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Nonexistence of global solutions for a fractional problems with a nonlinearity of the Fisher type

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Abstract

This paper deals with the Cauchy problem for a nonlinear hyperbolic equation

$$D_{0|t}^{1+\alpha}u + D_{0|t}^{\beta}u + (-\Delta)^{\frac{\gamma}{2}}u = h(t,x) \mid u \mid^{p} \mid 1-u \mid^{q},$$

posed in $Q = \mathbb{R}^{+} \times \mathbb{R}^{N}$, where $p_i, q_i > 1$, $-1 < \alpha < 1$, $0 < \beta < 2$, $0 < \gamma \leq 2$, and $\beta < 1+\alpha$ with given initial position and velocity $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$, and the Cauchy problem for a nonlinear hyperbolic system with initial data

$$\begin{cases} D_{0|t}^{1+\alpha_1}u + D_{0|t}^{\beta_1}u + (-\Delta)^{\frac{\gamma_1}{2}}u = h_1(t,x) \mid v \mid^{p_1} \mid 1-v \mid^{q_1}, & (t,x) \in Q \\ D_{0|t}^{1+\alpha_2}v + D_{0|t}^{\beta_2}v + (-\Delta)^{\frac{\gamma_2}{2}}v = h_2(t,x) \mid u \mid^{p_2} \mid 1-u \mid^{q_2}, & (t,x) \in Q \\ u(x,0) = u_0(x) \geq 0, \quad u_t(x,0) = u_1(x) \geq 0, & x \in \mathbb{R}^N \\ v(x,0) = v_0(x) \geq 0, \quad v_t(x,0) = v_1(x) \geq 0, & x \in \mathbb{R}^N \end{cases}$$

where $-1 < \alpha_i < 1$, $0 < \beta_i < 2$, $0 < \gamma_i \leq 2$, and $\beta_i < 1+\alpha_i$. $D^{\alpha_i}(i = 1,2)$ denote the time-derivative of arbitrary order α_i in the sense of Caputo.

We find a critical exponent of Fujita type in the case of the particular values of the fractional order and the separate terms $p_i, q_i (i = 1,2)$ and N.

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1. Introduction

In fractional calculus, we us derivatives and integrals of non integer order (see [9, 10]). Initial value fractional differential equations and systems was studied in several papers (see [3, 4, 5, 6, 7]) where was involved Riemann-Liouville fractional differential operator of order $\alpha \in (0, 1)$.

Kirane and Tatar in [6], considered the Cauchy problem for the hyperbolic fractional equation

$$u_{tt} + D^{\beta}_{0|t}u = \Delta u + h(t, x) \mid u \mid^{p}, \tag{1}$$

where $p > 1$ and $\beta \in (0, 1)$. This equation is used to describe anomalous diffusion fractal media, biological phenmena etc. (see [8]). The two authors cited above established that the conditions

$$1 < p \leq 1 + \frac{2\beta + \rho}{2 + N - 2\beta} \tag{2}$$

on the initial data arise, then solution of the last equation (1) doses not exist globally.

A large number of searcher treated the case when $\beta = 1$, so a lot of results of nonexistence has been proved, also global existence results has been found while using the fractional telegraph equation $D^{2\beta}u + D^{\beta}u = \Delta u$, $0 < \beta \leq 1$, or studying various other hyperbolic fractional equations as Brownian motions for example. (see also [2]) where Fuquin and Mingxin used a critical exponent while studying a huperbolic system of reaction-diffusion type form a point of view of existence and nonexistence of the solutions.

In [12] , Tatar studied the following fractional differential problem

$$\begin{cases} D^{1+\alpha}u + D^{\beta}u = \Delta u + h(t, x) \mid u \mid^{p}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ u(0, x) = u_0(x) \in L^1_{loc}(\mathbb{R}^N), \;\; u_t(0, x) = u_1(x) \in L^1_{loc}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{3}$$

where $-1 < \alpha < 1$ and $0 < \beta < 2$. He proved that for $u_0(x), u_1(x) \geq 0$, $0 < \alpha, \beta < 1$ and the function h satisfies $h(t, x) > 0$, $h^{1-q} \in L^1_{loc}(\mathbb{R}^+ \times \mathbb{R}^N)$ and $h(tR^2, xR^{\beta}) = R^{\rho}h(t, x)$ for some $\rho > 0$ and large $R > 0$. then, if $1 < p \leq 1 + \frac{2\beta + \rho}{2 + \beta N - 2\beta}$, the problem (3) does not admit nontrivial solutions global in time.

