Global Journal of Pure and Applied Mathematics.
ISSN 0973-1768 Volume 12, Number 2 (2016), pp. 1519-1535

http://www.ripublication.com/gjpam.htm

Two Step Hybrid Block Method with Two Generalized Off-step Points for Solving Second Ordinary Order Differential Equations Directly

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Abstract

In this paper, the new two step hybrid block method with two generalized off-step points for solving second order ordinary differential equation directly has been proposed. In the derivation of the method, power series of order six is used as basis function to obtain the continuous scheme through collocation and interpolation technique. As required by all numerical methods, the numerical properties of the developed block method which include consistent, zero stability, convergent and stability region are also established. the new method was found to compare favourably with the existing methods in term of error.

AMS subject classification: 65L05, 65L06, 65L20.
Keywords: Block method, Hybrid method, Second order differential equation, Collocation and Interpolation, Two off step points.

1. Introduction

This article considered the solution to the general second order initial value problem (IVPs) of the form

$$y^{''} = f(x, y, y^{'}), \;\; y(a) = \delta_0, \;\; y^{'}(a) = \delta_1. \; x \in [a, b]. \tag{1}$$

Equations (1) often arises in several areas of science and engineering an such as biology, physics and chemical. Generally, these equations have not exact solution. Thus, numerical methods become very crucial. In literature, numerous numerical methods for solving Equation (1) have been proposed, for example, Euler method, linear multistep method, Runge-Kutta, predictor - corrector method, hybrid method and block method (see [3], [5] and [2]). However, these methods have their setbacks which affects on their accuracy and efficiency. Recently, hybrid block methods for solving equations (1) directly have been proposed. In the latter, the researchers have tried to combine advantages of direct, block and hybrid methods (see [1], [4], [5], [6] and [7]) to overcome the zero stability problem in linear multistep as well as to avoid setbacks in reduction methods and generating numerical results concurrently.

2. Methodology

In this part, two step hybrid block method with two generalized off-step points $i.e\ x_{n+s}$ and x_{n+r} for solving (1) is derived.

Let the approximate solution of (1) to be the power series polynomial of the form:

$$y(x) = \sum_{i=0}^{q+d-1} a_i \left(\frac{x - x_n}{h}\right)^i. \tag{2}$$

where,

i $x \in [x_n, x_{n+1}]$ for $n = 0, 1, 2, \ldots, N - 1$,

ii q denotes of the number of interpolation points which is equal to the order of differential equation,

iii d represents the number of collocation points,

iv $h = x_n - x_{n-1}$ is constant step size of partition of interval $[a, b]$ which is given by $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$.

Differentiating (2) twice gives

$$y^{''}(x) = f(x, y, y^{'}) = \sum_{i=2}^{q+d-1} \frac{i(i-1)}{h^2} a_i \left(\frac{x - x_n}{h}\right)^{i-2}. \tag{3}$$

Interpolating (2) at x_{n+s}, x_{n+r} and collocating (3) at all points in the selected interval produces seven equations which can be written in matrix of the form:

$$\begin{pmatrix} 1 & s & s^2 & s^3 & s^4 & s^5 & s^6 \\ 1 & r & r^2 & r^3 & r^4 & r^5 & r^6 \\ 0 & 0 & \frac{2}{h^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{h^2} & \frac{6s}{h^2} & \frac{12s^2}{h^2} & \frac{20s^3}{h^2} & \frac{30s^4}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6r}{h^2} & \frac{12r^2}{h^2} & \frac{20r^3}{h^2} & \frac{30r^4}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{6}{h^2} & \frac{12}{h^2} & \frac{20}{h^2} & \frac{30}{h^2} \\ 0 & 0 & \frac{2}{h^2} & \frac{12}{h^2} & \frac{48}{h^2} & \frac{160}{h^2} & \frac{480}{h^2} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} y_{n+s} \\ y_{n+r} \\ f_n \\ f_{n+s} \\ f_{n+r} \\ f_{n+1} \\ f_{n+2} \end{pmatrix} \quad (4)$$

Employing Gaussian elimination method to (4) gives the unknown values of $a_i's, i = 0(1)6$ which are then substituted back into equation (2) to produce a continuous implicit scheme of the form

$$y(x) = \sum_{i=s,r} \alpha_i(x) y_{n+i} + \sum_{i=0,s,r,1} \beta_i(x) f_{n+i} \quad (5)$$

The first derivative of equation (5) gives

$$y'(x) = \sum_{i=s,r} \frac{\partial}{\partial x}\alpha_i(x) y_{n+i} + \sum_{i=0,s,r,1} \frac{\partial}{\partial x}\beta_i(x) f_{n+i} \quad (6)$$

where

$$\alpha_s = \frac{(x_n - x + hr)}{h(r-s)}$$
$$\alpha_r = \frac{(x - x_n - hs)}{h(r-s)}$$
$$\beta_0 = -\frac{(x_n - x + hs)(x_n - x + hr)}{(120rsh^4)}(h^4r^4 - h^4r^3s - 6h^4r^3 - h^4r^2s^2 + 9h^4r^2s$$
$$+10h^4r^2 - h^4rs^3 + 9h^4rs^2 - 30h^4rs + h^4s^4 - 6h^4s^3 + 10h^4s^2 + h^3r^3x - h^3r^3x_n$$
$$-h^3r^2sx + h^3r^2sx_n - 6h^3r^2x + 6h^3r^2x_n - h^3rs^2x + h^3rs^2x_n + 9h^3rsx + 10h^3rx$$

$$-9h^3rsx_n - 10h^3rx_n + h^3s^3x - h^3s^3x_n - 6h^3s^2x + 6h^3s^2x_n + 10h^3sx - 10h^3sx_n$$
$$-6h^2rx^2 + 12h^2rxx_n - 6h^2rx_n^2 + h^2s^2x^2 - 2h^2s^2xx_n + h^2s^2x_n^2 - 6h^2sx^2$$
$$+12h^2sxx_n + 20h^2xx_n - 10h^2x_n^2 + hrx^3 - 3hrx^2x_n + 3hrxx_n^2 - hrx_n^3$$
$$+hsx^3 - 3hsx^2x_n + 3hsxx_n^2 - hsx_n^3 - 6h^2sx_n^2 - 10h^2x^2 + h^2r^2x^2 - 2h^2r^2xx_n$$
$$+h^2r^2x_n^2 - h^2rsx^2 + 2h^2rsxx_n - h^2rsx_n^2 + 9hx^3 - 27hx^2x_n$$
$$+27hxx_n^2 - 9hx_n^3 - 2x^4 + 8x^3x_n - 12x^2x_n^2 + 8xx_n^3 - 2x_n^4)$$

