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Complex Fuzzy Group Based on Complex Fuzzy Space

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Abstract

In this paper the concept of complex fuzzy space and complex fuzzy binary operation are presented and developed. This concept generalises the concept of fuzzy space from the real range of membership function, [0,1], to complex range of membership function, unit disc in the complex plane. This generalisation leads us to introduce and study the approach of fuzzy group theory in the realm of complex numbers. A new theory of complex fuzzy group is introduced and developed.

Keywords: Complex fuzzy space, complex fuzzy subspace, complex fuzzy binary operation, complex fuzzy group, complex fuzzy subgroup.

1. Introduction

In the last four decades, the fuzzy group theory was developed as follows. Rosenfeld (1971) introduced the concept of fuzzy subgroup of a group. In 1975 Negoita and Ralescu considered a generalisation of Rosenfeld's definition in which the unit interval [0, l] was replaced by an appropriate lattice structure. In 1979 Anthony and Sherwood redefined fuzzy subgroup of a group using the concept of triangular norm previously defined by Schweizer and Sklar (1960). Several mathematicians followed the Rosenfeld-Anthony-Sherwood approach in investigating fuzzy group theory (Akgiil 1988; Anthony & Sherwood 1982; Das 1981; Liu 1982; Sessa 1984; Sherwood 1983). Youssef and Dib (1992) introduced a new approach to define fuzzy groupoid and fuzzy subgroupoid. In the absence of the concept of fuzzy universal set, formulation of the intrinsic definition for fuzzy group and fuzzy subgroup is not evident. This is considered as a purpose to introduce fuzzy universal set and fuzzy binary operation by Youssef and Dib (1992) and Dib (1994). The main difference between the Rosenfeld's approach and that of Youssef and Dib (1992) is the replacement of the t-norm f with a family of co-membership functions $\{f_{xy} : x, y \in X\}$. Dib (1994) introduced the concept of fuzzy space. Fathi and Salleh (2009) generalised the concept of fuzzy space to the concept of intuitionitic fuzzy space.This concept replaces the concept of universal set in the ordinary case. The fuzzy Cartesian product, fuzzy functions, and fuzzy binary operations in fuzzy spaces and fuzzy subspaces, as well, were defined by Dib (1994). After having defined fuzzy space and fuzzy binary operation, the concepts of fuzzy group, fuzzy subgroup, and fuzzy homomorphism were introduced and the theory of fuzzy groups was constructed in the same paper. In 1998, Dib and Hassan introduced the concept of fuzzy normal subgroup.

In 1989 Buckley incorporated the concepts of fuzzy numbers and complex numbers under the name fuzzy complex numbers. This concept has become a famous research topicand a goal for many researchers (Buckley 1989; Ramotet al. 2002; Qiu & Shu 2008). However, the concept given by Buckley has different range compared to the range of Ramot et al.'s definition for complex fuzzy set (CFS).Buckley's range goes to the interval $[0,1]$, while Ramot et al.'s range extends to the unit circle in the complex plane. Tamir et al. (2011, 2012) developed an axiomatic approach for propositional complex fuzzy logic. Besides, it explains some constraints in the concept of CFS that was given by Ramot et al. (2003). Moreover, many researchers have studied, developed, improved, employed and modified Ramot et al.'s approach in several fields (Tamir et al. 2011; Zhang et al. 2009; Jun et al. 2012).

In particular, the concept of fuzzy set spreads in many mathematical fields, such as topology, algebra and complex numbers. In this paper we incorporate two mathematical fields on fuzzy set, which are fuzzy algebra and complex fuzzy set theory to achieve a new algebraic system, called complex fuzzy group based on complex fuzzy space. We will use the approach of Youssef and Dib (1992) and generalise it to complex realm by following the approach of Ramot et al. (2002) and Tamir et al. (2011) to introduce the concept of complex fuzzy space. This concept generalises the concept of fuzzy space of Youssef and Dib. Having defined complex fuzzy spaces and complex fuzzy binary operations, the concepts of complex fuzzy group and complex fuzzy subgroup are introduced and the theory of complex fuzzy groups is developed.

In particular, the concept of fuzzy set spreads in many pure mathematical fields, such as topology (Lowen 1976), algebra (Hungerford 1974) and complex numbers (Buckley 1989). Our aim in this study is to incorporate two mathematical fields on fuzzy set, which are fuzzy algebra and complex fuzzy set theory to achieve a new algebraic system, called complex fuzzy group based on complex fuzzy space to generalise the concept of fuzzy space. We will use the approach of Yossef and Dib (1992) and generalise it to complex realm by following the approach of Ramot et al. (2002) and Tamir et al. (2011). Having defined complex fuzzy space and complex fuzzy binary operation, the concepts of complex fuzzy group and complex fuzzy subgroup are introduced and the theory of complex fuzzy group is developed.

2. Preliminaries

In this section we recall the definitions and related results which are needed in this work.

Definition 2.1 (Zadeh 1965) A *fuzzy set* A in a universe of discourse U is characterised by a membership function $\mu_A(x)$ that takes values in the interval $[0, 1]$.

