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Marcinkiewicz integrals on product spaces and extrapolation

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Abstract

In this paper, we prove sharp L^p estimates of Marcinkiewicz integral operators with rough kernels on product spaces. These estimates are used in an extrapolation arguments to obtain some new improved and extended results in Marcinkiewicz integrals.

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1. Introduction

Throughout this article, let $n, m \geq 2$, and let $\mathbf{S}^{N-1}$ ($N = n$ or m) be the unit sphere in $\mathbf{R}^N$ equipped with the normalized Lebesgue surface measure $d\sigma = d\sigma(\cdot)$. Also, let $x' = x/|x|$ for $x \in \mathbf{R}^n \setminus \{0\}$, $y' = y/|y|$ for $y \in \mathbf{R}^m \setminus \{0\}$. Let p' be denoted to the exponent conjugate to p.

For $\rho = a_1+ib_1, \tau = a_2+ib_2$ ($a_1, b_1, a_2, b_2 \in \mathbf{R}$ with $a_1, a_2 > 0$), let $K_{\Omega,h}(x, y) = \Omega(x', y')|x|^{\rho-n}|y|^{\tau-m}h(|x|, |y|)$, where h is a measurable function on $\mathbf{R}^+ \times \mathbf{R}^+$ and Ω

is a function on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ with $\Omega \in L^1(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ satisfying the cancellation conditions:

$$\int_{\mathbf{S}^{n-1}} \Omega(x', .)d\sigma(x') = \int_{\mathbf{S}^{m-1}} \Omega(., y')d\sigma(y') = 0. \tag{1.1}$$

For suitable mappings $\phi, \psi : \mathbf{R}^+ \to \mathbf{R}$, a measurable function h on $\mathbf{R}^+ \times \mathbf{R}^+$ and an Ω on $\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}$ satisfying (1.1), we define the Marcinkiewicz integral operator $\mathcal{M}^{\rho,\tau}_{\Omega,h,\phi,\psi}$ for $f \in \mathcal{S}(\mathbf{R}^n \times \mathbf{R}^m)$ by

$$\mathcal{M}^{\rho,\tau}_{\Omega,h,\phi,\psi} f(x, y) = \left(\int_0^\infty \int_0^\infty \frac{1}{t^\rho s^\tau} \left|F^{\phi,\psi}_{t,s} f(x, y)\right|^2 \frac{dtds}{ts}\right)^{1/2}, \tag{1.2}$$

where

$$F^{\phi,\psi}_{t,s} f(x, y) = \int_{|u|\le t} \int_{|v|\le s} f(x - \phi(|u|)u', y - \psi(|v|)v')K_{\Omega,h}(u, v)dudv. \tag{1.3}$$

If $\phi(t) = t$, $\psi(s) = s$, we denote $\mathcal{M}^{\rho,\tau}_{\Omega,h,\phi,\psi}$ by $\mathcal{M}^{\rho,\tau}_{\Omega,h}$. Also, when $\rho = \tau = 1$ and the function $h = 1$, the operator $\mathcal{M}^{\rho,\tau}_{\Omega,h}$ is the classical Marcinkiewicz integral operator on product domains which we shall denote by $\mathcal{M}_{\Omega,c}$.

The theory of Marcinkiewicz integral is an important part of analysis due to its powerful role in dealing with many significant problems arising in such parts of analysis as Poisson integrals, singular integrals and singular Radon transforms analysis. The study of Marcinkiewicz integral operators has attracted the attention of many authors for along time. For example, the author of [19] established the L^2 boundedness of $\mathcal{M}_{\Omega,c}$ if $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Subsequently, it was verified in [16] that $\mathcal{M}_{\Omega,c}$ is bounded for all $1 < p < \infty$ provided that $\Omega \in L(\log L)^2(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. For more information about the importance and the recent advances on the study of such operators, the readers are refereed (for instance to [2], [4], [15], [17], [18], [29], [30], as well as [31], and the references therein).