$$\beta_s = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^4s(s-1)(s-2)(r-s))}(h^4r^4 + h^4r^3s - 6h^4r^3 + h^4r^2s^2 - 6h^4r^2s$$
$$+10h^4r^2 + h^4rs^3 - 6h^4rs^2 + 10h^4rs - 2h^4s^4 + 9h^4s^3 - 10h^4s^2 + h^3r^3x$$
$$-h^3r^3x_n + h^3r^2sx - h^3r^2sx_n - 6h^3r^2x + 6h^3r^2x_n + h^3rs^2x - h^3rs^2x_n$$
$$-6h^3rsx + 6h^3rsx_n + 10h^3rx - 10h^3rx_n - 2h^3s^3x + 2h^3s^3x_n + 9h^3s^2x$$
$$-9h^3s^2x_n - 10h^3sx + 10h^3sx_n + h^2r^2x^2 + h^2r^2x_n^2 + h^2rsx^2 - 2h^2rsxx_n$$
$$+h^2rsx_n^2 - 6h^2rx^2 + 12h^2rxx_n - 6h^2rx_n^2 - 2h^2s^2x^2 + 9h^2sx^2 - 18h^2sxx_n$$
$$+9h^2sx_n^2 - 10h^2x^2 + 20h^2xx_n - 10h^2x_n^2 + hrx^3 - 3hrx^2x_n$$
$$+3hrxx_n^2 - hrx_n^3 - 2hsx^3 + 6hsx^2x_n - 6hsxx_n^2 + 2hsx_n^3 + 9hx^3 - 27hx^2x_n$$
$$+8xx_n^3 - 2h^2r^2xx_n + 4h^2s^2xx_n - 2h^2s^2x_n^2 + 27hxx_n^2 - 9hx_n^3 - 2x^4$$
$$+8x^3x_n - 12x^2x_n^2 - 2x_n^4)$$

$$\beta_r = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^4r(r-1)(r-2)(r-s))}(2h^4r^4 - h^4r^3s - 9h^4r^3 - h^4r^2s^2$$
$$+6h^4r^2s + 10h^4r^2 - h^4rs^3 + 6h^4rs^2 - 10h^4rs - h^4s^4 + 6h^4s^3$$
$$-10h^4s^2 + 6h^3rsx + 10h^3sx_n - 9h^2rx_n^2 + 2h^3r^3x - 2h^3r^3x_n - h^3r^2sx$$
$$+h^3r^2sx_n - 9h^3r^2x + 9h^3r^2x_n - h^3rs^2x + h^3rs^2x_n$$
$$-6h^3rsx_n + 10h^3rx - 10h^3rx_n - h^3s^3x + h^3s^3x_n + 6h^3s^2x - 6h^3s^2x_n$$
$$-10h^3sx + 2h^2r^2x^2 - 4h^2r^2xx_n + 2h^2r^2x_n^2 - h^2rsx^2 + 2h^2rsxx_n$$
$$-h^2rsx_n^2 - 9h^2rx^2 - 8xx_n^3 - h^2s^2x^2 + 2h^2s^2xx_n - h^2s^2x_n^2$$
$$+6h^2sx^2 - 12h^2sxx_n + 6h^2sx_n^2 + 10h^2x^2 - 20h^2xx_n + 10h^2x_n^2 + 2hrx^3$$
$$-6hrx^2x_n + 6hrxx_n^2 - 2hrx_n^3 - hsx^3 + 3hsx^2x_n - 3hsxx_n^2 + hsx_n^3$$
$$+18h^2rxx_n - 9hx^3 + 27hx^2x_n - 27hxx_n^2 + 9hx_n^3 + 2x^4 - 8x^3x_n$$
$$+12x^2x_n^2 + 2x_n^4)$$

$$\beta_1 = \frac{(x_n - x + hs)(x_n - x + hr)}{(60h^4(s-1)(r-1))}(h^4r^4 - h^4r^3s - 4h^4r^3 - h^4r^2s^2$$
$$+6h^4r^2s - h^4rs^3 + 6h^4rs^2 + h^4s^4 - 4h^4s^3 + h^3r^3x - h^3r^3x_n$$
$$-h^3r^2sx + h^3r^2sx_n - 4h^3r^2x + 4h^3r^2x_n$$

$$
\begin{aligned}
&-h^3rs^2x+h^3rs^2x_n+6h^3rsx-6h^3rsx_n+h^3s^3x-h^3s^3x_n-4h^3s^2x\\
&+4h^3s^2x_n-2h^2r^2xx_n+h^2r^2x_n^2-h^2rsx^2+2h^2rsxx_n-h^2rsx_n^2\\
&-4h^2rx^2+8h^2rxx_n-4h^2rx_n^2-2h^2s^2xx_n+h^2s^2x_n^2\\
&-4h^2sx^2+8h^2sxx_n-4h^2sx_n^2+hrx^3-3hrx^2x_n+3hrxx_n^2\\
&+hsx^3-3hsx^2x_n+3hsxx_n^2-hsx_n^3+6hx^3-18hx^2x_n+h^2s^2x^2-hrx_n^3\\
&+18hxx_n^2+h^2r^2x^2-6hx_n^3-2x^4+8x^3x_n-12x^2x_n^2+8xx_n^3-2x_n^4)
\end{aligned}
$$