Definition 2.2 (Dib 1994) A *fuzzy space* $(X, I = [0,1])$ is the set of all ordered pairs $(x, I), x \in X$ $(X, I) = \{(x, I) : x \in X\}$, where $(x, I) = \{(x, r) : r \in I\}$.

The ordered pair (x, I) is called a fuzzy element in the fuzzy space (X, I).

Definition 2.3 (Dib 1994) A *fuzzy subspace* U of the fuzzy space (X, I) is the collection of all ordered pairs (x, u_x), where $x \in U_0$ for some $U_0 \in X$ and u_x is a subset of I, which contains at least one element beside the zero element.

If it happens that $x \notin U_0$, then $u_x = 0$. An empty fuzzy subspace is defined as $\{(x, \varnothing_x) : x \notin U_0\}$.

Definition 2.4 (Dib 1994) A *fuzzy binary operation* $\underline{F} = (F, f_{xy})$ on the fuzzy space (X, I) is a fuzzy function from $(X, I) \square (X, I) \to (X, I)$, where $F : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ and $f_{xy} : I \square I \to I$ are functions (denoted by co-membership functions) that satisfy

(i) $f_{xy}(r, s) \neq 0$ if $r \neq 0$ and $s \neq 0$ and

(ii) f_{xy} are onto, *i.e;* $f_{xy}(I \square I) = I$ for all $(x, y) \in X \times X$.

Definition 2.5 (Dib1994) A fuzzy space (X, I) with fuzzy binary operation $\underline{F} = (F, f_{xy})$ is called a *fuzzy groupoid* and is denoted by $((X, I), \underline{F})$.

Definition 2.6 (Dib 1994) The ordered pair $(U; \underline{F})$ is called a *fuzzy subgroupoid* of the fuzzy groupoid $((X, I), \underline{F})$ if the fuzzy subspace $U = \{(x, u_x) : x \in U_o\}$ is closed under the fuzzy binary operation F.

Thus, fuzzy groupoid $((X, I), \underline{F})$ is a fuzzy group iff for every $(x, I), (y, I), (z, I) \in (X, I)$ the following conditions are satisfied:

(i) The binary operation is associative:

$$((x, I)\underline{F}(y, I))\underline{F}(z, I) = (x, I)\underline{F}((y, I)\underline{F}(z, I)),$$

$$((xFy)Fz, I) = (xF(yFz), I).$$

(ii) There is an identity element (e, I) in $((X, I); \underline{F})$ such that

$$(x, I)\underline{F}(e, I) = (e, I)\underline{F}(x, I) = (x, I), (xFe, I) = (eFx, I) = (x, I).$$

(iii) Every fuzzy element (x, I) has an inverse $(x, I)^{-1}$
$(x, I)\underline{F}(x, I)^{-1} = (x, I)^{-1}\underline{F}(x, I) = (e, I)$.

Denote $(x, I)^{-1} = (y, I)$, then we have $(xFy, I) = (yFx, I) = (e, I)$.
From (i), (ii) and (iii), it follows that (*X, F*) is an ordinary group.
Therefore, we can write $x^{-1} = y$ and then $(x, I)^{-1} = (y, I)$.
A fuzzy group $((X, I), \underline{F})$ is called a uniform fuzzy group if $\underline{F} = (F, f_{xy})$ is a uniform fuzzy binary operation, *i.e.,* $f_{xy}(r, s) = f(r, s)$, for all $x, y \in X$.

Definition 2.7 (Dib 1994) A fuzzy group $((X, I), \underline{F})$ is called a *commutative or abelian fuzzy group* if $(x, I)\underline{F}(y, I) = (y, I)\underline{F}(x, I)$, for all fuzzy elements (x, I) and (y, I) of the fuzzy space (X, I).
It is clear that $((X, I), \underline{F})$ is a commutative fuzzy group iff (*X, F*) is an ordinary commutative group.

Definition 2.8 (Ramot et al. 2002) A *complex fuzzy set (CFS) A*, defined on a universe of discourse *U*, is characterised by a membership function $\mu_A(x)$, that assigns to any element $x \in U$ a complex-valued grade of membership in *A*. By definition, the values of $\mu_A(x)$, may receive all lying within the unit circle in the complex plane, and are thus of the form $\mu_A(x) = r_A(x)e^{i\,\omega_A(x)}$, where $i = \sqrt{-1}$, each of $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$. The CFS *A* may be represented as the set of ordered pairs

$$A = \{ (x, \mu_A(x)) : x \in U \}$$
$$= \left\{ (x, r_A(x)\, e^{i\,\omega_A(x)}) : x \in U \right\}.$$

Definition 2.9 (Ramot et al. 2002) Let *A* and *B* be two complex fuzzy sets on *U* and *V* respectively, where

$A = \left\{\left\langle x, \mu_A(x) = r_A(x)e^{i\arg_{r_A}(x)}\right\rangle : x \in U\right\}$ and $B = \left\{\left\langle y, \mu_B(x) = r_B(x)e^{i\arg_{r_B}(x)}\right\rangle : x \in U\right\}$.

