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Modified degenerate tangent numbers and polynomials

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Abstract

In this paper, we introduce the modified degenerate tangent numbers and polynomials. We also obtain some explicit formulas for modified degenerate tangent numbers and polynomials.

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1. Introduction

Recently, many mathematicians have studied in the area of degenerate Bernoulli polynomials and Euler polynomials (see [1, 2, 3, 4, 6]). For example, in [2], L. Carlitz introduced the degenerate Bernoulli polynomials. In [3], Feng Qi *et al.* studied the partially degenerate Bernoull polynomials of the first kind in p-adic field. In this paper, we introduce the modified degenerate tangent numbers and polynomials. We also establish some interesting properties for modified degenerate tangent numbers and polynomials. Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers. The Tangent numbers are defined by means of the following generating function:

$$\sum_{n=0}^{\infty} T_n \frac{t^n}{n!} = \frac{2}{e^{2t}+1} \qquad \left(|t| < \frac{\pi}{2}\right). \tag{1.1}$$

In [5], we defined Tangent polynomials by multiplying e^{xt} on the right side of the Eq. (1.1) as follows:

$$\sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right) e^{xt} \qquad \left(|t| < \frac{\pi}{2}\right). \tag{1.2}$$

For more theoretical properties of the tangent numbers and polynomials, the readers may refer to [4, 6, 7]. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations (see [7])

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \text{ and } x^n = \sum_{k=0}^{n} S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order n. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{1.3}$$

We also have

$$\sum_{n=m}^{\infty} S_1(n,m)\frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{1.4}$$

The generalized falling factorial $(x|\lambda)_n$ with increment λ is defined by

$$(x|\lambda)_n = \prod_{k=0}^{n-1}(x-\lambda k) \tag{1.5}$$

for positive integer n, with the convention $(x|\lambda)_0 = 1$; we may also write

$$(x|\lambda)_n = \sum_{k=0}^{n} S_1(n,k)\lambda^{n-k}x^k. \tag{1.6}$$

Note that $(x|\lambda)$ is a homogeneous polynomials in λ and x of degree n, so if $\lambda \neq 0$ then $(x|\lambda)_n = \lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n = x^n$. We also need the binomial theorem: for a variable x,

$$(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}. \tag{1.7}$$

For a variable t, we consider the degenerate tangent polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda t)^{2/\lambda}+1}(1+\lambda t)^{x/\lambda} = \sum_{n=0}^{\infty}\mathcal{T}_n(x,\lambda)\frac{t^n}{n!}. \tag{1.8}$$

When $x=0$, $\mathcal{T}_n(0,\lambda) = \mathcal{T}_n(\lambda)$ are called the degenerate tangent numbers(see [6]). Note that $(1+\lambda t)^{1/\lambda}$ tends to e^t as $\lambda \to 0$. In [6], we investigated some properties which are related to degenerate tangent numbers $\mathcal{T}_n(\lambda)$ and polynomials $\mathcal{T}_n(x,\lambda)$.

2. On modified degenerate tangent numbers and polynomials

In this section, we define the modified degenerate tangent numbers and polynomials, and we obtain explicit formulas for them. For a variable t, we consider the modified degenerate tangent polynomials which are given by the generating function to be

$$\frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} = \sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.1}$$

When $x = 0$, $\mathbf{T}_{n,\lambda}(0) = \mathbf{T}_{n,\lambda}$ are called the modified degenerate tangent numbers. Note that $(1+\lambda)^{t/\lambda}$ tends to e^t as $\lambda \to 0$.

From (2.1) and (1.2), we note that

$$\begin{aligned}\sum_{n=0}^{\infty} \lim_{\lambda\to 0} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!} &= \lim_{\lambda\to 0} \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} \\ &= \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}.\end{aligned}$$

Thus, we have

$$\lim_{\lambda\to 0} \mathbf{T}_{n,\lambda}(x) = T_n(x), \ (n \geq 0).$$

From (2.1) and (1.5), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} \\ &= \left(\sum_{m=0}^{\infty} \mathbf{T}_{m,\lambda}\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n x^n\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l\right)\frac{t^n}{n!}.\end{aligned} \tag{2.2}$$

Therefore, by (2.2), we obtain the following theorem.