$$
\begin{aligned}
\beta_2 \;=\; &-\frac{(x_n-x+hs)(x_n-x+hr)}{(120h^4(s-2)(r-2))}(h^4r^4-h^4r^3s-2h^4r^3-h^4r^2s^2\\
&+3h^4r^2s-h^4rs^3+3h^4rs^2+h^4s^4-2h^4s^3+h^3r^3x-h^3r^3x_n-h^3r^2sx\\
&+h^3r^2sx_n-2h^3r^2x+2h^3r^2x_n-h^3rs^2x+h^3rs^2x_n+3h^3rsx-3h^3rsx_n\\
&+h^3s^3x-h^3s^3x_n-2h^3s^2x+2h^3s^2x_n+h^2r^2x^2\\
&-2h^2r^2xx_n+h^2r^2x_n^2-h^2rsx^2+2h^2rsxx_n-h^2rsx_n^2-2h^2rx^2\\
&+4h^2rxx_n-2h^2rx_n^2-2h^2s^2xx_n+h^2s^2x_n^2-2h^2sx^2+4h^2sxx_n\\
&-2h^2sx_n^2+hrx^3-3hrx^2x_n+3hrxx_n^2\\
&+hsx^3-3hsx^2x_n+3hsxx_n^2-hsx_n^3+3hx^3-9hx^2x_n+9hxx_n^2-3hx_n^3\\
&-2x^4+8x^3x_n+h^2s^2x^2-hrx_n^3-12x^2x_n^2+8xx_n^3-2x_n^4)
\end{aligned}
$$

Equation (5) is evaluated at the non-interpolatig point x_{n+1} and x_{n+2} while Equation (6) is evaluated at all points to give the discrete schemes and its derivative. The discrete scheme and its derivatives are combined i a matrix form as below

$$
A^{[2]_2}Y_M = B^{[2]_2}R_1^{[2]_2} + h^2\left[C^{[2]_2}R_2^{[2]_2} + D^{[2]_2}R_3^{[2]_2}\right] \tag{7}
$$

$$
A^{[2]_2} = \begin{pmatrix}
\frac{-r}{(r-s)} & \frac{s}{(r-s)} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{-(r-1)}{(r-s)} & \frac{(s-1)}{(r-s)} & 1 & 0 & 0 & 0 & 0 & 0\\
\frac{-(r-2)}{(r-s)} & \frac{(s-2)}{(r-s)} & 0 & 1 & 0 & 0 & 0 & 0\\
\frac{1}{(h(r-s))} & \frac{-1}{(h(r-s))} & 0 & 0 & 0 & 0 & 0 & 0\\
\frac{1}{(h(r-s))} & \frac{-1}{(h(r-s))} & 0 & 0 & 1 & 0 & 0 & 0\\
\frac{1}{(h(r-s))} & \frac{-1}{(h(r-s))} & 0 & 0 & 0 & 1 & 0 & 0\\
\frac{1}{(h(r-s))} & \frac{-1}{(h(r-s))} & 0 & 0 & 0 & 0 & 1 & 0\\
\frac{1}{(h(r-s))} & \frac{-1}{(h(r-s))} & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},\quad
Y_M = \begin{pmatrix}
y_{n+s}\\ y_{n+r}\\ y_{n+1}\\ y_{n+2}\\ y'_{n+s}\\ y'_{n+r}\\ y'_{n+1}\\ y'_{n+2}
\end{pmatrix},
$$

$$B^{[2]_2} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$R_1^{[2]_2} = \begin{pmatrix} y_n \\ y_n^{'} \end{pmatrix}, \; C^{[2]_2} = \begin{pmatrix} C_{11}^{[2]_2} \\ C_{21}^{[2]_2} \\ C_{31}^{[2]_2} \\ C_{41}^{[2]_2} \\ C_{51}^{[2]_2} \\ C_{61}^{[2]_2} \\ C_{71}^{[2]_2} \\ C_{81}^{[2]_2} \end{pmatrix}, \; R_2^{[2]_2} = \begin{pmatrix} f_n \end{pmatrix},$$

$$D^{[2]_2} = \begin{pmatrix} D_{11}^{[2]_2} & D_{12}^{[2]_2} & D_{13}^{[2]_2} & D_{14}^{[2]_2} \\ D_{21}^{[2]_2} & D_{22}^{[2]_2} & D_{23}^{[2]_2} & D_{24}^{[2]_2} \\ D_{31}^{[2]_2} & D_{32}^{[2]_2} & D_{33}^{[2]_2} & D_{34}^{[2]_2} \\ D_{41}^{[2]_2} & D_{42}^{[2]_2} & D_{43}^{[2]_2} & D_{44}^{[2]_2} \\ D_{51}^{[2]_2} & D_{52}^{[2]_2} & D_{53}^{[2]_2} & D_{54}^{[2]_2} \\ D_{61}^{[2]_2} & D_{62}^{[2]_2} & D_{63}^{[2]_2} & D_{64}^{[2]_2} \\ D_{71}^{[2]_2} & D_{72}^{[2]_2} & D_{73}^{[2]_2} & D_{74}^{[2]_2} \\ D_{81}^{[2]_2} & D_{82}^{[2]_2} & D_{83}^{[2]_2} & D_{84}^{[2]_2} \end{pmatrix}$$

$$R_3^{[1]_2} = \begin{pmatrix} f_{n+s} \\ f_{n+r} \\ f_{n+1} \\ f_{n+2} \end{pmatrix}$$

$$C_{11}^{[2]_2} = \frac{-h^2}{120}(r^4 - r^3s - 6r^3 - r^2s^2 + 9r^2s + 10r^2 - rs^3 + 9rs^2 - 30rs + s^4 -6s^3 + 10s^2)$$

$$C_{21}^{[2]_2} = \frac{-h^2(r-1)(s-1)}{120rs)}(r^4 - r^3s - 5r^3 - r^2s^2 + 8r^2s + 5r^2 - rs^3 + 8rs^2 -22rs + 5r + s^4 - 5s^3 + 5s^2 + 5s - 3)$$

$$C_{31}^{[2]_2} = \frac{-h^2(r-2)(s-2)}{120rs}(r^4 - r^3s - 4r^3 - r^2s^2 + 7r^2s + 2r^2 - rs^3 + 7rs^2 -16rs + 4r + s^4 - 4s^3 + 2s^2 + 4s)$$