The *complex fuzzy union* of *A* and *B,* denoted by $A \oplus B$, is specified by

$$A \oplus B = \{\langle x, \mu_{A\oplus B}(x)\rangle : x \in U\},$$

where $\mu_{A\oplus B}(x) = r_{A\oplus B}(x)e^{i\arg_{r_{A\oplus B}}(x)} = \max(r_A(x), r_B(x))\, e^{i\max(\arg_{r_A}(x),\, \arg_{r_B}(x))}$.
The *complex fuzzy intersection* of *A* and *B*, denoted by $A \otimes B$, is specified by

$$A \otimes B = \{\langle x, \mu_{A\otimes B}(x)\rangle : (x) \in U\},$$

where $\mu_{A\otimes B}(x)=r_{A\otimes B}(x)e^{i\arg_{r_{A\otimes B}}(x)}=\min(r_A(x),r_B(x))e^{i\min(\arg_{r_A}(x),\arg_{r_B}(x))}$.

Definition 2.10 (Zhang et al. 2009) Let A and B be two complex fuzzy sets on X, and let $\mu_A(x)=r_A(x).e^{i\arg_A(x)}$ and $\mu_B(x)=r_B(x).e^{i\arg_B(x)}$ their membership functions, respectively. We say that A is greater than B, denoted by $A\supseteq B$ or $B\subseteq A$, if for any $x\in X$, $r_A(x)\le r_B(x)$, and $\arg_A(x)\le\arg_B(x)$.

3. Main Results

In this section we generalisethe notion of a fuzzy space of Dib (1994) to the case of a complex fuzzy space and discuss its properties. We also discuss some related results.

Let X be a given nonempty set and let A be a complex fuzzy set of X. The complex fuzzy set A can be identified with its membership function $\mu_A : X\rightarrow\{a\in\square : |a|\le 1\}$ defined by $\mu_A(x)=r_A(x)\,e^{i\,\alpha\,\omega_A(x)}$, where $i=\sqrt{-1}$, each of $r_A(x)$ and $\omega_A(x)$ are both real-valued, and $\omega_A(x)$, $r_A(x)\in[0,1]$, $\alpha\in[0,2\pi]$ if $x\in A$ and $A(x)=0\times e^{i0}$ if $x\notin A$. In our notation, we can write $A=\{(x,r_x e^{i\theta_x})\}$ where the membership values r_x take their values from the set $[0,1]$, $\theta\in[0,2\pi]$ and the set $A=\{(x,r_x e^{i\theta})\}$ is a complex subset of $X\times E^2$, where E^2 is the unit disc with the usual order of complex numbers. The scaling factor $\alpha\in(0,2\pi]$, is used to confine the performance of the phases within the interval $(0,2\pi]$, and the unit circle. Hence, the phase term θ may represent the fuzzy set information, and phase term values in this representation case belong to the interval [0, 1] and satisfiesfuzzy set restriction. So, α in this paper will always be considered equal to 2π.

Let E^2 be the unit disc.Then $E^2\square E^2$ is the Cartesain product $E^2\times E^2$ with partial order defined by:

(i) $(r_1e^{i\theta_{r1}},r_2e^{i\theta_{r2}})\le(s_1e^{i\theta_{s1}},\ s_2e^{i\theta_{s2}})$ iff $r_1\le s_1$ and $r_2\le s_2$, $\theta_{r_1}\le\theta_{s_1}$ and $\theta_{r_2}\le\theta_{s_2}$ whenever $s_1\ne 0$ and $s_1\ne 0$ for all $r_1,s_1,\theta_{r_1},\theta_{S_1}\in E^2$ and $r_2,s_2,\theta_{r_2},\theta_{s_2}\in E^2$.

(ii) $\left(0e^{i\theta_1},0e^{i\theta_2}\right)=(s_1e^{i\theta s_1},s_2e^{i\theta_{s2}})$ whenever $s_1=0$ or$s_2=0$ and $\theta_1=0$ or $\theta_2=0$ for every $s_1,\theta_1\in E^2$ and $s_2,\theta_2\in E^2$.

By mentioning the E^2-complex fuzzy subset, the associated membership functions are meant to take their values from the circle E^2. The complex fuzzy subset A will be denoted by $\{(x,A(x));x\in X\}$ or simply $\{(x,A(x))\}$. A complex fuzzysingleton of X with support $x\in X$ and value $r_1e^{i\theta_1}\in E^2$, $r_1\ne 0$, $\theta_1\ne 0$ may be denoted by $[x,r_1e^{i\alpha\theta_1}]$. Throughout this paper the notation $(x,r_1e^{i\theta_1})\in A$, where $A\in E^2$, means that $A(x)=r_1e^{i\theta_1}$.