We point out, in the one parameter case, the study of parametric Marcinkiewicz integral operator was initiated by Hörmander in [23] in which he showed that $\mathcal{M}^{\rho}_{\Omega,1}$ is bounded on $L^p(\mathbf{R}^n)$ for $1 < p < \infty$ when $\rho > 0$ and $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$ with $\alpha > 0$, and subsequently by Sakamoto and Yabuta in [24] (for the corresponding results in the one parameter cases, see for instance [3], [6], [8], [9], [12], [13], [14], [20], [21], [26], and [28]).

The main concern in this work is to establish L^p estimates of $\mathcal{M}^{\rho,\tau}_{\Omega,h,\phi,\psi}$ for various functions ϕ, ψ and h; satisfying conditions similar to that found in [5], and then use these estimates in the extrapolation argument used in [7] to obtain new improved results. The main result of this paper is described in the following theorem.

Theorem 1.1. Let $\Omega \in L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $1 < q \le 2$, $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $\gamma > 1$. Suppose that ϕ, ψ are $C^2([0, \infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Then for any $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with p satisfying $|1/p - 1/2| <$

$\min\{1/2, 1/\gamma'\}$, there exists a constant C_p (independent of Ω, h, γ, and q) such that

$$\left\|\mathcal{M}^{\rho,\tau}_{\Omega,\phi,\psi,h} f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p \frac{A(\gamma)}{q-1} \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)},$$

where

$$A(\gamma) = \begin{cases} \gamma & \text{if } \gamma > 2, \\ (\gamma-1)^{-1} & \text{if } 1 < \gamma \leq 2; \end{cases}$$

and $\Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ (for $\gamma \geq 1$) denotes the collection of all measurable functions $h : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{C}$ satisfying

$$\sup_{R_1,R_2>0} \left(\frac{1}{R_1, R_2} \int_0^{R_1} \int_0^{R_2} |h(t,s)|^\gamma \, dt ds\right)^{1/\gamma} < \infty.$$

The power of our theorem lies in using its conclusion and the extrapolation arguments found in [7] to obtain improved results. In particular, Theorem 1.1 and extrapolation lead to the following theorem.

Theorem 1.2. Suppose that $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $\gamma > 1$, and Ω satisfies (1.1). Let ϕ, ψ be $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. (i) If $\Omega \in B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ for some $q > 1$, then

$$\left\|\mathcal{M}^{\rho,\tau}_{\Omega,\phi,\psi,h} f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p A(\gamma) \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \times \left(1 + \|\Omega\|_{B_q^{(0,0)}(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\right)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$;
(ii) If $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$, then

$$\left\|\mathcal{M}^{\rho,\tau}_{\Omega,\phi,\psi,h} f\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p A(\gamma) \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \times \left(1 + \|\Omega\|_{L(logL)(\mathbf{S}^{n-1}\times\mathbf{S}^{n-1})}\right)$$

for $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$.

Remarks 1.3.

(1) If Ω belongs to the block space $B_q^{(0,0)}(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$ and $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $q, \gamma > 1$, then the L^p boundedness of $\mathcal{M}^{\rho,\tau}_{\Omega,\phi,\psi,h}$ was obtained in [10] for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$ provided that ϕ, ψ are $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$.

(2) The authors of [4] established the L^p $(1 < p < \infty)$ boundedness of $\mathcal{M}^{1,1}_{\Omega,c}$ under the condition $\Omega \in L(\log L)(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1})$. Furthermore, they proved that the exponent 1 is the best possible.

(3) In the one parameter case, it was proved in [1] that if $\Omega \in B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$ and $h \in \Delta_\gamma(\mathbf{R}^+)$ for some $1 < \gamma \leq 2$, then $\mathcal{M}^{1}_{\Omega,\phi,h}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$, where ϕ is $C^2([0,\infty))$, convex and increasing function with $\phi(0) = 0$. However, Ali in [9] used the extrapolation arguments to improve the results of [1]. In fact, he showed that if Ω belongs to the class $\Omega \in L(\log L)^{1/2}(\mathbf{S}^{n-1})$ or to the class $B_q^{(0,-1/2)}(\mathbf{S}^{n-1})$ for some $q > 1$, then $\mathcal{M}^{\rho}_{\Omega,\phi,h}$ is bounded on $L^p(\mathbf{R}^n)$ for any p satisfying $|1/p-1/2| < \min\{1/2, 1/\gamma'\}$.