In [11], Saoudi and Haouam considered the following fractional differential system

$$\begin{cases} D^{1+\alpha_1}_{0|t}u + D^{\beta_1}_{0|t}u + (-\Delta)^{\frac{\gamma_1}{2}}u = \mid v \mid^{p}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ D^{1+\alpha_2}_{0|t}v + D^{\beta_2}_{0|t}v + (-\Delta)^{\frac{\gamma_2}{2}}v = \mid u \mid^{q}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ u(x, 0) = u_0(x) \geq 0, \;\; u_t(x, 0) = u_1(x) \geq 0, & x \in \mathbb{R}^N \\ v(x, 0) = v_0(x) \geq 0, \;\; v_t(x, 0) = v_1(x) \geq 0, & x \in \mathbb{R}^N \end{cases} \tag{4}$$

Where $p, q > 1$, $-1 < \alpha_i < 1$, $0 < \beta_i < 2$ and $0 < \beta_i < 1 + \alpha_i$ $(i = 1, 2)$. They proved that for $p, q > 1$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$ $(i = 1, 2)$. If

$$\frac{N}{2} \leq \max\left\{\frac{1 + pq(\beta_2 - 1) + \beta_1 p}{\beta_1(p - 1) + \beta_2(q - 1)p}, \frac{1 + pq(\beta_1 - 1) + \beta_2 q}{\beta_2(q - 1) + \beta_1(p - 1)q}\right\} \quad for\ N \geq 1,$$

the problem (4) does not admit nontrivial global weak solutions.

In this paper, we consider two problems. The first problem is

$$\begin{cases} D^{1+\alpha}_{0|t} u + D^{\beta}_{0|t} u + (-\Delta)^{\frac{\gamma}{2}} u = h(t, x) \mid u \mid^{p} \mid 1 - u \mid^{q}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ u(0, x) = u_0(x) \in L^1_{loc}(\mathbb{R}^N), \ \ u_t(0, x) = u_1(x) \in L^1_{loc}(\mathbb{R}^N), & x \in \mathbb{R}^N \end{cases} \tag{5}$$

with given initial data and where $p, q > 1$, $-1 < \alpha < 1$, $0 < \beta < 2$, $\gamma < 2$ and $\beta < 1 + \alpha$. D^{α}, D^{β} denote respectively the time-derivatives of arbitrary order α and β in the sens of Caputo, $(-\Delta)^{\frac{\gamma}{2}}$ is the fractional power of the Laplacien $-\Delta_x$ in the x variable defined by

$$(-\Delta)^{\frac{\gamma}{2}} u(t, x) = \mathcal{F}^{-1}(\mid \xi \mid^{\gamma} \mathcal{F}(u)(\xi))(t, x),$$

where $\mathcal{F}$ is the Fourier transform and $\mathcal{F}^{-1}$ its inverse. And the second one is

$$\begin{cases} D^{1+\alpha_1}_{0|t} u + D^{\beta_1}_{0|t} u + (-\Delta)^{\frac{\gamma_1}{2}} u = h_1(t, x) \mid v \mid^{p_1} \mid 1 - v \mid^{q_1}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ D^{1+\alpha_2}_{0|t} v + D^{\beta_2}_{0|t} v + (-\Delta)^{\frac{\gamma_2}{2}} v = h_2(t, x) \mid u \mid^{p_2} \mid 1 - u \mid^{q_2}, & (t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{N} \\ u(0, x) = u_0(x) \in L^1_{loc}(\mathbb{R}^N), \ \ u_t(0, x) = u_1(x) \in L^1_{loc}(\mathbb{R}^N), & x \in \mathbb{R}^N \\ v(0, x) = v_0(x) \in L^1_{loc}(\mathbb{R}^N), \ \ v_t(0, x) = v_1(x) \in L^1_{loc}(\mathbb{R}^N), & x \in \mathbb{R}^N)) \end{cases} \tag{6}$$

where $-1 < \alpha_i < 1$, $0 < \beta_i < 2$, $0 < \gamma_i \leq 2$, and $\beta_i < 1 + \alpha_i$.

2. Organization and Aim

Our paper is organized as follows:

- In section 3, we present the definitions of the fractional derivative in the sens of Riemann-Liouville and the fractional derivative in the sens of Caputo and the relationship between these two definitions.
- We also give the definition of a week solution of the cited problems.
- section 4, is devoted to a result of nonexistence of solutions for the fractional system (5)
- In section 5, we establish a result of nonexistence of solutions for the fractional system (6).

Remark 2.1. Especially the second term in equation (5) and in the system (6) are taken in a Fisher type form (see [1]), which interpret a mathematical model for the simulation growth and spread of a particular bacterial population in an unbounded domain R.

Remark 2.2. In the case $q = 0$ and $\gamma = 2$, the problem (5) reduces to the Cauchy problem (3) studied in [12].

Remark 2.3. In the case $h_1(t, x) = h_2(t, x) = 1, q_1 = q_2 = 0$ and $\gamma_1 = \gamma_2 = 2$ the system (6) reduces to the system (4) studied in [11].

3. Preliminaries

In this section, we present two different definitions of fractional derivatives, some of their properties and the definition of weak solutions to our problem (5).