Definition 3.1 A *complex fuzzy function* from X to Y is defined as a function $\underline{F}$ from E^2 to E^2 characterised by the ordered pair $(F, \{f_x\}_{x\in X})$, where $F: X \to Y$ is a function from X to Y and $\{f_x\}_{x\in X}$ is a family of functions $f_x : E^2 \to E^2$ satisfying the conditions

(i) f_x is nondecreasing on E^2

(ii) $f_x(0e^{i\theta}) = 0$ if $\theta = 1$ and $f_x(1e^{i\theta}) = 1$ if $\theta = 0$ such that the image of any complex fuzzy subset A of X under $\underline{F}$ is the complex fuzzy subset $\underline{F}(A)$ of Y defined by

$$\underline{F}(re^{i\theta_r})y = \begin{cases} \bigvee_{x\in F^{-1}(y)} f_x(re^{i\theta_r}) & \text{if } F^{-1}(y) \neq \emptyset \\ 0 & \text{if } F^{-1}(y) = \emptyset \end{cases} \qquad \text{for any } y \in Y.$$

We write $\underline{F} = (F, f_x): X \to Y$ to denote a *complex fuzzy function* from X to Y and we call the functions $f_x, x \in X$ the comembership functions associated to $\underline{F}$. A complex fuzzy function $\underline{F} = (F, f_x)$ is said to be uniform if the comembership functions f_x are identical for all $x \in X$ i.e. $f_x = f$ for $x \in X$.

Theorem 3.1 *Two complex fuzzy functions* $\underline{F} = (F, f_x)$ *and* $\underline{G} = (G, g_x)$ *from* X *to* Y *are equal if* $F = G$ *and* $f_x = g_x$ *for all* $x \in X$.

Proof. It is clear that if $F = G$ and $f_x = g_x$, for every $x \in X$, then $\underline{F} = \underline{G}$.
Conversely, let $\underline{F} = \underline{G}$. If $F \neq G$, then there exists an element $x_0 \in X$. such that $F(x_0) \neq G(x_0)$. Consider the complex fuzzy subset A of X defined by:

$$A(x) = \begin{cases} 1e^{2\pi i} & \text{if } x = x_0 \\ 0\,e^{0i} & \text{if } x \neq x_0 \end{cases}$$

We have

$$\underline{F}(A)y = \begin{cases} 1e^{2\pi i} & \text{if } y = F(x_0) \\ 0e^{i0} & \text{if } x \neq F(x_0) \end{cases} \quad \text{and} \quad \underline{G}(A)y = \begin{cases} 1e^{2\pi i} & \text{if } y = G(x_0) \\ 0e^{i0} & \text{if } y \neq G(x_0). \end{cases}$$

Now, if $F(x_0) \neq G(x_0)$. then $\underline{F}(A) \neq \underline{G}(B)$, which contradicts the fact that $\underline{F} = \underline{G}$.

On the other hand, if $f_x \neq g_x$, then there exist $x_0 \in X$. and $r_0 e^{i\theta_0} \in E^2$. such that $f_{x_0}(r_0 e^{i\theta_0}) \neq g_{x_0}(r_0 e^{i\theta_0})$.

If we consider the complex fuzzy subset of (X, E^2)

$$B(x) = \begin{cases} r_0 e^{i\theta_0} & \text{if } x = x_0 \\ 0e^{i0} & \text{if } x \neq x_0 \end{cases}$$

then $F = G$ and $f_{x_0}(r_0 e^{i\theta_0}) \neq g_{x_0}(r_0 e^{i\theta_0})$. This implies that $\underline{F}(B) \neq \underline{G}(B)$. The theorem is thus proved.

Definition 3.2 A complex fuzzy function from $X \times Y$ to Z is a function $\underline{F}$ from the complex fuzzy Cartesian product $X \times Y = (E^2 \times E^2)^{X \times Y}$ of X and Y to the set E^2 of complex fuzzy subsets of Z, characterised by the ordered pair $(F, \{f_{xy}\}_{(x,y)\in X\times Y})$, where $F: X \times Y \to Z$ is a function and $\{f_{xy}\}_{(x,y)\in X\times Y}$ is a family of functions $f_{xy}: E^2 \times E^2 \to E^2$ satisfying the conditions

(i) f_{xy} is nondecreasing on $E^2 \times E^2$ and

(ii) $f_{xy}(0e^{i\theta}, 0e^{i\theta}) = 0$ if $\theta = 0$ and $f_{xy}(1e^{i\theta}, 1e^{i\theta}) = 1$ if $\theta = 0$, such that the image of any $E^2 \times E^2$-complex fuzzy subset C of $X \times Y$ under $\underline{F}$ the complex fuzzy is subset $\underline{F}(C)$ of Z defined by

$$\underline{F}(C)z = \begin{cases} \bigvee_{(x,y)\in F^{-1}(z)} f_{xy}\left(C(x,y)\right) & \text{if } F^{-1}(z) \neq \emptyset \\ 0 & \text{if } F^{-1}(z) = \emptyset \end{cases}$$

for every $z \in Z$.

The complex fuzzy binary operation $\underline{F} = (F, f_{xy})$ on a set X is a complex fuzzy function from $X \times X$ to X and is said to be uniform if F is a uniform complex fuzzy function.

Definition 3.3 A *complex fuzzy space*, denoted by (X, E^2), where E^2 is the unit disc, is a set of all ordered pairs (x, E^2), $x \in X$, *i.e.*, $(X, E^2) = \{(x, E^2) : x \in X\}$. We can write $(x, E^2) = \{(x, re^{i\theta}) : re^{i\theta} \in E^2\}$, where $i = \sqrt{-1}$, $r \in [0, 1]$, and $\theta \in [0, 2\pi]$. The ordered pair (x, E^2) is called a complex fuzzy element in the complex fuzzy space (X, E^2).