$$C_{41}^{[2]_2} = \frac{h}{120rs}(r^5 - r^4s - 6r^4 - r^3s^2 + 9r^3s + 10r^3 - r^2s^3 + 9r^2s^2 - 30r^2s -rs^4 + 9rs^3 - 30rs^2 + s^5 - 6s^4 + 10s^3)$$

$$C_{51}^{[2]_2} = \frac{h}{120rs}(r^5 - r^4s - 6r^4 - r^3s^2 + 9r^3s + 10r^3 - r^2s^3 + 9r^2s^2 - 30r^2s -18rs^5 + 4rs^4 - 21rs^3 + 30rs^2 - 18s^6 - 68s^5 + 9s^4 - 10s^3)$$

$$C_{61}^{[2]_2} = \frac{-h}{120rs}(18r^6 + 18r^5s + 68r^5 - 4r^4s - 9r^4 + r^3s^2 + 21r^3s + 10r^3 + r^2s^3 -9r^2s^2 - 30r^2s + rs^4 - 9rs^3 + 30rs^2 - s^5 + 6s^4 - 10s^3)$$

$$C_{71}^{[2]_2} = \frac{h}{120rs}(r^5 - r^4s - 6r^4 - r^3s^2 + 9r^3s + 10r^3 - r^2s^3 + 9r^2s^2 - 30r^2s -rs^4 + 9rs^3 - 30rs^2 + 50rs - 33r + s^5 - 6s^4 + 10s^3 - 33s - 59)$$

$$C_{81}^{[2]_2} = \frac{h}{120rs}(r^5 - r^4s - 6r^4 - r^3s^2 + 9r^3s + 10r^3 - r^2s^3 + 9r^2s^2 - 30r^2s -rs^4 + 9rs^3 - 30rs^2 + 40rs - 576r + s^5 - 6s^4 + 10s^3 -576s - 2128)$$

$$D_{11}^{[2]_2} = \frac{(h^2r}{(60(r-s)(s-1)(s-2))}(r^4 + r^3s - 6r^3 + r^2s^2 - 6r^2s + 10r^2 + rs^3 -6rs^2 + 10rs - 2s^4 + 9s^3 - 10s^2))$$

$$D_{12}^{[2]_2} = \frac{(h^2s}{(60(r-s)(r-1)(r-2))}(2r^4 - r^3s - 9r^3 - r^2s^2 + 6r^2s + 10r^2 -rs^3 + 6rs^2 - 10rs - s^4 + 6s^3 - 10s^2))$$

$$D_{13}^{[2]_2} = \frac{(h^2rs}{(60(r-1)(s-1))}(r^4 - r^3s - 4r^3 - r^2s^2 + 6r^2s - rs^3 +6rs^2 + s^4 - 4s^3))$$

$$D_{14}^{[2]_2} = \frac{-(h^2rs}{(120(s-2)(r-2))}(r^4 - r^3s - 2r^3 - r^2s^2 + 3r^2s - rs^3 +3rs^2 + s^4 - 2s^3))$$

$$D_{21}^{[2]_2} = \frac{(h^2(r-1)}{(60s(r-s)(s-2))}(r^4+r^3s-5r^3+r^2s^2-5r^2s+5r^2+rs^3 -5rs^2+5rs+5r-2s^4+7s^3-3s^2-3s-3))$$

$$D_{22}^{[2]_2} = \frac{(h^2(s-1)}{(60r(r-s)(r-2))}(2r^4-r^3s-7r^3-r^2s^2+5r^2s+3r^2-rs^3 +5rs^2-5rs+3r-s^4+5s^3-5s^2-5s+3))$$

$$D_{23}^{[2]_2} = \frac{(h^2}{60}(r^4-r^3s-3r^3-r^2s^2+5r^2s-3r^2-rs^3+5rs^2+5rs-3r +s^4-3s^3-3s^2-3s+4))$$

$$D_{24}^{[2]_2} = \frac{-(h^2(s-1)(r-1)}{(120(r-2)(s-2))}(r^4-r^3s-r^3-r^2s^2+2r^2s-r^2-rs^3 +2rs^2+2rs-r+s^4-s^3-s^2-s+1))$$

$$D_{31}^{[2]_2} = \frac{(h^2(r-2)}{(60s(r-s)(s-1))}(r^4+r^3s-4r^3+r^2s^2-4r^2s+2r^2+rs^3 -4rs^2+2rs+4r-2s^4+5s^3))$$

$$D_{32}^{[2]_2} = \frac{(h^2(s-2)}{(60r(r-s)(r-1))}(2r^4-r^3s-5r^3-r^2s^2+4r^2s-rs^3+4rs^2 -2rs-s^4+4s^3-2s^2-4s))$$

$$D_{33}^{[2]_2} = \frac{(h^2(r-2)(s-2)}{(60(r-1)(s-1))}(r^4-r^3s-2r^3-r^2s^2+4r^2s-4r^2-rs^3 +4rs^2+8rs-8r+s^4-2s^3-4s^2-8s+16))$$

$$D_{34}^{[2]_2} = \frac{-(h^2}{120}(r^4-r^3s-r^2s^2+r^2s-rs^3+rs^2+2rs+s^4-8))$$

$$D_{41}^{[2]_2} = \frac{-h}{(60s(r-s)(s-1)(s-2))}(r^5+r^4s-6r^4+r^3s^2-6r^3s+10r^3 +r^2s^3-6r^2s^2+10r^2s+rs^4-6rs^3+10rs^2-2s^5+9s^4-10s^3))$$

$$D_{42}^{[2]_2} = \frac{-h}{(60r(r-s)(r-1)(r-2))}(2r^5-r^4s-9r^4-r^3s^2+6r^3s+10r^3-r^2s^3 +6r^2s^2-10r^2s-rs^4+6rs^3-10rs^2-s^5+6s^4-10s^3)$$

$$D_{43}^{[2]_2} = \frac{-h}{(60(r-1)(s-1))}r^5-r^4s-4r^4-r^3s^2+6r^3s-r^2s^3+6r^2s^2 -rs^4+6rs^3+s^5-4s^4)$$