We define the left-handed derivative and the right-handed derivative in the Riemann-Liouville sense respectively as follows:

$$D^{\gamma}_{0|t} f(t) = \frac{1}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_0^t (t-\tau)^{n-\gamma-1} f(\tau)d\tau, \quad n = [\gamma]+1, \;\; \gamma > 0.$$

$$D^{\gamma}_{t|T} f(t) = \frac{(-1)^n}{\Gamma(n-\gamma)} \left(\frac{d}{dt}\right)^n \int_t^T (\tau-t)^{n-\gamma-1} f(\tau)d\tau, \quad n = [\gamma]+1, \;\; \gamma > 0.$$

the Caputo derivative, in a general case, is given by

$$\mathbf{D}^{\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{n-\gamma-1} f^{(n)}(\tau)d\tau, \quad n = [\gamma]+1, \;\; \gamma > 0.$$

Therefore the Caputo derivative is related to the left-handed Riemann-Liouville derivative (see [9]) as follows:

$$D^{\gamma}_{t|T} f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)t^{k-\gamma}}{\Gamma(1+k-\gamma)} + \mathbf{D}^{\gamma} f(t).$$

We have also the following formula of integration by parts,(see [10])

$$\int_0^T f(t) D^{\gamma}_{0|t} g(t)dt = \int_0^T g(t) D^{\gamma}_{t|T} f(t)dt, \quad 0 < \gamma < 1.$$

Remark 3.1. The above defined integrals are assumed to be convergent and the solution is called global if $T = +\infty$.

Denoting by Q_T the set $Q_T = (0, T) \times \mathbb{R}^N$ and by $L^p_{loc}(Q_T, hdtdx)$ the space of all functions $v : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$ such that $\int_K | v |^p \; h(t, x)dtdx < +\infty$ for any compact K in $\mathbb{R}^+ \times \mathbb{R}^N$.

Definition 3.1. Let $0 < \alpha < 1$ and $0 < \beta < 1$. A weak solution of (5) is a locally integrable function u such that $u \in L^p_{loc}(Q_T, hdtdx)$ and

$$\begin{aligned} & \int_{Q_T} \varphi h \mid u \mid^p \mid 1 - u \mid^q \, dtdx \\ = & \int_{Q_T} u(t,x) D^{\alpha+1}_{t|T} \varphi dtdx - \int_{Q_T} u_1(x) D^{\alpha}_{t|T} \varphi dtdx \\ - & \int_{\mathbb{R}^N} u_0(x) D^{\alpha}_{t|T} \varphi(0) dx + \int_{Q_T} [u(t,x) - u_0(x)] D^{\beta}_{t|T} \varphi dtdx \\ + & \int_{Q_T} u(t,x)(-\Delta)^{\frac{\gamma}{2}} \varphi dtdx \end{aligned}$$

holds for any $\varphi \in C^2_0(Q_T)$, $\varphi \geq 0$ and satisfying $\varphi(T,x) = D^{\alpha}_{t|T}\varphi(T,x) = 0$.

Definition 3.2. Suppose that $0 < \alpha < 1$, $1 < \beta < 2$ and $\beta \leq 1 + \alpha$. A weak solution of (5) is a locally integrable function u such that $u \in L^p_{loc}(Q_T, hdtdx)$ and

$$\begin{aligned} \int_{Q_T} \varphi h \mid u \mid^p \mid 1 - u \mid^q \, dtdx = & \int_{Q_T} u(t,x) D^{\alpha+1}_{t|T} \varphi dtdx - \int_{Q_T} u_1(x) D^{\alpha}_{t|T} \varphi dtdx \\ - & \int_{\mathbb{R}^N} u_0(x) D^{\alpha}_{t|T} \varphi(0) dx + \int_{Q_T} u(t,x) D^{\beta}_{t|T} \varphi dtdx \\ - & \int_{Q_T} u_1(x) D^{\beta-1}_{t|T} \varphi dtdx - \int_{\mathbb{R}^N} u_0(x) D^{\beta-1}_{t|T} \varphi(0) dx \\ - & \int_{Q_T} u(t,x)(-\Delta)^{\frac{\gamma}{2}} \varphi dtdx \end{aligned}$$

holds for any $\varphi \in C^2_0(Q_T)$, $\varphi \geq 0$ and satisfying

$$\varphi(T,x) = D^{\alpha}_{t|T}\varphi(T,x) = D^{\beta-1}_{t|T}\varphi(T,x) = 0.$$

Remark 3.2. In order to get weak formulation in the above definitions, we used some added properties as:

$$D^{1+\alpha}_{0|t} f = D.D^{\alpha}_{0|t} f \quad \text{and} \quad D^{1+\alpha}_{t|T} f = -D.D^{\alpha}_{t|T} f$$

and the exponent property

$$\mathbf{D}^{n+\alpha} f(t) = \mathbf{D}^n \mathbf{D}^{\alpha} f(t), \quad 0 < \alpha < 1, \ \ n = 1, 2, \ldots$$

4. Nonexistence result

Here we consider only the case $0 < \alpha < 1$ and $0 < \beta < 1$. The other cases can be treated similarly using the appropriate definition.

We announce our first result as a theorem.

Theorem 4.1. Suppose that $u_0(x), u_1(x) \geq 0$, $0 < \alpha, \beta < 1$, $u \neq 1$ and the function h satisfies $h(t,x) > 0$ and $h(tR^2, xR^\beta) = R^\rho h(t,x)$ for some $\rho > 0$ and large $R > 0$. Then, if $1 < p \leq 1 + \dfrac{\beta\gamma + \rho}{2 + \beta N - \beta\gamma}$, the problem (5) does not admit nontrivial global solutions in time.