Here and henceforth, the letter C denotes a bounded positive constant that may vary at each occurrence but independent of the essential variables.

2. Main Lemmas

In this section, we present and establish some lemmas used in the sequel. The first lemma of this section can be found in [9], which has its roots in [25].

Lemma 2.1. Suppose that ϕ is $C^2([0,\infty))$, convex and increasing function with $\phi(0) = 0$. Let $\mathcal{M}_{\phi,u}f$ be the maximal function of f in the direction u defined by

$$\mathcal{M}_{\phi,u}f(x) = \sup_{t\in\mathbf{R}^+} \frac{1}{t}\left|\int_{t/2}^{t} f(x-\phi(r)u)dr\right|.$$

Then, there exists a constant C_p such that

$$\left\|\mathcal{M}_{\phi,u}(f)\right\|_{L^p(\mathbf{R}^N)} \leq C_p \|f\|_{L^p(\mathbf{R}^N)}$$

for any $f \in L^p(\mathbf{R}^N)$ with $1 < p \leq \infty$.

Lemma 2.2. Suppose that ϕ, ψ are $C^2([0,\infty))$, convex and increasing functions with $\phi(0) = \psi(0) = 0$. Let $\mathcal{M}_{\phi,\psi,u,v}f$ be the maximal function of f defined by

$$\mathcal{M}_{\phi,\psi,u,v}f(x,y) = \sup_{t,s\in\mathbf{R}^+} \frac{1}{ts}\left|\int_{t/2}^{t}\int_{s/2}^{s} f(x-\phi(r)u, y-\psi(k)v)drdk\right|.$$

Then, there exists a constant C_p such that

$$\left\|\mathcal{M}_{\phi,\psi,u,v}(f)\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

for any $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with $1 < p \leq \infty$.

The proof of Lemma 2.2 follows immediately by using Lemma 2.1 and the inequality $\mathcal{M}_{\phi,\psi,u,v}f(x,y) \leq \mathcal{M}_{\psi,v}\circ\mathcal{M}_{\phi,u}f(x,y)$, where $\circ$ denotes the composition of operators and $\mathcal{M}_{\phi,u}f(x,y) = \mathcal{M}_{\phi,u}f(\cdot,y)(x)$, $\mathcal{M}_{\psi,v}f(x,y) = \mathcal{M}_{\psi,v}f(x,\cdot)(y)$.

For suitable functions ϕ, ψ on $\mathbf{R}^+$, $\theta \geq 2$, a measurable function $h : \mathbf{R}^+ \times \mathbf{R}^+ \to \mathbf{C}$, and $\Omega : \mathbf{S}^{n-1} \times \mathbf{S}^{m-1} \to \mathbf{R}$, we define the family of measures $\{\sigma_{\Omega,\phi,\psi,h,t,s} : t, s \in \mathbf{R}^+\}$ and the corresponding maximal operators $\sigma^*_{\Omega,\phi,\psi,h}$ and $M_{\Omega,\phi,\psi,h,\theta}$ on $\mathbf{R}^n \times \mathbf{R}^m$ by

$$\begin{aligned}
\int_{\mathbf{R}^n\times\mathbf{R}^m} f d\sigma_{\Omega,\phi,\psi,h,t,s} &= t^{-\rho} s^{-\tau} \int_{1/2t\leq|u|\leq t} \int_{1/2s\leq|v|\leq s} \\
&\quad \times f(\phi(|u|)u', \psi(|v|)v') K_{\Omega,h}(u,v) du dv, \\
\sigma^*_{\Omega,\phi,\psi,h} f(x,y) &= \sup_{t,s\in\mathbf{R}^+} ||\sigma_{\Omega,\phi,\psi,h,t,s}| * f(x,y)|, \\
M_{\Omega,\phi,\psi,h,\theta} f(x,y) &= \sup_{i,j\in\mathbf{Z}} \int_{\theta^i}^{\theta^{i+1}} \int_{\theta^j}^{\theta^{j+1}} ||\sigma_{\Omega,\phi,\psi,h,t,s}| * |f(x,y)| \frac{dtds}{ts},
\end{aligned}$$

where $|\sigma_{\Omega,\phi,h,t}|$ is defined in the same way as $\sigma_{\Omega,\phi,h,t}$, but with replacing Ω, h by $|\Omega|$, $|h|$, respectively. ■