Theorem 2.1. For $n \geq 0$, we have

$$\mathbf{T}_{n,\lambda}(x) = \sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l.$$

By (1.4), (2.1), and using Cauchy product, we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} \\
&= \left(\sum_{k=0}^{\infty} \mathbf{T}_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\left(\frac{tx}{\lambda}\right)_m x^m \frac{\lambda^n}{n!}\right) \\
&= \left(\sum_{k=0}^{\infty} \mathbf{T}_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{m=0}^{\infty}\sum_{l=0}^{m} S_1(m,l)\left(\frac{tx}{\lambda}\right)^l \frac{\lambda^m}{m!}\right) \\
&= \left(\sum_{k=0}^{\infty} \mathbf{T}_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\sum_{m=l}^{\infty} S_1(m,l)\left(\frac{tx}{\lambda}\right)^l \frac{\lambda^m}{m!}\right) \\
&= \left(\sum_{k=0}^{\infty} \mathbf{T}_{k,\lambda}\frac{t^k}{k!}\right)\left(\sum_{l=0}^{\infty}\left(\sum_{m=l}^{\infty} S_1(m,l)\left(\frac{x}{\lambda}\right)^l \frac{\lambda^m}{m!}l!\right)\frac{t^l}{l!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\sum_{m=l}^{\infty} S_1(m,l)\left(\frac{x}{\lambda}\right)^l l!\frac{\lambda^m}{m!}\mathbf{T}_{n-l,\lambda}\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}
$$

Comparing the coefficients of $\dfrac{t^l}{l!}$ in (2.3), we obtain the following theorem.

Theorem 2.2. For $n \geq 0$, we have

$$
\mathbf{T}_{n,\lambda}(x) = \sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}\mathbf{T}_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l l!\frac{\lambda^m}{m!}.
$$

By theorem 2.1 and Theorem 2.2, we have the following corollary.

Corollary 2.3. For $n \geq 0$, we have

$$
\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}\mathbf{T}_{n-l,\lambda}S_1(m,l)\left(\frac{x}{\lambda}\right)^l l!\frac{\lambda^m}{m!} = \sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l.
$$

From (2.1), we can derive the following recurrence relation:

$$\begin{aligned}
2 &= ((1+\lambda)^{2t/\lambda}+1)\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}\frac{t^n}{n!}\\
&= (1+\lambda)^{2t/\lambda}\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}\frac{t^n}{n!}\\
&= \left(\sum_{l=0}^{\infty}2^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l\frac{t^l}{l!}\sum_{m=0}^{\infty}\mathbf{T}_{m,\lambda}\frac{t^m}{m!}\right)+\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}\frac{t^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}2^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l\mathbf{T}_{n-l,\lambda}+\mathbf{T}_{n,\lambda}\right)\frac{t^n}{n!}.
\end{aligned}\tag{2.4}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.4), we have the following theorem.

Theorem 2.4. For $n \in \mathbb{Z}_+$, we have

$$\sum_{l=0}^{n}\binom{n}{l}2^l\left(\frac{\log(1+\lambda)}{\lambda}\right)^l\mathbf{T}_{n-l,\lambda}+\mathbf{T}_{n,\lambda}=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}$$

Also, we can obtain similar recurrence relation:

$$\sum_{l=0}^{n}\sum_{m=l}^{\infty}\binom{n}{l}\mathbf{T}_{n-l,\lambda}S_1(m,l)\frac{2^l l!\lambda^{m-l}}{m!}+\mathbf{T}_{n,\lambda}=\begin{cases}2, & \text{if } n=0,\\ 0, & \text{if } n\neq 0.\end{cases}$$

By (2.1), we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}(x+2)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathbf{T}_{n,\lambda}\frac{t^n}{n!}\\
&= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{(x+2)t/\lambda}+\frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda}\\
&= 2(1+\lambda)^{xt/\lambda}\\
&= 2\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n x^n\frac{t^n}{n!}.
\end{aligned}\tag{2.5}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.4), we obtain the following theorem.

Theorem 2.5. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x+2) + \mathbf{T}_{n,\lambda}(x) = 2\left(\frac{\log(1+\lambda)}{\lambda}\right)^n x^n.$$

By (2.1), we get

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda)^{-2t/\lambda}+1}(1+\lambda)^{(xt-2t)/\lambda} \\ &= \frac{2}{(1+\lambda)^{-2t/\lambda}+1}(1+\lambda)^{(2-x)t/\lambda} \\ &= \sum_{n=0}^{\infty}(-1)^n \mathbf{T}_{n,\lambda}(2-x)\frac{t^n}{n!}.\end{aligned} \tag{2.6}$$

By comparing of the coefficients $\frac{t^n}{n!}$ on the both sides of (2.6), we have the following theorem.