$$D_{44}^{[2]_2} = \frac{h}{(120(s-2)(r-2))}(r^5-r^4s-2r^4-r^3s^2+3r^3s-r^2s^3+3r^2s^2-rs^4 +3rs^3+s^5-2s^4))$$

$$D_{51}^{[2]_2} = \frac{-h}{(60s(r-s)(s-1)(s-2))}(r^5+r^4s-6r^4+r^3s^2-6r^3s+10r^3+r^2s^3 -6r^2s^2+10r^2s-18rs^5-14rs^4+54rs^3-50rs^2 -56s^5-36s^4+30s^3)$$

$$D_{52}^{[2]_2} = \frac{-h}{(60r(r-s)(r-1)(r-2))}(2r^5-r^4s-9r^4-r^3s^2+6r^3s+10r^3-r^2s^3 +6r^2s^2-10r^2s-rs^4+6rs^3-10rs^2+18s^6+68s^5-9s^4+10s^3)$$

$$D_{53}^{[2]_2} = \frac{-h}{(60(r-1)(s-1))}(r^5-r^4s-4r^4-r^3s^2+6r^3s-r^2s^3+6r^2s^2-18rs^5 +4rs^4-14rs^3-18s^6-50s^5+6s^4)$$

$$D_{54}^{[2]_2} = \frac{h)}{(120(s-2)(r-2)}(r^5-r^4s-2r^4-r^3s^2+3r^3s-r^2s^3+3r^2s^2-18rs^5 +4rs^4-7rs^3-18s^6-32s^5+3s^4)$$

$$D_{61}^{[2]_2} = \frac{h}{(60s(r-s)(s-1)(s-2))}(18r^6+68r^5-r^4s-9r^4-r^3s^2+6r^3s+10r^3 -r^2s^3+6r^2s^2-10r^2s-rs^4+6rs^3-10rs^2+2s^5-9s^4+10s^3)$$

$$D_{62}^{[2]_2} = \frac{-h)}{(60r(r-s)(r-1)(r-2))}(18r^5s+56r^5+14r^4s+36r^4-r^3s^2-54r^3s -30r^3-r^2s^3+6r^2s^2+50r^2s-rs^4+6rs^3-10rs^2-s^5 +6s^4-10s^3)$$

$$D_{63}^{[2]_2} = \frac{h}{(60(r-1)(s-1))}(18r^6+18r^5s+50r^5-4r^4s-6r^4+r^3s^2+14r^3s +r^2s^3-6r^2s^2+rs^4-6rs^3-s^5+4s^4)$$

$$D_{64}^{[2]_2} = \frac{-h}{(120(s-2)(r-2))}(18r^6+18r^5s+32r^5-4r^4s-3r^4+r^3s^2+7r^3s +r^2s^3-3r^2s^2+rs^4-3rs^3-s^5+2s^4)$$

$$D_{71}^{[2]_2} = \frac{-h}{(60s(r-s)(s-1)(s-2))}(r^5+r^4s-6r^4+r^3s^2-6r^3s+10r^3+r^2s^3 -6r^2s^2+10r^2s+rs^4-6rs^3+10rs^2-33r-2s^5+9s^4-10s^3-59)$$

$$D_{72}^{[2]_2} = \frac{-h}{(60r(r-s)(r-1)(r-2))}(2r^5-r^4s-9r^4-r^3s^2+6r^3s+10r^3-r^2s^3 +6r^2s^2-10r^2s-rs^4+6rs^3-10rs^2-s^5+6s^4-10s^3+33s+59)$$

$$D_{73}^{[2]_2} = \frac{-h}{(60(r-1)(s-1))}(r^5-r^4s-4r^4-r^3s^2+6r^3s-r^2s^3+6r^2s^2-rs^4 +6rs^3-40rs+7r+s^5-4s^4+7s-66)$$

$$
\begin{aligned}
D_{74}^{[2]_2} &= \frac{h}{(120(s-2)(r-2))}(r^5 - r^4s - 2r^4 - r^3s^2 + 3r^3s - r^2s^3 + 3r^2s^2 - rs^4 \\
&\quad +3rs^3 - 10rs - 13r + s^5 - 2s^4 - 13s - 33) \\
D_{81}^{[2]_2} &= \frac{-h}{(60s(r-s)(s-1)(s-2))}(r^5 + r^4s - 6r^4 + r^3s^2 - 6r^3s + 10r^3 + r^2s^3 \\
&\quad -6r^2s^2 + 10r^2s + rs^4 - 6rs^3 + 10rs^2 - 576r - 2s^5 + 9s^4 - 10s^3 - 2128) \\
D_{82}^{[2]_2} &= \frac{-h}{(60r(r-s)(r-1)(r-2))}(2r^5 - r^4s - 9r^4 - r^3s^2 + 6r^3s + 10r^3 - r^2s^3 \\
&\quad +6r^2s^2 - 10r^2s - rs^4 + 6rs^3 - 10rs^2 - s^5 + 6s^4 - 10s^3 + 576s + 2128) \\
D_{83}^{[2]_2} &= \frac{-h}{(60(r-1)(s-1))}(r^5 - r^4s - 4r^4 - r^3s^2 + 6r^3s - r^2s^3 + 6r^2s^2 - rs^4 \\
&\quad +6rs^3 - 80rs - 496r + s^5 - 4s^4 - 496s - 1632) \\
D_{84}^{[2]_2} &= \frac{h}{(120(s-2)(r-2))}(r^5 - r^4s - 2r^4 - r^3s^2 + 3r^3s - r^2s^3 + 3r^2s^2 - rs^4 \\
&\quad +3rs^3 + 40rs - 656r + s^5 - 2s^4 - 656s - 816)
\end{aligned}
$$