Our method in proving Theorem 1.1 relies heavily on certain maximal functions and certain Fourier transform estimates. So, in order to establish our results, we need to prove the following lemmas.

Lemma 2.3. Let $\Omega \in L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $1 < q \leq 2$ and $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $\gamma > 1$. Assume that and ϕ, ψ are given as in Theorem 1.1. Then for any $f \in L^p(\mathbf{R}^n \times \mathbf{R}^m)$ with $\gamma' < p \leq \infty$, there exists a constant C_p such that

$$\|\sigma^*_{\Omega,\phi,\psi,h}(f)\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \leq C_p \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}.$$

Proof. It is clear that, by using Hölder's inequality, we get

$$\begin{aligned}
||\sigma_{\Omega,\phi,\psi,h}| * f(x,y)| &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \\
&\quad \times \left(\frac{1}{ts} \int_{\frac{t}{2}}^{t} \int_{\frac{s}{2}}^{s} \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u,v)|\right. \\
&\quad \times \left. |f(x-\phi(r)u, y-\psi(k)v)|^{\gamma'} d\sigma(u) d\sigma(v) dr dk\right)^{1/\gamma'}.
\end{aligned}$$

By using Minkowski's inequality for integrals, we have

$$\begin{aligned}
\|\sigma^*_{\Omega,\phi,\psi,h} f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} &\leq C\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^{1/\gamma}_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \\
&\quad \times \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} |\Omega(u,v)|\right. \\
&\quad \times \left. \left(\|\mathcal{M}_{\psi,\psi,u,v}\left(|f|^{\gamma'}\right)\|_{L^{(p/\gamma')}(\mathbf{R}^n\times\mathbf{R}^m)}\right) d\sigma(u) d\sigma(v)\right)^{1/\gamma'}.
\end{aligned}$$

Hence, we finish the proof of lemma 2.3 by using the las inequality and Lemma 2.2. ■

Lemma 2.4. Let $\theta \geq 2, \Omega \in L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $q > 1$ and $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $\gamma > 1$. Suppose that ϕ, ψ are given as in Lemma 2.2. Then there are constants C and α with $0 < \alpha < \dfrac{1}{2q'}$ such that

$$\left\|\sigma_{\Omega,\phi,\psi,h,t,s}\right\| \leq C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\,\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}; \tag{2.1}$$

$$\int_{\theta^i}^{\theta^{i+1}}\int_{\theta^j}^{\theta^{j+1}} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \leq C\ln^2(\theta)\,\|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\,\|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}$$
$$\times \begin{cases} \left(\phi(\theta^{j-1})\,|\xi|\right)^{-\frac{2\alpha}{q'\omega}}\left(\psi(\theta^{i-1})\,|\eta|\right)^{-\frac{2\alpha}{q'\omega}}; \\ \left(\phi(\theta^{j+1})\,|\xi|\right)^{\frac{2\alpha}{q'\omega}}\left(\psi(\theta^{i+1})\,|\eta|\right)^{\frac{2\alpha}{q'\omega}}; \\ \left(\phi(\theta^{j+1})\,|\xi|\right)^{\frac{2\alpha}{q'\omega}}\left(\psi(\theta^{i-1})\,|\eta|\right)^{-\frac{2\alpha}{q'\omega}}; \\ \left(\phi(\theta^{j-1})\,|\xi|\right)^{-\frac{2\alpha}{q'\omega}}\left(\psi(\theta^{i+1})\,|\eta|\right)^{\frac{2\alpha}{q'\omega}} \end{cases} \tag{2.2}$$

hold for all $i, j \in \mathbf{Z}$, where $\omega = \max\{2, \gamma'\}$. The constant C is independent of i, j, ξ, η, q, and θ.