Therefore, the complex fuzzy space is an (ordinary) set of ordered pairs. In each pair the first component indicates the (ordinary) element and the second component indicates a set of possible complex membershisp values $re^{i\theta}$ where *r* represents an amplitude term and θ represents aphaseterm.

Definition 3.4. The *complex fuzzy subspace* U of the complex fuzzy space (X, E^2) is the collection of all ordered pairs $(x, r_x e^{i\theta_x})$ where $x \in U_0$ for some $U_0 \subset X$

and $r_x e^{i\theta_x}$ is a subset of E^2, which contains at least one element beside the zero element. If it happens that $x \notin U_0$, then $r_x = 0$ and $\theta_x = 0$. The complex fuzzy subspace U is denoted by $U = \{(x, r_x e^{i\theta_x}) : x \in U_0\}$. Let U_0 denote the support of U, that is $U_0 = \{x \in U : r > 0 \text{ and } \theta > 0\}$ and denoted by $SU = U_0$. Any empty complex fuzzy subspace is defined as $\{(x, \varnothing_x = 0_x e^{i0_x}) : x \in \varnothing\}$. *i.e.*, $S\varnothing = \varnothing$.

4. Algebra of Comlex Fuzzy Subspaces

By using the definitions of union and intersection of Ramot (2002), we introduce the following definitions:

Definition 4.1 Let $U = \{(x, r_1 e^{i\theta_{r1}}) : x \in U_o\}$ and $V = \{(x, r_2 e^{i\theta_{r2}}) : x \in V_o\}$ be complex fuzzy subspaces of the complex fuzzy space (X, E^2). The union $U \cup V$ and the intersection $U \cap V$ of complex fuzzy subspaces are defined as follows:

$U \cup V = \{(x, r_1 \cup r_2 \ e^{i(\theta_{r1} \cup \theta_{r2})}) : x \in U_o \cup V_0\}$

$U \cap V = \{(x, r_1 \cap r_2 \ e^{i(\theta_{r1} \cap \theta_{r2})}) : x \in U_o \cap V_0\}$

The support of these complex fuzzy subspaces satisfies the following:

$S(U \cup V) = S(U) \cup S(V) = U_0 \cup V_0$

$S(U \cap V) = S(U) \cap S(V) = U_0 \cap V_0$.

Definition 4.2 The *difference* $U - V$ between the complex fuzzy subspaces $U = \{(x, r_1 e^{i\theta_{r1}}) : x \in U_o\}$ and $V = \{(x, r_2 e^{i\theta_{r2}}) : x \in V_o\}$ is defined by $U - V = (r_1 - r_2)\, e^{i(\theta_{r1} - \theta_{r2})} \cup \{0\}$.

Notice that $S(U - V) \supset U_0 - V_0$ and equality holds if $r_1 \subset r_2$, $\theta_{r_1} \subset \theta_{r_2}$ for all $x \in U_0 \cup V_0$.

Definition 4.3 The complex fuzzy subspace $V = \{(x, r_1 e^{i\theta_{r1}}) : x \in V_o\}$ is *contained* in the complex fuzzy subspace $U = \{(x, r_2 e^{i\theta_{r2}}) : x \in U_o\}$ and denoted by $V \subset U$, if $V_0 \subset U_0, r_1 < r_2$ and $\theta_{r_1} < \theta_{r_1}$ for all $x \in V_0$.

Note. It is clear that the empty complex fuzzy subspace is contained in any complex fuzzy subspace U.

Now let (X, E^2) be a complex fuzzy space and let A be a complex fuzzy subset of X. Denote by A_o the subset of X containing all the elements with non-zero complex membership values in A, *i.e.*, the complex fuzzy subset A induces the following complex fuzzy subspaces:

1. The lower complex fuzzy subspace, induced by A, is given as:
   $\underline{H}(A) = \{(x, [0, r_1 e^{i\theta_{r_1}}]) : x \in A_o\}$
2. The upper complex fuzzy subspace, induced by A, is given as:
   $\overline{H}(A) = \{(x, \{0\} \cup [r_1 e^{i\theta_{r_1}}, 1]) : x \in A_o\}$
3. The finite complex fuzzy subspace, induced by A, is given as:
   $H_o(A) = \{(x, \{0, r_1 e^{i\theta_{r_1}}\}) : x \in A_o\}$.

Let (X, E^2) and (Y, E^2) be complex fuzzy spaces.

Definition 4.4 The *Cartesian product* of the complex fuzzy spaces (X, E^2) and (Y, E^2) is a complex fuzzy space, denoted by $(X, E^2))\ (Y, E^2)$, defined by

$$(X, E^2))\ (Y, E^2) = (X \times Y,\ E^2 \square E^2)$$
$$= \{((x, y),\ E^2 \square E^2) : x \in X \wedge y \in Y\}.$$

Every $E^2 \times E^2$-complex fuzzy subset of $X \times Y$, $A : X \times Y \to E^2 \square E^2$, belongs to the complex fuzzy space $(X \times Y,\ E^2 \square E^2)$ and $((x, y), E^2 \square E^2)$ is the complex fuzzy element of this space.
The Cartesian product of two complex fuzzy elements (x, E^2) and (y, E^2) is defined by $(x, E^2))\ (y, E^2) = ((x, y), E^2 \square E^2)$.
By using this relation, we can write
$(X, E^2))\ (Y, E^2) = \{(x, E^2))\ (y, E^2) : x \in X\ ,\ y \in Y\}$.