Proof. Proceed by contradiction that a solution exists for all time $t > 0$. and let us consider the solution u on $(0, T_\star)$ and let T and R be two positive constants such that $0 < TR^2 < T_\star$. As a test function, we consider

$$\varphi(t,x) = \varphi_0\left(\frac{t^{2\beta} + |\, x \,|^4}{R^{4\beta}}\right)$$

such that $\varphi(TR^2, x) = D^\alpha_{t|TR^2}\varphi(t,x)\Big|_{TR^2} = 0$. The function $\varphi_0 \in C_0^2(R_+)$ is nonnegative, nonincreasing and satisfying

$$\varphi_0(z) = \begin{cases} 1 & if \;\; 0 \leq z \leq 1, \\ 0 & if \;\; z \geq 2, \end{cases}$$

and $0 \leq \varphi_0 \leq 1$.

From definition 3.1, the weak formulation of solution to our problem is

$$\begin{aligned} &\int_{Q_{TR^2}} \varphi h \,|\, u \,|^p |\, 1 - u \,|^q \, dtdx + \int_{Q_{TR^2}} u_1(x) D^\alpha_{t|TR^2}\varphi dtdx \\ &+ \int_{Q_{TR^2}} u_0(x) D^\beta_{t|TR^2}\varphi dtdx + \int_{\mathbb{R}^N} u_0(x) D^\alpha_{t|T}\varphi(0) dx \\ &= \int_{Q_{TR^2}} u(t,x) D^{\alpha+1}_{t|TR^2}\varphi dtdx \\ &+ \int_{Q_{TR^2}} u(t,x) D^\beta_{t|TR^2}\varphi dtdx + \int_{Q_{TR^2}} u(t,x)(-\Delta)^{\frac{\gamma}{2}}\varphi dtdx. \end{aligned} \tag{7}$$

It is clear from the definitions of the test function and the derivative function that $D^\alpha_{t|T}\varphi \geq 0$ and $D^\beta_{t|T}\varphi \geq 0$, then

$$\begin{aligned} \int_{Q_{TR^2}} \varphi h \,|\, u \,|^p |\, 1 - u \,|^q \, dtdx \;\leq\; & \int_{Q_{TR^2}} u(t,x) D^{\alpha+1}_{t|TR^2}\varphi dtdx \\ & + \int_{Q_{TR^2}} u(t,x) D^\beta_{t|TR^2}\varphi dtdx \\ & + \int_{Q_{TR^2}} u(t,x)(-\Delta)^{\frac{\gamma}{2}}\varphi dtdx. \end{aligned} \tag{8}$$

Now, to follow the proof, the test function φ is chosen so that

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\alpha+1}_{t|TR^2}\varphi \mid^{\frac{P}{p-1}} dtdx < \infty,$$

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\beta}_{t|TR^2}\varphi \mid^{\frac{P}{p-1}} dtdx < \infty,$$

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}}\varphi \mid^{\frac{P}{p-1}} dtdx < \infty.$$

By the ε-Young inequality, we have

$$\begin{aligned}
&\int_{Q_{TR^2}} uD^{\alpha+1}_{t|TR^2}\varphi dtdx \\
&= \int_{Q_{TR^2}} u(1-u)^{\frac{q}{p}}(\varphi h)^{\frac{1}{p}}(1-u)^{-\frac{q}{p}}(D^{\alpha+1}_{t|TR^2}\varphi)(\varphi h)^{\frac{-1}{p}} dtdx \\
&\leq \varepsilon \int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q dxdt \\
&+C_\varepsilon \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} \mid D^{\alpha+1}_{t|TR^2}\varphi \mid^{\frac{P}{p-1}} (\varphi h)^{-\frac{1}{p-1}} dtdx.
\end{aligned} \tag{9}$$

Similarly,

$$\begin{aligned}
&\int_{Q_{TR^2}} uD^{\beta}_{t|TR^2}\varphi dtdx \\
&= \int_{Q_{TR^2}} u(1-u)^{\frac{q}{p}}(\varphi h)^{\frac{1}{p}}(1-u)^{-\frac{q}{p}}(D^{\beta}_{t|TR^2}\varphi)(\varphi h)^{\frac{-1}{p}} dtdx \\
&\leq \varepsilon \int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q dxdt \\
&+C_\varepsilon \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} \mid D^{\beta}_{t|TR^2}\varphi \mid^{\frac{P}{p-1}} (\varphi h)^{-\frac{1}{p-1}} dtdx.
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\int_{Q_{TR^2}} u(-\Delta)^{\frac{\gamma}{2}}\varphi dtdx \\
&= \int_{Q_{TR^2}} u(1-u)^{\frac{q}{p}}(\varphi h)^{\frac{1}{p}}(1-u)^{-\frac{q}{p}}((-\Delta)^{\frac{\gamma}{2}}\varphi)(\varphi h)^{\frac{-1}{p}} dtdx \\
&\leq \varepsilon \int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q dxdt \\
&+C_\varepsilon \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}}\varphi \mid^{\frac{P}{p-1}} (\varphi h)^{-\frac{1}{p-1}} dtdx.
\end{aligned} \tag{11}$$

Taking into account (9)-(11) in (8) we infer, for $\varepsilon < \dfrac{1}{3}$ that

$$\int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q \, dxdt \le C_\varepsilon \Big[A_1 + A_2 + A_3 \Big]. \tag{12}$$

Where

$$A_1 = \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\alpha+1}_{t|TR^2} \varphi \mid^{\frac{P}{p-1}} dtdx \tag{13}$$

$$A_2 = \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\beta}_{t|TR^2} \varphi \mid^{\frac{P}{p-1}} dtdx \tag{14}$$

$$A_3 = \int_{Q_{TR^2}} \mid 1-u \mid^{-\frac{q}{p-1}} (\varphi h)^{-\frac{1}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}} \varphi \mid^{\frac{P}{p-1}} dtdx. \tag{15}$$

Now, we estimate the right hand of (12). For $u > 1$, $(u \neq 1)$ we distingue two cases.