Proof. Since $L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right) \subseteq L^2\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for $q \geq 2$, it is enough to prove this lemma for $1 < q \leq 2$. By using the definition of $\sigma_{\Omega,\phi,\psi,h,t,s}$, it is trivial to establish the inequality (2.1). By a simple change of variables and Hölder's inequality, we get that

$$\begin{aligned} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| &\leq C\int_{1/2}^{1}\int_{1/2}^{1} |h(tr,ks)| \left|\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-i\{\phi(tr)x\cdot\xi+\psi(sk)y\cdot\eta\}}\right. \\ &\times \left.\Omega(x,y)\,d\sigma(x)\,d\sigma(y)\right| \frac{drdk}{rk} \\ &\leq C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\left(\int_{1/2}^{1}\int_{1/2}^{1} \left|G_{t,s}(r,k)\right|^{\gamma'}\frac{drdk}{rk}\right)^{1/\gamma'}, \end{aligned}$$

where

$$G_{t,s}(r,k) = \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}} e^{-i\{\phi(tr)x\cdot\xi+\psi(sk)y\cdot\eta\}}\Omega(x,y)\,d\sigma(x)\,d\sigma(y).$$

When $\gamma \in (2,\infty)$, then by using Hölder's inequality, we reach

$$\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| \leq \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\left(\int_{1/2}^{1}\int_{1/2}^{1} \left|G_{t,s}(r,k)\right|^{2}\frac{drdk}{rk}\right)^{1/2}.$$

and when $\gamma \in (1, 2]$, we conclude

$$\begin{aligned}&\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|\\&\quad\le \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{(1-2/\gamma')}\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|G_{t,s}(r,k)\right|^2\frac{drdk}{rk}\right)^{1/\gamma'}.\end{aligned}$$

Thus, in either case we deduce that

$$\begin{aligned}&\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|\\&\le \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{(\max\{2,\gamma'\}-2)/\gamma'}\left(\int_{1/2}^{1}\int_{1/2}^{1}\left|G_{t,s}(r,k)\right|^2\frac{drdk}{rk}\right)^{1/\max\{2,\gamma'\}}\\&\le C\,\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\,\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{(\omega-2)/\gamma'}\\&\quad\times\left(\int_{\mathbf{S}^{m-1}}\left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}}J(\xi,x,u)\Omega(x,y)\,\overline{\Omega(u,y)}d\sigma(x)\,d\sigma(u)\right)d\sigma(y)\right)^{1/\omega},\end{aligned}$$

where $\omega = \max\{2, \gamma'\}$ and $J(\xi, x, u) = \int_{1/2}^{1} e^{-i\phi(tr)\xi\cdot(x-u)}\frac{dr}{r}$. Write $J(\xi, x, u) = \int_{1/2}^{1} Y_t'(r)\frac{dr}{r}$, where

$$Y_t(r) = \int_{1/2}^{r} e^{-i\phi(tz)\xi\cdot(x-u)}dz,\ 1/2 \le z \le r \le 1.$$

By the assumptions on ϕ and the mean value theorem we have that

$$\frac{d}{dz}\left(\phi(tz)\right) = t\phi'(tz) \ge \frac{\phi(tz)}{z} \ge \frac{\phi(t/2)}{r}\ \text{for}\ 1/2 \le z \le r \le 1.$$