Definition 4.5 The *Cartesian product* of the complex fuzzy subspaces $U = \{(x, re^{i\theta_r}) : x \in U_o\}$ and $V = \{(y, se^{i\theta_s}) : y \in V_o\}$ of the complex fuzzy spaces (X, E^2) and (Y, E^2), respectively, is a complex fuzzy subspace of the complex fuzzy Cartesian product $(X \times Y,\ E^2 \square E^2)$, which is denoted by $U)\ V$:

$$U)\ V = \{((x, y),\ (r \square s) e^{i(\theta_r \square \theta_s)}) : (x, y) \in U_o \times V_o\}.$$

5. Complex Fuzzy Functions on Complex Fuzzy Spaces

In this section we introduce the definition of a complex fuzzy function on complex fuzzy spaces.

Definition 5.1 Let (X, E^2) and (Y, E^2) be complex fuzzy spaces. The *complex fuzzy function* $\underline{F}$ from the complex fuzzy space (X, E^2) into the complex fuzzy space (Y, E^2) is defined as an ordered pair $\underline{F} = (F, \{f_x\}_{x \in X})$, where $F: X \to Y$ is a function and $\{f_x\}_{x\in X}$ is a family of functions $f_x : E^2 \to E^2$ satisfying the conditions

(i) f_x is nondecreasing on E^2

(ii) $f_x(0e^{i\theta}) = 0$ if $\theta = 1$ and $f_x(1e^{i\theta}) = 1$ if $\theta = 0$, such that the image of any E^2-complex fuzzy subset A of X under $\underline{F}$ is the complex fuzzy subset $\underline{F}(A)$ of Y defined by

$$\underline{F}(A)\,y = \begin{cases} \bigvee_{re^{i\theta}\in F^{-1}(y)} f_x(re^{i\theta}) & \text{if } F^{-1}(y) \neq \emptyset \\ 0 & \text{if } F^{-1}(y) = \emptyset \end{cases}$$ for any $F^{-1}(y) \in E^2, y \in Y$.

We write $\underline{F} = (F, f_x) : (X, E^2) \to (Y, E^2)$ to denote the *complex fuzzy function* from (X, E^2) to (Y, E^2), and we call the functions f_x, $x \in X$ the comembership functions associated to $\underline{F}$. A complex fuzzy function $\underline{F} = (F, f_x)$ is said to be uniform if the comembership functions f_x are identical for all $x \in X$.

Every complex fuzzy function $\underline{F} = (F, f_x) : (X, E^2) \to (Y, E^2)$ acts on the complex fuzzy element (x, E^2) of (X, E^2) as follows:

$\underline{F}(x, E^2) = (F(x), f_x(E^2)) = (F(x), E^2)$.

Definition 5.2 Let $(X, E^2), (Y, E^2)$ and (Z, E^2) be complex fuzzy spaces**.** The *complex fuzzy function* $\underline{F}$ from $(X, E^2) \times (Y, E^2) = (X \times Y, E^2 \times E^2)$ into (Z, E^2) is defined by the ordered pair $(F, \{f_{xy}\}_{(x,y)\in X\times Y})$ where $F: X \times Y \to Z$ is a function and $\{f_{xy}\}_{(x,y)\in X\times Y}$ is a family of functions $f_{xy} : E^2 \times E^2 \to E^2$ satisfying the conditions:

(i) f_{xy} is non decreasing on $E^2 \times E^2$,

(ii) $f_{xy}(0e^{i\theta}, 0e^{i\theta}) = 0$ if $\theta = 0$ and $f_{xy}(1e^{i\theta}, 1e^{i\theta}) = 1$ if $\theta = 0$

such that the image of any $E^2 \times E^2$-complex fuzzy subset C of $X \times Y$ under $\underline{F}$ is the complex fuzzy subset $\underline{F}(C)$ of Z defined by

$$\underline{F}(C)\,z = \begin{cases} \bigvee_{(x,y)\in F^{-1}(z)} f_{xy}(C(x,y)), & \text{if } F^{-1}(z) \neq \emptyset \\ 0, & \text{if } F^{-1}(z) = \emptyset \end{cases}$$ for every $z \in Z$.

We write $\underline{F}=(F,\ f_{xy}):(X\times Y\ ,\ E^2\square E^2)\to(Z\ ,E^2)$ and f_{xy} are called the comembership functions associated to $\underline{F}$.