Multiplying Equation (7) by the inverse of $A^{[1]_2}$ gives

$$
IY_m = \bar{B}^{[2]_2}R_1^{[2]_2} + h^2\left[\bar{C}^{[2]_2}R_2^{[2]_2} + \bar{D}^{[2]_2}R_3^{[2]_2}\right] \tag{8}
$$

where I is 8×8 identity matrix and

$$
\bar{B}^{[1]_2} = \begin{pmatrix} 1 & hs \\ 1 & 1 \\ 1 & rh \\ 1 & 2h \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad
\bar{C}^{[2]_2} = \begin{pmatrix} \bar{C}_{11}^{[2]_2} \\ \bar{C}_{21}^{[2]_2} \\ \bar{C}_{31}^{[2]_2} \\ \bar{C}_{41}^{[2]_2} \\ \bar{C}_{51}^{[2]_2} \\ \bar{C}_{61}^{[2]_2} \\ \bar{C}_{71}^{[2]_2} \\ \bar{C}_{81}^{[2]_2} \end{pmatrix}, \quad
\bar{E}^{[1]_2} = \begin{pmatrix}
\bar{D}_{11}^{[2]_2} & \bar{D}_{12}^{[2]_2} & \bar{D}_{13}^{[2]_2} & \bar{D}_{14}^{[2]_2} \\
\bar{D}_{21}^{[2]_2} & \bar{D}_{22}^{[2]_2} & \bar{D}_{23}^{[2]_2} & \bar{D}_{24}^{[2]_2} \\
\bar{D}_{31}^{[2]_2} & \bar{D}_{32}^{[2]_2} & \bar{D}_{33}^{[2]_2} & \bar{D}_{34}^{[2]_2} \\
\bar{D}_{41}^{[2]_2} & \bar{D}_{42}^{[2]_2} & \bar{D}_{43}^{[2]_2} & \bar{D}_{44}^{[2]_2} \\
\bar{D}_{51}^{[2]_2} & \bar{D}_{52}^{[2]_2} & \bar{D}_{53}^{[2]_2} & \bar{D}_{54}^{[2]_2} \\
\bar{D}_{61}^{[2]_2} & \bar{D}_{62}^{[2]_2} & \bar{D}_{63}^{[2]_2} & \bar{D}_{64}^{[2]_2} \\
\bar{D}_{71}^{[2]_2} & \bar{D}_{72}^{[2]_2} & \bar{D}_{73}^{[2]_2} & \bar{D}_{74}^{[2]_2} \\
\bar{D}_{81}^{[2]_2} & \bar{D}_{82}^{[2]_2} & \bar{D}_{83}^{[2]_2} & \bar{D}_{84}^{[2]_2}
\end{pmatrix}
$$

with

$$\bar{C}_{11}^{[2]_2} = \frac{(h^2s^2(40r-10s-15rs+2rs^2+6s^2-s^3))}{(120r)}$$
$$\bar{C}_{21}^{[2]_2} = \frac{-(h^2r^2(10r-40s+15rs-2r^2s-6r^2+r^3))}{(120s)}$$
$$\bar{C}_{31}^{[2]_2} = \frac{(h^2(35rs-8s-8r+3))}{(120rs)}$$
$$\bar{C}_{41}^{[2]_2} = \frac{(2h^2(5rs-s-r))}{(15rs)}$$
$$\bar{C}_{51}^{[2]_2} = \frac{-(hs(20s-60r+30rs-5rs^2+18rs^3-15s^2+69s^3+18s^4))}{(120r)}$$
$$\bar{C}_{61}^{[2]_2} = \frac{-(hr(20r-60s+30rs-5r^2s+18r^3s-15r^2+69r^3+18r^4))}{(120s)}$$
$$\bar{C}_{71}^{[2]_2} = \frac{(h(50rs-33s-33r-59))}{120rs)}$$
$$\bar{C}_{81}^{[2]_2} = \frac{(h(5rs-72s-72r-266))}{(15rs)}$$
$$\bar{E}_{11}^{[2]} = \frac{(h^2s^2(20r-10s-15rs+3rs^2+9s^2-2s^3))}{(60(s-1)(s-2)(r-s))}$$
$$\bar{E}_{12}^{[2]} = \frac{(h^2s^4(s^2-6s+10))}{(60r(r-1)(r-2)(r-s))}$$
$$\bar{E}_{13}^{[2]} = \frac{-(h^2s^4(4s-10r+2rs-s^2))}{(60(r-1)(s-1))}$$
$$\bar{E}_{14}^{[2]} = \frac{(h^2s^4(2s-5r+2rs-s^2))}{(120(r-2)(s-2))}$$
$$\bar{E}_{21}^{[2]} = \frac{(h^2r^4(r^2-6r+10))}{(60s(s-1)(s-2)(r-s))}$$
$$\bar{E}_{22}^{[2]} = \frac{(h^2r^2(10r-20s+15rs-3r^2s-9r^2+2r^3))}{(60(r-1)(r-2)(r-s))}$$
$$\bar{E}_{23}^{[2]} = \frac{(h^2r^4(10s-4r-2rs+r^2))}{(60(s-1)(r-1))}$$
$$\bar{E}_{24}^{[2]} = \frac{-(h^2r^4(5s-2r-2rs+r^2))}{(120(s-2)(r-2))}$$

$$\bar{E}_{31}^{[2]} = \frac{(h^2(8r-3))}{(60s(s-1)(s-2)(r-s))}$$

$$\bar{E}_{32}^{[2]} = \frac{-(h^2(8s-3))}{(60r(r-1)(r-2)(r-s))}$$

$$\bar{E}_{33}^{[2]} = \frac{(h^2(15rs-7s-7r+4))}{(60(s-1)(r-1))}$$

$$\bar{E}_{34}^{[2]} = \frac{-(h^2(5rs-2s-2r+1))}{(120(s-2)(r-2))}$$

$$\bar{E}_{41}^{[2]} = \frac{(4h^2r)}{(15s(s-1)(s-2)(r-s))}$$

$$\bar{E}_{42}^{[2]} = \frac{-(4h^2s)}{(15r(r-1)(r-2)(r-s))}$$

$$\bar{E}_{43}^{[2]} = \frac{(4h^2(5rs-4s-4r+4))}{(15(s-1)(r-1))}$$

$$\bar{E}_{44}^{[2]} = \frac{-(2h^2(r+s-2))}{(15(s-2)(r-2))}$$

$$\bar{E}_{51}^{[2]} = \frac{(hs(60r-40s-60rs+15rs^2+18rs^3+45s^2+54s^3))}{(60(s-1)(s-2)(r-s))}$$