- **First case**: If $0 < u < 1$, then $\exists\, r > 0$: $0 < u < r < 1$ and we have $\mid 1-u \mid^{-\frac{q^2}{p-1}} < C_{p,q}$.
- **Second case**: If $u > 1$, then $\exists\, r > 0$: $u > r > 1$ that is $\mid 1-u \mid^{-\frac{q^2}{p-1}} < C_{p,q}$. So, we have

$$\forall u > 0, (u \neq 1) : \mid 1-u \mid^{-\frac{q^2}{p-1}} < C_{p,q}. \tag{16}$$

Using (16) and (12), we can write

$$\begin{aligned} &\int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q \, dtdx \le \\ &C\Big[\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\alpha+1}_{t|TR^2} \varphi \mid^{\frac{P}{p-1}} dtdx \\ &+ \int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\beta}_{t|TR^2} \varphi \mid^{\frac{P}{p-1}} dtdx \\ &+ \int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}} \varphi \mid^{\frac{P}{p-1}} dtdx \Big]. \end{aligned} \tag{17}$$

For some generic positive constant C.

Next, we introduce the scaled variables $t = R^2 \tau$ and $x = R^\beta y$, we define the set Ω and the function χ by

$$\Omega = \{(\tau, y) \in \mathbb{R}^+ \times \mathbb{R}^N : \tau^{2\beta} + \mid y \mid^4 \le 2\}$$

and

$$\chi(\tau, y) = \varphi(R^2\tau, R^\beta y) = \varphi(t, x).$$

Clearly, we have

$$dtdx = R^{2+\beta N} d\tau dy,$$
$$D^{\alpha+1}_{t|TR^2}\varphi = R^{-2(\alpha+1)} D^{\alpha+1}_{\tau|T}\chi,$$
$$D^{\beta}_{t|TR^2}\varphi = R^{-2\beta} D^{\beta}_{\tau|T}\chi,$$

and

$$(-\Delta\varphi)^{\frac{\gamma}{2}} = R^{-\beta\gamma}(-\Delta\chi)^{\frac{\gamma}{2}}.$$

Substitution gives:

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\alpha+1}_{t|TR^2}\varphi \mid^{\frac{p}{p-1}} dtdx = R^{\beta N+2-\frac{2(\alpha+1)p}{p-1}-\frac{\rho}{p-1}} \int_{\Omega} (\chi h)^{-\frac{1}{p-1}} \mid D^{\alpha+1}_{\tau|T}\chi \mid^{\frac{p}{p-1}} d\tau dy \tag{18}$$

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid D^{\beta}_{t|TR^2}\varphi \mid^{\frac{p}{p-1}} dtdx = R^{\beta N+2-\frac{2(\alpha+1)p}{p-1}-\frac{\rho}{p-1}} \int_{\Omega} (\chi h)^{-\frac{1}{p-1}} \mid D^{\beta}_{\tau|T}\chi \mid^{\frac{p}{p-1}} d\tau dy \tag{19}$$

$$\int_{Q_{TR^2}} (\varphi h)^{-\frac{1}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}}\varphi \mid^{\frac{p}{p-1}} dtdx = R^{\beta N+2-\frac{\beta\gamma p}{p-1}-\frac{\rho}{p-1}} \int_{\Omega} (\chi h)^{-\frac{1}{p-1}} \mid (-\Delta)^{\frac{\gamma}{2}}\chi \mid^{\frac{p}{p-1}} d\tau dy. \tag{20}$$

These relations (18)-(20) together with (17) imply that

$$\int_{Q_{TR^2}} \varphi h \mid u \mid^p \mid 1-u \mid^q dtdx \le C R^{\beta N+2-\frac{\beta\gamma p}{p-1}-\frac{\rho}{p-1}} \tag{21}$$

$$\int_{\Omega} (\chi h)^{-\frac{1}{p-1}} \left[\mid D^{\alpha+1}_{\tau|T}\chi \mid^{\frac{p}{p-1}} + \mid D^{\beta}_{\tau|T}\chi \mid^{\frac{p}{p-1}} + \mid (-\Delta)^{\frac{\gamma}{2}}\chi \mid^{\frac{p}{p-1}} \right] d\tau dy \le C R^{\beta N+2-\frac{\beta\gamma p}{p-1}-\frac{\rho}{p-1}}. \tag{22}$$

Observe that $\beta N + 2 - \dfrac{\beta\gamma p}{p-1} - \dfrac{\rho}{p-1} \le 0$ is equivalent to our assumption $p \le 1 + \dfrac{\beta\gamma+\rho}{2+\beta N-\beta\gamma}$.

