graphs, we were able to prove several necessary conditions for a regular graph to be cube complementary and characterized all circulant cube-complementary graphs with number of vertices equal $9k$ where k is any integer.

Results obtained in this paper motivate a further study of regular cubeco graphs. Since a complete characterization of regular cubeco graphs seems perhaps too challenging, we pose the following:

Open problem:

- Is it true that for circulant graphs if $C_n(D)$ is cubeco, then, n is a multiple of 9 or n is a prime?

- In regular cubeco graphs, the $girth$ can be 3, 4, or, 9, but it can not be 8. Is there a regular cubeco graph of $girth$ 5, 6, or 7?

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