Every complex fuzzy function $\underline{F}=(F,\ f_{xy}):(X\times Y\ ,\ E^2\square E^2)\to(Z,E^2)$ acts on the complex fuzzy element $((x\ ,y),\ E^2\square E^2)$ of $(X\times Y\ ,\ E^2\square E^2)$ as follows:

$$\underline{F}\left((x\ ,\ y\ ),\ E^{\,2}\square\ E^{\,2}\right)=\left(F\ (x\ ,\ y\ ),\ f_{xy}\ (E^{\,2}\square E^{\,2}\ )\right)$$
$$=\left(F\ (x\ ,y\ ),\ E^{\,2}\right).$$

Definition 5.3 A *complex fuzzy binary operation* $\underline{F}=(F,\ f_{xy})$ on the complex fuzzy space $(X\ ,E^2)$ is a complex fuzzy function from $\underline{F}:(X,E^2)\times(X,E^2)\to(X,E^2)$ with comembership functions f_{xy} satisfying:

(i) $f_{xy}(re^{i\theta_r},\ se^{i\theta_s})\neq 0$ iff $r\neq 0,\ s\neq\ 0\ ,\theta_s\neq 0$ and $\theta_r\neq 0$,

(ii) f_{xy} are onto, *i.e.,* $f_{xy}(E^{\,2}\square\ E^{\,2})=E^{\,2},\ x\ ,y\in X$.

The complex fuzzy binary operation $\underline{F}=(F,f_{xy})$ on a set X is a complex fuzzy function from $X\times X$ to X and is said to be uniform if $\underline{F}$ is a uniform complex fuzzy function.

6. Complex Fuzzy Groupoids

From now on, we consider the complex fuzzy space $(X\ ,\ E^2)$.

Definition 6.1 Acomplex fuzzy space $(X\ ,\ E^2)$ with complex fuzzy binary operation $\underline{F}=(F,\ f_{xy})$ is called a *complex fuzzy groupoid* and is denoted by $\left((X\ ,\ E^{\,2}),\ \underline{F}\right)$.

For every complex fuzzy elements $(x,\ E^2)$ and $(y\ ,\ E^{\,2}),$ we can write

$(x\ ,\ E^{\,2})\ )\quad (\mathrm{y},\ E^{\,2})\ =\left((x\ ,y\ ),\ E^{\,2}\square\ E^{\,2}\right),$

which is a complex fuzzy element of $\left(X\times X,\ E^2\square\ E^2\right)$. The action of $\underline{F}=(F,\ f_{xy})$ on this complex fuzzy element is given by

$$\left(x\ ,\ E^2\right)\underline{F}\left(y\ ,\ E^2\ \right)=\underline{F}\left(\left((x\ ,\ E^2)\ )\quad (y\ ,\ E^2)\right)\right)$$
$$=\underline{F}\left(\left((x\ ,y\ ),\ (E^2\square\ E^2)\right)\right)$$
$$=\left(F(x\ ,y\ ),\ f_{xy}\ (E^2\square E^2)\right)$$
$$=\left(F(x\ ,y),\ E^2\right).$$

If we write $F(x, y) = x F y$, then we have
$(x, E^2) \underline{F} (y, E^2) = (x F y, E^2)$.

Theorem 6.1 *Associated to each complex fuzzy groupoid* $((X, E^2); \underline{F})$, *where* $\underline{F} = (F, f_{xy})$, *is a fuzzy groupoid* $((X, I); \underline{F})$ *which is isomorphic to the complex fuzzy groupoid* $((X, E^2); \underline{F})$ *by the correspondence* $(x, E^2) \leftrightarrow (x, I)$.

Proof. Let $((X, E^2); \underline{F})$ be a given complex fuzzy groupoid. Now redefine $\underline{F} = (F, f_{xy})$ to be $\mathbf{F} = (F, f_{xy})$. Since f_{xy} satisfies the axioms of fuzzy comembershisp function, $\mathbf{F} = (F, f_{xy})$ will be a fuzzy binary operation in the sense of Dib. That is $((X, I); \mathbf{F})$ will define a fuzzy groupoid in the sense of Dib.

Theorem 6.2 *To each complex fuzzy groupoid* $((X, E^2); \underline{F})$ *there is an associated (ordinary) groupoid* (X, F) *which is isomorphic to the complex fuzzy groupoid* $((X, E^2); \underline{F})$ *by the correspondence* $(x, E^2) \leftrightarrow x$.

Proof. Consider the complex fuzzy groupoid $((X, E^2); \underline{F})$. Now using the isomorphism $(x, E^2) \leftrightarrow x$. we can redefine the complex fuzzy function $\underline{F} = (F, f_{xy})$ to be $\underline{F} = F : X \times X \to X$. That is F defines an ordinary binary operation over X. Thus, (X, F) is the associated ordinary groupoid.
A complex fuzzy binary operation $\underline{F} = (F, f_{xy})$ on (X, E^2) is said to be uniform if the associated comembership functions f_{xy} are identical, i.e., $f_{xy} = f$ for all $x, y \in X$. A complex fuzzy groupoid $((x, E^2); \underline{F})$ is called a uniform complex fuzzy groupoid if $\underline{F}$ is uniform.

Definition 6.2 The ordered pair $(U ; \underline{F})$ is called a *complex fuzzy subgroupoid* of the complex fuzzygroupoid $((X, E^2); \underline{F})$, if the complex fuzzy subspace $U = \{(x, r_x e^{i\theta_x}) : x \in U_o\}$ is closed under the complex fuzzy binary operation *F*.