Theorem 2.6. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x) = (-1)^n \mathbf{T}_{n,\lambda}(2-x),$$

In particular,

$$\mathbf{T}_{n,\lambda} = (-1)^n \mathbf{T}_{n,\lambda}(2).$$

For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} \\ &= \frac{2}{(1+\lambda)^{2dt/\lambda}+1}(1+\lambda)^{xt/\lambda}\sum_{l=0}^{d-1}(-1)^l(1+\lambda)^{2lt/\lambda} \\ &= \sum_{n=0}^{\infty}\left(d^n \sum_{l=0}^{d-1}(-1)^l \mathbf{T}_{n,\lambda}\left(\frac{2l+x}{d}\right)\right)\frac{t^n}{n!}.\end{aligned}$$

By comparing coefficients of $\frac{t^m}{m!}$ in the above equation, we have the following theorem:

Theorem 2.7. For $d \in \mathbb{N}$ with $d \equiv 1 (\mathrm{mod}\ 2)$ and $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x) = d^n \sum_{l=0}^{d-1}(-1)^l T_{n,\lambda}\left(\frac{2l+x}{d}\right).$$

In particular,

$$\mathbf{T}_{n,\lambda} = d^n \sum_{l=0}^{d-1} (-1)^l T_{n,\lambda}\left(\frac{2l}{d}\right).$$

From (2.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x+y)\frac{t^n}{n!} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{(x+y)t/\lambda} \\ &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda}(1+\lambda)^{yt/\lambda} \\ &= \left(\sum_{n=0}^{\infty} \mathbf{T}_{n,\lambda}(x)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\left(\frac{\log(1+\lambda)}{\lambda}\right)^n y^n \frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l}\mathbf{T}_{n-l,\lambda}(x)\left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l\right)\frac{t^n}{n!}.\end{aligned} \tag{2.7}$$

Therefore, by (2.7), we have the following theorem.

Theorem 2.8. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{n,\lambda}(x+y) = \sum_{l=0}^{n}\binom{n}{l}\left(\frac{\log(1+\lambda)}{\lambda}\right)^l x^l \mathbf{T}_{n-l,\lambda}(x).$$

By replacing t by $(e-1)t$ and λ by $e-1$ in (2.1), we obtain

$$\begin{aligned}\frac{2}{e^{2t}+1}e^{xt} &= \sum_{n=0}^{\infty} \mathbf{T}_{n,e-1}(x)\frac{(e-1)^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathbf{T}_{n,e-1}(x)(e-1)^n\frac{t^n}{n!}.\end{aligned} \tag{2.8}$$

Thus, by (2.8) and (1.2), we obtain the following theorem.

Theorem 2.9. For $n \in \mathbb{Z}_+$, we have

$$T_m(x) = \mathbf{T}_{m,e-1}(x)(e-1)^m.$$

By replacing t by $\log(1+\lambda)^{t/\lambda}$ in (1.2), we have

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(x)\left(\log(1+\lambda)^{t/\lambda}\right)^n \frac{1}{n!} &= \frac{2}{(1+\lambda)^{2t/\lambda}+1}(1+\lambda)^{xt/\lambda} \\ &= \sum_{m=0}^{\infty} \mathbf{T}_{m,\lambda}(x)\frac{t^m}{m!}.\end{aligned} \tag{2.9}$$

and

$$\sum_{n=0}^{\infty} T_n(x) \left(\log(1+\lambda)^{t/\lambda}\right)^n \frac{1}{n!} = \sum_{n=0}^{\infty} T_n(x) \left(\frac{\log(1+\lambda)}{\lambda}\right)^n \frac{t^n}{n!}. \tag{2.10}$$

Thus, by (2.9) and (2.10), we have the following theorem.

Theorem 2.10. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{m,\lambda}(x) = \left(\frac{\log(1+\lambda)}{\lambda}\right)^n T_n(x).$$

By (1.8) and (2.1), we have the following theorem.

Theorem 2.11. For $n \in \mathbb{Z}_+$, we have

$$\mathbf{T}_{m,e-1}(x)(e-1)^m = \sum_{n=0}^{m} \mathcal{T}_{n,\lambda}(x)\lambda^{m-n} S_2(m,n).$$

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