$$\bar{E}_{52}^{[2]} = \frac{-(hs^3(18s^3+69s^2-15s+20))}{(60r(r-1)(r-2)(r-s))}$$

$$\bar{E}_{53}^{[2]} = \frac{(hs^3(20r-10s-5rs+18rs^2+51s^2+18s^3))}{(60(s-1)(r-1))}$$

$$\bar{E}_{54}^{[2]} = \frac{-(hs^3(10r-5s-5rs+18rs^2+33s^2+18s^3))}{(120(s-2)(r-2))}$$

$$\bar{E}_{61}^{[2]} = \frac{(hr^3(18r^3+69r^2-15r+20))}{(60s(s-1)(s-2)(r-s))}$$

$$\bar{E}_{62}^{[2]} = \frac{-(hr(60s-40r-60rs+15r^2s+18r^3s+45r^2+54r^3))}{(60(r-1)(r-2)(r-s))}$$

$$\bar{E}_{63}^{[2]} = \frac{(hr^3(20s-10r-5rs+18r^2s+51r^2+18r^3))}{(60(s-1)(r-1))}$$

$$\bar{E}_{64}^{[2]} = \frac{-(hr^3(10s-5r-5rs+18r^2s+33r^2+18r^3))}{(120(s-2)(r-2))}$$

$$\bar{E}_{71}^{[2]} = \frac{(h(33r+59))}{(60s(s-1)(s-2)(r-s))}$$

$$\bar{E}_{72}^{[2]} = \frac{-(h(33s+59))}{(60r(r-1)(r-2)(r-s))}$$

$$\bar{E}_{73}^{[2]} = \frac{(h(40rs-7s-7r+66))}{(60(s-1)(r-1))}$$

$$\bar{E}_{74}^{[2]} = \frac{-(h(13r+13s+10rs+33))}{(120(s-2)(r-2))}$$

$$\bar{E}_{81}^{[2]} = \frac{(4h(36r+133))}{(15s(s-1)(s-2)(r-s))}$$

$$\bar{E}_{82}^{[2]} = \frac{-(4h(36s+133))}{(15r(r-1)(r-2)(r-s))}$$

$$\bar{E}_{83}^{[2]} = \frac{(4h(31r+31s+5rs+102))}{(15(s-1)(r-1))}$$

$$\bar{E}_{84}^{[2]} = \frac{(h(5rs-82s-82r-102))}{(15(s-2)(r-2))}$$

3. Analysis of the Method

3.1. Order of the Methhod

The linear difference operator L associated with (8) is defined as

$$L[y(x);h] = IY_M - \bar{B}^{[2]_2}R_1^{[2]_2} - h^2\left[\bar{C}^{[2]_2}R_2^{[2]_2} + \bar{D}^{[2]_2}R_3^{[2]_2}\right] \qquad (9)$$

where $y(x)$ is an arbitrary test function continuously differentiable on [a,b]. Y_M and $R_3^{[2]_2}$ components are expanded in Taylor's series respectively and its terms are collected in powers of h to give

$$L[y(x),h] = \bar{C}_0 y(x) + \bar{C}_1 h y^{'}(x) + \bar{C}_2 h y^{''}(x) + \cdots \qquad (10)$$

Definition 3.1. Hybrid block method (8) and associated linear operator (9) are said to be of order p, if $\bar{C}_0 = \bar{C}_1 = \bar{_2} = \cdots = \bar{C}_{p+2} = 0$ and $\bar{C}_{p+2} \neq 0$ with error vector constants $\bar{C}_{p+2}$.

Expanding (8) in Taylor series about x_n gives

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(s)^j h^j}{j!}y_n^j - y_n - (sh)y_n^{'} - \frac{(s^2(40r-10s-15rs+2rs^2+6s^2-s^3))}{(120r)}y_n^{''} \\ -\frac{(s^2(20r-10s-15rs+3rs^2+9s^2-2s^3))}{(60(s-1)(s-2)(r-s))}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(s^4(s^2-6s+10))}{(60r(r-1)(r-2)(r-s))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ +\frac{(s^4(4s-10r+2rs-s^2))}{(60(r-1)(s-1))}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(s^4(2s-5r+2rs-s^2))}{(120(r-2)(s-2))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ \sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - (h)y_n^{'} - \frac{-(r^2(10r-40s+15rs-2r^2s-6r^2+r^3))}{(120s)}y_n^{''} \\ -\frac{(r^4(r^2-6r+10))}{(60s(s-1)(s-2)(r-s))}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(r^2(10r-20s+15rs-3r^2s-9r^2+2r^3))}{(60(r-1)(r-2)(r-s))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(r^4(10s-4r-2rs+r^2))}{(60(s-1)(r-1))}\sum_{j=0}^{\infty}\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{-(r^4(5s-2r-2rs+r^2))}{(120(s-2)(r-2))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ \sum_{j=0}^{\infty}\frac{(r)^j h^j}{j!}y_n^j - y_n - (rh)y_n^{'} - \frac{((35rs-8s-8r+3))}{(120rs)}y_n^{''} \\ -\frac{(r^4(r^2-6r+10))}{(60s(s-1)(s-2)(r-s))}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{-(h^2(8s-3))}{(60r(r-1)(r-2)(r-s))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(h^2(15rs-7s-7r+4))}{(60(s-1)(r-1))}\sum_{j=0}^{\infty}-\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{-(h^2(5rs-2s-2r+1))}{(120(s-2)(r-2))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ \sum_{j=0}^{\infty}\frac{(2)^j h^j}{j!}y_n^j - y_n - (2h)y_n^{'} - \frac{(2h^2(5rs-s-r))}{(15rs)}y_n^{''} \\ -\frac{(4h^2r)}{(15s(s-1)(s-2)(r-s))}\sum_{j=0}^{\infty}\frac{(s)^j h^{j+2}}{j!}y_n^{j+2} \\ -\frac{-(4h^2s)}{(15r(r-1)(r-2)(r-s))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \\ -\frac{(4h^2(5rs-4s-4r+4))}{(15(s-1)(r-1))}\sum_{j=0}^{\infty}-\frac{(r)^j h^{j+2}}{j!}y_n^{j+2} - \frac{-(2h^2(r+s-2))}{(15(s-2)(r-2))}\sum_{j=0}^{\infty}\frac{h^{j+2}}{j!}y_n^{j+2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By comparing the coefficient of h, we obtain the order of the method to be $[5, 5, 5, 5]^T$ with error constant vector

$$\bar{C}_7 = \begin{bmatrix} \dfrac{s^4(70r - 28s - 42rs + 7rs^2 + 21s^2 - 4s^3)}{50400} \\ \dfrac{r^4(70s - 28r - 42rs + 7r^2s + 21r^2 + 46r^3)}{50400} \\ \dfrac{(56rs - 21s - 21r + 60)}{50400} \\ \dfrac{(7rs + 396)}{3150} \end{bmatrix}$$