First case:

If $p < 1 + \dfrac{\beta\gamma+\rho}{2+\beta N-\beta\gamma}$, then $\lim_{R\to+\infty} \int_{QTR^2} h \mid u \mid^p \mid 1-u \mid^q = 0$. This implies that

$u = 0$, Since $h(t,x) > 0$ on $R^+ \times R^N$ and $u \neq 1$. This is a contradiction.

Second case:
If $p = 1 + \dfrac{\beta\gamma + \rho}{2 + \beta N - \beta\gamma}$, then from (21), we have

$$\int_{Q_\infty} h \mid u \mid^p \mid 1 - u \mid^q \leq C. \tag{23}$$

Applying Hölder inequality to all three terms in the right-hand side of (8), we find

$$\begin{aligned}
&\int_{QTR^2} \varphi h \mid u \mid^p \mid 1 - u \mid^q \\
\leq\ &\left(\int_{C_R} \varphi h \mid u \mid^p \mid 1 - u \mid^q\right)^{\frac{1}{p}} \\
&.\left(\int_{C_R} \mid 1 - u \mid^{-\frac{q}{pp'}} (\varphi h)^{-\frac{p'}{p}} \left[\mid D^{\alpha+1}_{\tau|T} \chi \mid^{p'} + \mid D^{\beta}_{\tau|T} \chi \mid^{p'} + \mid (-\Delta)^{\frac{\gamma}{2}} \chi \mid^{p'}\right]\right)^{\frac{1}{p'}},
\end{aligned}$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1 \ \ and \quad C_R = \{(t,x) \in R^+ \times R^N : 0 \leq t^{2\beta} + \mid x \mid^4 \leq 2R^{4\beta}\}.$$

passing to the limit as $R \to \infty$, and using the convergence of the integral in (22), we get

$$\int_{Q_\infty} h \mid u \mid^p \mid 1 - u \mid^q = 0, i.e\ \ u = 0 \ \ (\text{since } \ h(t,x) > 0, u \neq 1).$$

We conclude that there cannot exist nontrivial global solutions. ■

Remark 4.1. If $\gamma = 2, q = 0$, we obtain the critical exponent $p \leq 1 + \dfrac{2\beta + \rho}{2 + \beta N - 2\beta}$ of the problem (3) treated by Nasser-eddine Tatar in [12].

5. System of fractional equations

In this section we consider the Cauchy problem (6) for a nonlinear hyperbolic fractional system with initial data, so we are able now to give our second result.

Theorem 5.1. Let $N > 1$, $p > 1, q > 1$, $0 < \alpha_i < 1$, $0 < \beta_i < 1$, for $i = 1, 2$, then if

$$N \leq \max \left\{ \frac{2 + 2p_1p_2(\beta_2 - 1) + 2\beta_1 p_1 + \rho(p_1 + 1)}{\beta_1(p_1 - 1) + \beta_2 p_1(p_2 - 1)}, \frac{2 + 2p_1p_2(\beta_1 - 1) + 2\beta_2 p_2 + \rho(p_2 + 1)}{\beta_2(p_2 - 1) + \beta_1 p_2(p_1 - 1)} \right\}$$

for $N \geq 1$. Then the system (6) does not admit nontrivial global weak solutions.

Proof. We proceed always by contradiction. Suppose that the nontrivial nonnegative solution $u \neq 1$ exists for all time $t > 0$ in $(0, T^\star)$, with arbitrary $T^\star > 0$.

Let T and R be two positive constants such that $0 < TR^2 < T^\star$. We consider the test function

$$\varphi_j(t,x) = \varphi_0\left(\frac{t^{2\beta_j} + \mid x \mid^4}{R^{4\beta_j}}\right), \quad j = 1, 2$$

such that $\varphi_j(TR^2, x) = D_{t|TR^2}\varphi_j(t,x)\Big|_{TR^2} = 0$.

The function $\varphi_0 \in C_0^2(R_+)$ is nonnegative, nonincreasing and satisfying

$$\varphi_0(z) = \begin{cases} 1 & if \;\; 0 \leq z \leq 1, \\ 0 & if \;\; z \geq 2, \end{cases}$$

and $0 \leq \varphi_0 \leq 1$.

From the definition 3.1 the weak formulation of solution to our problem is

$$\begin{aligned} & \int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1} \mid 1 - v \mid^{q_1} dtdx + \int_{Q_{TR^2}} u_1(x) D^{\alpha_1}_{t|TR^2}\varphi_1 dtdx \\ + & \int_{Q_{TR^2}} u_0(x) D^{\beta_1}_{t|TR^2}\varphi_1 dtdx + \int_{\mathbb{R}^N} u_0(x) D^{\alpha_1}_{t|TR^2}\varphi_1(0) dx \\ = & \int_{Q_{TR^2}} u(t,x) D^{\alpha_1+1}_{t|TR^2}\varphi_1 dtdx \\ + & \int_{Q_{TR^2}} u(t,x) D^{\beta_1}_{t|TR^2}\varphi_1 dtdx + \int_{Q_{TR^2}} u(t,x)(-\Delta)^{\frac{\gamma_1}{2}}\varphi_1 dtdx \end{aligned} \tag{24}$$