Hence, by Van der Corput's lemma we get $|Y_t(r)| \le r\,|\phi(t/2)\xi|^{-1}\left|\xi'\cdot(x-u)\right|^{-1}$, and then by integration by parts, we conclude

$$|J(\xi,x,u)| \le C\,|\phi(t/2)\xi|^{-1}\left|\xi'\cdot(x-u)\right|^{-1}.$$

Combine the last estimate with the trivial estimate $|J(\xi, x, y)| \le C$, and choose $0 < 2\alpha q' < 1$, we get

$$|J(\xi,x,y)| \le C\,|\phi(t/2)\xi|^{-\alpha}\left|\xi'\cdot(x-y)\right|^{-\alpha},$$

which leads to

$$\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| \le C\left|\phi(t/2)\xi\right|^{\frac{-\alpha}{q'\omega}} \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{(\omega-2)/\gamma'}$$

$$\times \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}^{(2/\omega)} \left(\int_{\mathbf{S}^{n-1}\times\mathbf{S}^{n-1}} \left|\xi'\cdot(x-y)\right|^{-\alpha q'} d\sigma(x)d\sigma(y)\right)^{1/q'\omega}.$$

By the assumption of ϕ, and since the last integral is finite, we reach

$$\int_{\theta^i}^{\theta^{i+1}}\int_{\theta^j}^{\theta^{j+1}} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \le C\ln^2(\theta)\left|\xi(\phi(\theta^{j-1}))\right|^{\frac{-2\alpha}{q'\omega}} \times \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}. \quad (2.3)$$

Similarly, we derive

$$\int_{\theta^i}^{\theta^{i+1}}\int_{\theta^j}^{\theta^{j+1}} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \le C\ln^2(\theta)\left|\eta(\psi(\theta^{i-1}))\right|^{\frac{-2\alpha}{q'\omega}} \times \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}. \quad (2.4)$$

The other estimates in (2.2) can be reached by using the cancelation property of Ω. By a change of variable, we have that

$$\begin{aligned}\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| &\le \int_{\mathbf{S}^{n-1}\times\mathbf{S}^{m-1}}\int_{1/2}^{1}\int_{1/2}^{1} |e^{-i\phi(tr)\xi\cdot x} \\ &\quad -1||\Omega(x,y)||h(tr,ks)|\frac{drdk}{rk}d\sigma\,(x)\,d\sigma\,(y) \\ &\le |\xi|\,\|\Omega\|_{L^1(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\int_{\frac{1}{2}}^{1}\int_{\frac{1}{2}}^{1}|h(tr,ks)||\phi(tr)|\frac{drdk}{rk}.\end{aligned}$$

Since $\gamma > 1$, $\frac{1}{2} < r < 1$ and $\phi\,(t)$ is increasing, we achieve that

$$\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| \le C\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\left|\xi\phi(\theta^{j+1})\right|,$$

which when combined with the trivial estimate $\left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right| \le C$, we derive

$$\int_{\theta^i}^{\theta^{i+1}}\int_{\theta^j}^{\theta^{j+1}} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \le C\ln^2(\theta)\left|\xi(\phi(\theta^{j+1}))\right|^{\frac{2\alpha}{q'\omega}} \times \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}. \quad (2.5)$$

In the same manner, we obtain that

$$\int_{\theta^i}^{\theta^{i+1}}\int_{\theta^j}^{\theta^{j+1}} \left|\hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta)\right|^2 \frac{dtds}{ts} \le C\ln^2(\theta)\left|\eta(\psi(\theta^{i+1}))\right|^{\frac{2\alpha}{q'\omega}} \times \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}. \tag{2.6}$$

Therefore, by (2.3)-(2.6), the proof of this lemma is complete. ■

The following lemma can be obtained by applying the same arguments (with only minor modifications) used in the proof of [Lemma 8, [6]], which have their roots in [7] and [10].

Lemma 2.5. Let $h \in \Delta_\gamma\left(\mathbf{R}^+ \times \mathbf{R}^+\right)$ for some $\gamma > 1$, $\Omega \in L^q\left(\mathbf{S}^{n-1} \times \mathbf{S}^{m-1}\right)$ for some $1 < q \le 2$ and $\theta = 2^{q'\gamma'}$. Assume that ϕ, ψ are given as in Lemma 2.2. Then for any p satisfying $|1/p - 1/2| < \min\{1/2, 1/\gamma'\}$, there exists a positive constant C_p such that

$$\left\|\left(\sum_{i,j\in\mathbf{Z}}\int_{\theta^j}^{\theta^{j+1}}\int_{\theta^i}^{\theta^{i+1}}\left|\sigma_{\Omega,\phi,\psi,h,t,s} * g_{i,j}\right|^2\frac{dsdt}{st}\right)^{\frac{1}{2}}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \le C_p\frac{A(\gamma)}{q-1}\|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)}\|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})}\left\|\left(\sum_{i,j\in\mathbf{Z}}\left|g_{i,j}\right|^2\right)^{1/2}\right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)}$$

holds for arbitrary functions $\{g_{i,j}(\cdot,\cdot),\ i,j\in\mathbf{Z}\}$ on $\mathbf{R}^n \times \mathbf{R}^m$.