3.2. Zero Stability

The hybrid block method (8) is said to be zero stable if the first characteristic polynomial $\pi(x)$ having roots such that $|x_z|1$, and if $|x_z| = 1$, then, the multiplicity of x_z must not greater than two.

In order to find the zero-stability of main block $\hat{B}^{[2]_2} = [y_{n+s}, y_{n+r}, y_{n+1}, y_{n+2}]^T$ in (8), we only consider the solution of the first characteristic polynomial where I is 4×4 identity matrix, that is

$$\begin{aligned} \Pi(r) &= |x\, I - \hat{B}^{[2]_2}| \\ &= \left| x \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\ &= x^3(x-1) \end{aligned}$$

which implies $x = 0, 0, 1, 1$. Hence, our method is zero stable for all $s, r \in (0, 1)$

3.3. Consistency

The one step hybrid block method (8) is said to be consistent if its order greater than or equal one i.e. $P \geq 1$. This proves that our method is consistent for all $s, r \in (0, 1)$.

3.4. Convergence

Theorem 3.2. [Henrici, 1962] Consistency and zero stability are sufficient conditions for a linear multistep method to be convergent.

Since the method is consistent and zero stable, it implies the method is convergent for all $s, r \in (0, 1)$.

3.5. Numerical Results

In finding the accuracy of our methods, the following second order ODEs are examined. The new block methods solved the same problems the existing methods solved in order to compare results in terms of error.

Table 1: Comparison of the new method with Kayode and Adeyeye (2013) for solving Problem 1, $h = \dfrac{1}{10}$

x	exact solution	computed solution in new method	error in our method, $s = \frac{1}{4}, r = \frac{3}{4}$	errors in [2]
0.1	-0.10517091807564771	-0.10517091930555557	$1.229908e^{-9}$	$8.17176e^{-7}$
0.2	-0.22140275816016985	-0.22140275555555555	$2.604614e^{-9}$	$3.10356e^{-6}$
0.3	-0.34985880757600318	-0.34987534374244883	$1.653617e^{-9}$	$6.56957e^{-6}$
0.4	-0.49182469764127035	-0.49185950563199765	$3.480799e^{-5}$	$1.14380e^{-5}$
0.5	-0.64872127070012819	-0.64879648045757798	$7.520976e^{-5}$	$1.79656e^{-5}$

Table 2: Comparison of the new method with Awoyemi et al. (2011) for solving problem 2, $h = \dfrac{1}{320}$

x	exact solution	computed solution in new method $s = \frac{1}{3}$	error in our method, $s = \frac{1}{4}, r = \frac{3}{4}$	errors in [3]
0.1	1.0500417292784914	1.0500417293648956	$8.640422e^{-11}$	$6.5650e^{-11}$
0.2	1.1003353477310753	1.1003353480963718	$3.652960e^{-10}$	$5.4803e^{-10}$
0.3	1.1511404359364665	1.1511404368001816	$8.637149e^{-10}$	$1.9256e^{-9}$
0.4	1.2027325540540816	1.2027325556864228	$1.632341e^{-9}$	$4.8029e^{-9}$
0.5	1.2554128118829946	1.2554128146373926	$2.754398e^{-9}$	$1.0006e^{-8}$

Problem 2: $y'' - y' = 0, \quad y(0) = 0, \quad y'(0) = -1, \; h = 0.1.$
Exact solution: $y(x) = 1 - e^x$

Problem 1: $y'' - x(y')^2 = 0, \; y(0) = 1, \; y'(0) = \frac{1}{2}, \; h = \frac{1}{320}.$
Exact solution: $y(x) = 1 + \frac{1}{2} ln\left(\frac{2+x}{2-x}\right)$

4. Conclusion

A two step hybrid block method with two generalized off-step points was developed. The method was tested to be convergent with order five for all general off-step point belong to selected interval. The derived method was applied to solve both non-linear and linear second ODEs problems without converting to the equivalents system of first order ODEs. New generated results confirm the accuracy of the new methods in terms of error.

References

[1] S. N. Jator, Solving second order initial value problems by a hybrid multistep method without predictors, *Applied Mathematics and Computation*, (2010), **217**(8), 4036–4046.

[2] D. O. Awoyemi, E. A. Adebile, A. O. Adesanya & T. A. Anake, Modified block method for the direct solution of second order ordinary differential equations, *International Journal of Applied Mathematics and Computation*, (2011)., **3**(3), 181–188.

[3] Kayode, & O. Adeyey, Two-step two-point hybrid methods for general second order differential equations, *African Journal of Mathematics and Computer Science Research*, (2013), 6(10), 191–196.

[4] T. A. Anake, D. O. Awoyemi and A. A. Adesanya, A one step method for the solution of general second order ordinary differential equations, *International Journal of science and Technology*, (2012b), **2**(4), 159–163.

[5] J.D. Lambert, *Computational methods in ordinary differential equations*, (1973).

[6] T. A Anake, D. O. Awoyemi and A. A. Adesanya, One-step implicit hybrid block method for the direct solution of general second order ordinary differential equations, *IAENG International Journal of Applied Mathematics*, (2012a), **42**(4), 224–228.

[7] A. Sagir, An accurate computation of block hybrid method for solving stiff ordinary differential equations, *Journal of Mathematics*, (2012), **4**, 18–21.

[8] Henrici, P. (1962). Discrete variable methods in ordinary differential equations.