and

$$\begin{aligned} & \int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1 - u \mid^{q_2} dtdx + \int_{Q_{TR^2}} v_1(x) D^{\alpha_2}_{t|TR^2}\varphi_2 dtdx \\ + & \int_{Q_{TR^2}} v_0(x) D^{\beta_2}_{t|TR^2}\varphi_2 dtdx + \int_{\mathbb{R}^N} v_0(x) D^{\alpha_2}_{t|TR^2}\varphi_2(0) dx \\ = & \int_{Q_{TR^2}} v(t,x) D^{\alpha_2+1}_{t|TR^2}\varphi_2 dtdx \\ + & \int_{Q_{TR^2}} v(t,x) D^{\beta_2}_{t|TR^2}\varphi_2 dtdx + \int_{Q_{TR^2}} v(t,x)(-\Delta)^{\frac{\gamma_2}{2}}\varphi_2 dtdx. \end{aligned} \tag{25}$$

From (24) and (25), while $D^{\alpha_i}_{t|TR^2}\varphi_i \geq 0$ and $D^{\beta_i}_{t|TR^2}\varphi_i \geq 0$, $i, j = 1, 2$ then we obtain the following estimates

$$\begin{aligned}
&\int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1} \mid 1 - v \mid^{q_1} \, dtdx \\
\leq & \int_{Q_{TR^2}} u(t,x) D^{\alpha_1+1}_{t|TR^2} \varphi_1 dtdx + \int_{Q_{TR^2}} u(t,x) D^{\beta_1}_{t|TR^2} \varphi_1 dtdx \\
+ & \int_{Q_{TR^2}} u(t,x) (-\Delta)^{\frac{\gamma_1}{2}} \varphi_1 dtdx. \qquad (26)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1 - u \mid^{q_2} \, dtdx \\
\leq & \int_{Q_{TR^2}} v(t,x) D^{\alpha_2+1}_{t|TR^2} \varphi_2 dtdx + \int_{Q_{TR^2}} v(t,x) D^{\beta_2}_{t|TR^2} \varphi_2 dtdx \\
+ & \int_{Q_{TR^2}} v(t,x) (-\Delta)^{\frac{\gamma_2}{2}} \varphi_2 dtdx. \qquad (27)
\end{aligned}$$

Now, we estimate the quantities which are in the second parts from (26) and (27). By using the Hölder inequality we get

$$\begin{aligned}
\int_{Q_{TR^2}} u(t,x) D^{\alpha_1+1}_{t|TR^2} \varphi_1 \leq & \left(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1 - u \mid^{q_2} \right)^{\frac{1}{p_2}} \\
& . \left(\int_{Q_{TR^2}} \mid 1 - u \mid^{-\frac{q_2 p_2'}{p_2}} \mid D^{\alpha_1+1}_{t|TR^2} \varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}} \right)^{\frac{1}{p_2'}} \\
\leq & C \left(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1 - u \mid^{q_2} \right)^{\frac{1}{p_2}} \\
& \left(\int_{Q_{TR^2}} \mid D^{\alpha_1+1}_{t|TR^2} \varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}} \right)^{\frac{1}{p_2'}} \qquad (28)
\end{aligned}$$

and

$$\begin{aligned}
\int_{Q_{TR^2}} u(t,x) D^{\beta_1}_{t|TR^2} \varphi_1 \leq & C \left(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1 - u \mid^{q_2} \right)^{\frac{1}{p_2}} \\
& . \left(\int_{Q_{TR^2}} \mid D^{\beta_1}_{t|TR^2} \varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}} \right)^{\frac{1}{p_2'}} \qquad (29)
\end{aligned}$$

we also have:

$$\int_{Q_{TR^2}} u(t,x)(-\Delta)^{\frac{\gamma_1}{2}}\varphi_1 \leq C\Big(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2}\mid 1-u \mid^{q_2}\Big)^{\frac{1}{p_2}} .\Big(\int_{Q_{TR^2}} \mid (-\Delta)^{\frac{\gamma_1}{2}}\varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}}\Big)^{\frac{1}{p_2'}}. \qquad (30)$$

Consequently,

$$\int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1}\mid 1-v \mid^{q_1} \leq C\Big(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2}\mid 1-u \mid^{q_2}\Big)^{\frac{1}{p_2}}.\mathcal{A} \qquad (31)$$

where

$$\mathcal{A} = \Big(\int_{Q_{TR^2}} \mid D^{\alpha_1+1}_{t|TR^2}\varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}}\Big)^{\frac{1}{p_2'}} +\Big(\int_{Q_{TR^2}} \mid D^{\beta_1}_{t|TR^2}\varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}}\Big)^{\frac{1}{p_2'}} +\Big(\int_{Q_{TR^2}} \mid (-\Delta)^{\frac{\gamma_1}{2}}\varphi_1 \mid^{p_2'} (\varphi_2 h)^{-\frac{p_2'}{p_2}}\Big)^{\frac{1}{p_2'}}.$$