3. Proof of Theorem 1.1

We prove Theorem 1.1 by following the same approaches that the authors of [9] and [11] used, and have their roots in [3] as well as [22]. Assume that $h \in \Delta_\gamma(\mathbf{R}^+ \times \mathbf{R}^+)$ for some $\gamma > 1$; and ϕ, ψ are $C^2([0,\infty))$, convex and increasing functions with $\phi(0) =$

$\psi(0)=0$. Thanks to Minkowski's inequality, we get that

$$\begin{aligned}\mathcal{M}^{\rho,\tau}_{\Omega,h,\phi,\psi}f(x,y) &= \Bigg(\int_{\mathbf{R}^+\times\mathbf{R}^+}\Bigg|\sum_{i,j=0}^{\infty}\frac{1}{t^\rho s^\tau}\int_{2^{-i-1}t<|u|\le 2^{-i}t}\int_{2^{-j-1}s<|v|\le 2^{-j}s}\\ &\times\ f(x-\phi(|u|)u',y-\psi(|v|)v')K_{\Omega,h}(u,v)dudv\Bigg|^2\frac{dtds}{ts}\Bigg)^{1/2}\\ &\le \sum_{i,j=0}^{\infty}\Bigg(\int_{\mathbf{R}^+\times\mathbf{R}^+}\Bigg|\frac{1}{t^\rho s^\tau}\int_{2^{-i-1}t<|u|\le 2^{-i}t}\int_{2^{-j-1}s<|v|\le 2^{-j}s}\\ &\times\ f(x-\phi(|u|)u',y-\psi(|v|)v')K_{\Omega,h}(u,v)dudv\Bigg|^2\frac{dtds}{ts}\Bigg)^{1/2}\\ &\le \frac{2^{a_1+a_2}}{(2^{a_1}-1)(2^{a_2}-1)}\Bigg(\int_{\mathbf{R}^+\times\mathbf{R}^+}\left|\sigma_{\Omega,\phi,\psi,h,t,s}*f(x,y)\right|^2\frac{dtds}{ts}\Bigg)^{1/2}.\end{aligned} \tag{3.1}$$

Let $\theta=2^{q'\gamma'}$, and for $i\in\mathbf{Z}$, let $\{\Gamma_{i,\phi}\}_{-\infty}^{\infty}$ be a smooth partition of unity in $(0,\infty)$ adapted to the interval $\mathcal{I}_{i,\phi}=[\phi(\theta^{i+1})^{-1},\phi(\theta^{i-1})^{-1}]$. More precisely, we require the following:

$$\begin{aligned}\Gamma_{i,\phi} &\in C^\infty,\ 0\le\Gamma_{i,\phi}\le 1,\ \sum_i\Gamma_{i,\phi}(t)=1,\\ \operatorname{supp}\Gamma_{i,\phi} &\subseteq \mathcal{I}_{i,\phi},\ \ and\ \ \left|\frac{d^k\Gamma_{i,\phi}(t)}{dt^k}\right|\le\frac{C_k}{t^k},\end{aligned}$$

where C_k is independent of the lacunary sequence $\{\phi(\theta^i):i\in\mathbf{Z}\}$. Define the multiplier operators $M_{i,j}$ on $\mathbf{R}^n\times\mathbf{R}^m$ by $(\widehat{M_{i,j}f})(\xi,\eta)=\Gamma_{i,\phi}(|\xi|)\Gamma_{j,\psi}(|\eta|)\hat{f}(\xi,\eta)$. Then for any $f\in\mathcal{S}(\mathbf{R}^n\times\mathbf{R}^m)$ and $i,j\in\mathbf{Z}$, we have $f(x,y)=\sum\limits_{d,l\in\mathbf{Z}}M_{i+d,j+l}(f)(x,y)$. Therefore, by Minkowski's inequality we obtain