Similarly, we have the estimate

$$\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2}\mid 1-u \mid^{q_2} \leq C\Big(\int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1}\mid 1-v \mid^{q_1}\Big)^{\frac{1}{p_1}}.\mathcal{B} \qquad (32)$$

where

$$\mathcal{B} = \Big(\int_{Q_{TR^2}} \mid D^{\alpha_2+1}_{t|TR^2}\varphi_2 \mid^{p_1'} (\varphi_1 h)^{-\frac{p_1'}{p_1}}\Big)^{\frac{1}{p_1'}} +\Big(\int_{Q_{TR^2}} \mid D^{\beta_2}_{t|TR^2}\varphi_2 \mid^{p_1'} (\varphi_1 h)^{-\frac{p_1'}{p_1}}\Big)^{\frac{1}{p_1'}} +\Big(\int_{Q_{TR^2}} \mid (-\Delta)^{\frac{\gamma_2}{2}}\varphi_2 \mid^{p_1'} (\varphi_1 h)^{-\frac{p_1'}{p_1}}\Big)^{\frac{1}{p_1'}}.$$

Inequalities (31) and (32) imply that

$$\Big(\int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1}\mid 1-v \mid^{q_1}\Big)^{1-\frac{1}{p_1p_2}} \leq C\mathcal{B}^{\frac{1}{p_2}}.\mathcal{A}$$

and

$$\left(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1-u \mid^{q_2}\right)^{1-\frac{1}{p_1 p_2}} \leq C\mathcal{A}^{\frac{1}{p_1}}.\mathcal{B}.$$

In $\mathcal{A}$ and $\mathcal{B}$, we use the scaled variables $t = R^2\tau$ and $x = R^{\beta_i} y$, $i = 1, 2$ we obtain

$$\left(\int_{Q_{TR^2}} \varphi_1 h \mid v \mid^{p_1} \mid 1-v \mid^{q_1}\right)^{1-\frac{1}{p_1 p_2}} \leq C_1 (R^{k_1})^{\frac{1}{p_2}} R^{k_2}$$

and

$$\left(\int_{Q_{TR^2}} \varphi_2 h \mid u \mid^{p_2} \mid 1-u \mid^{q_2}\right)^{1-\frac{1}{p_1 p_2}} \leq C_1 (R^{k_2})^{\frac{1}{p_1}} R^{k_1}.$$

Where

$$k_1 = \frac{2+\beta_1 N}{p_1'} - 2\beta_1 - \frac{\rho}{p_1} \quad and \quad k_2 = \frac{2+\beta_2 N}{p_2'} - 2\beta_2 - \frac{\rho}{p_2}.$$

Noting that

$$\frac{k_1}{p_2} + k_2 \iff N \leq \frac{2+2p_1p_2(\beta_2-1)+2\beta_1 p_1 + \rho(p_1+1)}{\beta_1(p_1-1)+\beta_2 p_1(p_2-1)}$$

and

$$\frac{k_1}{p_2} + k_2 \iff N \leq \frac{2+2p_1p_2(\beta_1-1)+2\beta_2 p_2 + \rho(p_2+1)}{\beta_2(p_2-1)+\beta_1 p_2(p_1-1)},$$

while $\dfrac{1}{p_1} + \dfrac{1}{p_1'} = 1$ and $\dfrac{1}{p_2} + \dfrac{1}{p_2'} = 1$. We obtain from a sufficient assumption as a critical exponent:

$$N \leq \max \left\{ \frac{2+2p_1p_2(\beta_2-1)+2\beta_1 p_1 + \rho(p_1+1)}{\beta_1(p_1-1)+\beta_2 p_1(p_2-1)}, \frac{2+2p_1p_2(\beta_1-1)+2\beta_2 p_2 + \rho(p_2+1)}{\beta_2(p_2-1)+\beta_1 p_2(p_1-1)} \right\}$$

for $N \geq 1$.

Letting $R \to \infty$ in

$$\Omega_i = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N : 0 \leq t^{2\beta_i} + \mid x \mid^4 \leq 2R^{4\beta_i}\}$$

and with the convergence of certain integrals, we conclude that this brings us to

$$\int_{\mathbb{R}^+\times\mathbb{R}^N} h \mid u \mid^{p_2} \mid 1-u \mid^{q_2} = 0 \;\; i.e. \;\; \int_{\mathbb{R}^+\times\mathbb{R}^N} \mid u \mid^{p_2} = 0,$$

so this leads to $u = 0$. Then the nontrivial global solutions cannot exist. This completes the proof. ■

Remark 5.1. When $\rho = 0$, and $q_1 = q_2 = 0$ we recover the system studied by Saoudi and Haouam [11], consequently we get the same estimate found by them, i.e.

$$\frac{N}{2} \leq \max\left\{\frac{1 + p_1 p_2(\beta_2 - 1) + \beta_1 p_1}{\beta_1(p_1 - 1) + \beta_2 p_1(p_2 - 1)}, \frac{1 + p_1 p_2(\beta_1 - 1) + \beta_2 p_2}{\beta_2(p_2 - 1) + \beta_1 p_2(p_1 - 1)}\right\},$$

for $N \geq 1$.

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