$$\mathcal{M}^{\rho,\tau}_{\Omega,\phi,\psi,h}f(x,y)\le C\sum_{d,l\in\mathbf{Z}}S_{d,l}f(x,y), \tag{3.2}$$

where

$$S_{d,l}f(x,y)=\left(\int_0^\infty\int_0^\infty\left|Y_{d,l}(x,y,t,s)\right|^2\frac{dtds}{ts}\right)^{1/2}.$$

$$Y_{d,l}(x,y,t,s)=\sum_{i,j\in\mathbf{Z}}\sigma_{\Omega,\phi,\psi,h,t,s}*M_{i+d,j+l}f(x,y)\chi_{[\theta^i,\theta^{i+1})\times[\theta^j,\theta^{j+1})}(t,s).$$

On one hand, we compute the L^p-norm of $S_{d,l}(f)$ for the any p satisfying $|1/p-1/2|<\min\{1/\gamma',1/2\}$ with $p\neq 2$. By applying the Littlewood-Paley theory and Theorem 1.2

along with the remark that follows its statement in [[27], p. 96], plus using Lemma 2.5, we have

$$\left\| S_{d,l}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \le C_p \frac{A(\gamma)}{q-1} \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} . \tag{3.3}$$

On the other hand, by Plancherel's theorem, Fubinis theorem, Lemma 2.4, and the approaches used [10]-[11], there is $0 < \varepsilon < 1$ such that

$$\begin{aligned} &\left\| S_{d,l}(f) \right\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)} \\ \le\ & \sum_{i,j\in\mathbf{Z}} \int_{\Delta_{i+d,j+l,\mu}} \left(\int_{\theta^i}^{\theta^{i+1}} \int_{\theta^j}^{\theta^{j+1}} \left| \hat{\sigma}_{\Omega,\phi,\psi,h,t,s}(\xi,\eta) \right|^2 \frac{dtds}{ts} \right) \left| \hat{f}(\xi,\eta) \right|^2 d\xi d\eta \\ \le\ & C_p \ln^2(\theta) 2^{-\varepsilon(|l|+|d|)} \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \sum_{i,j\in\mathbf{Z}} \int_{\Delta_{i+s,j+d}} \left| \hat{f}(\xi,\eta) \right|^2 d\xi d\eta \\ \le\ & C_p \left(\frac{A(\gamma)}{q-1} \right)^2 2^{-\varepsilon(|l|+|d|)} \|h\|^2_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|^2_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|^2_{L^2(\mathbf{R}^n\times\mathbf{R}^m)} , \end{aligned} \tag{3.4}$$

where $\Delta_{i,j} = \left\{ (\xi,\eta) \in \mathbf{R}^n \times \mathbf{R}^m : (|\xi|, |\eta|) \in \mathcal{I}_{i,\phi} \times \mathcal{I}_{j,\psi} \right\}$.

Interpolation between (3.3) and (3.4), we conclude

$$\begin{aligned} &\left\| S_{d,l}(f) \right\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \\ \le\ & C_p \frac{A(\gamma)}{q-1} 2^{-\frac{\kappa}{2}(|l|+|d|)} \|h\|_{\Delta_\gamma(\mathbf{R}^+\times\mathbf{R}^+)} \|\Omega\|_{L^q(\mathbf{S}^{n-1}\times\mathbf{S}^{m-1})} \|f\|_{L^p(\mathbf{R}^n\times\mathbf{R}^m)} \end{aligned} \tag{3.5}$$

holds for any p with $|1/p - 1/2| < \min\{1/\gamma', 1/2\}$ for some $0 < \kappa < 1$. Consequently, by (3.1), (3.2) and (3.5), we finish the proof of Theorem 1.1. ■

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