Theorem 6.3 *The complex fuzzy subspace* $U = \{(x, r_x e^{i\theta_x}) : x \in U_o\}$ *with the complex fuzzy binary operation* $\underline{F} = (F, f_{xy})$ *is a complex fuzzy subgroupoid of* $((X, E^2), \underline{F})$ *ifffor all* $x, y \in U_o$:

(i) $xFy \in U_o$

(ii) $f_{xy}\left((r \square s) e^{i(\theta_x \square \theta_y)} \right) = r_{xFy} e^{i\theta_{xFy}}$.

7. Complex Fuzzy Groups

Having defined the concept of complex fuzzy groupoid and complex fuzzy subgroupoid, we are now in a position to define the concept of complex fuzzy group.

Definition 7.1 A *complex fuzzy semigroup* is a complex fuzzy groupoid that is associative. A *complex fuzzymonoid* is a complex fuzzy semigroup that admits an identity.

After defining the concept of complex fuzzygroupiod, complex fuzzy semigroup and complex fuzzy monoid, we now introduce the concept of complex fuzzy group.

We call the pair $((X, E^2), \underline{F})$ a complex fuzzy algebraic system.

Definition 7.2 A complex fuzzy algebraic system $((X, E^2), \underline{F})$ is called a *complex fuzzy group* if and only if for every $(x, E^2), (y, E^2), (z, E^2) \in (X, E^2)$ the following conditions are satisfied:

(i) Associativity:

$$((x, E^2)\underline{F}(y, E^2))\underline{F}(z, E^2) = (x, E^2)\underline{F}((y, E^2)\underline{F}(z, E^2)),$$

$$i.e., ((xFy)Fz, E^2) = (xF(yFz), E^2).$$

(ii) There exists an identity element (e, E^2), for which

$$(x, E^2)\underline{F}(e, E^2) = (e, E^2)\underline{F}(x, E^2) = (x, E^2),$$

$$i.e., (x\underline{F}e, E^2) = (e\underline{F}x, E^2) = (x, E^2).$$

(iii) Every complex fuzzy element (x, E^2) has an inverse $(x, E^2)^{-1}$ such that

$$(x, E^2)\underline{F}(x, E^2)^{-1} = (x, E^2)^{-1}\underline{F}(x, E^2) = (e, E^2).$$

Denote $(x, E^2)^{-1} = (y, E^2)$, then we have $(xFy, E^2) = (yFx, E^2) = (e, E^2)$.

From (i), (ii) and (iii), it follows that (X, F) is a fuzzy group.

Therefore, we can write $x^{-1} = y$ and then $(x, E^2)^{-1} = (y, E^2)$. From the preceding discussion, we have the following theorem.

Theorem 7.1 *Associated to each complex fuzzy group* $((X, E^2); \underline{F})$, *where* $\underline{F} = (F, f_{xy})$, *is a fuzzy group* $((X, I); \mathbf{F})$, *where* $\mathbf{F} = (F, f_{xy})$, *which is isomorphic to the complex fuzzy group* $((X, E^2); \underline{F})$ *by the correspondence* $(x, E^2) \leftrightarrow (x, I)$.

Proof. The proof is similar to that of Theorem 6.1.

Theorem 7.2 *To each complex fuzzy group* $\left((X, E^2); \underline{F}\right)$ *there is an associated (ordinary) group* (X, F), *which is isomorphic to the complex fuzzy group* $\left((X, E^2); \underline{F}\right)$ *by the correspondence* $(x, E^2) \leftrightarrow x$.

Proof. Similar to the proof of Theorem 6.2.

A complex fuzzy group $\left((X, E^2), \underline{F}\right)$ is called a uniform complex fuzzy group if $\underline{F} = (F, f_{xy})$ is a uniform complex fuzzy binary operation, *i.e.,*

$f_{xy}(re^{i\theta_r}, se^{i\theta_s}) = f(re^{i\theta_r}, se^{i\theta_s})$, for all $x, y \in X$.

Definition 7.3 A complex fuzzy group $\left((X, E^2), \underline{F}\right)$ is called a *commutative or abelian complex fuzzy group* if $(x, E^2)\,\underline{F}\,(y, E^2) = (y, E^2)\,\underline{F}\,(x, E^2)$, for all complex fuzzy elements (x, E^2) and (y, E^2) of the complex fuzzy space (X, E^2). It is clear that $\left((X, E^2); \underline{F}\right)$ is a commutative complex fuzzy group iff $\left(X, F\right)$ is a commutative group.

Example 7.1 Consider the set $G = \{a\}$. Define the complex fuzzy binary operation $\underline{F} = (F, f_{xy})$ over the complex fuzzy space $\left(G, E^2\right)$ such that: $F(a, a) = a$ and $f_{aa}(re^{i\theta_r}, se^{i\theta_s}) = (r \wedge s)e^{i(\theta_r \wedge \theta_s)}$. Thus, the complex fuzzy space $\left(G, E^2\right)$ together with $\underline{F}$ define a (trivial) complex fuzzy group $\left((G, E^2), \underline{F}\right)$.

Example 7.2 Consider the set $\mathbb{Z}_3 = \{0, 1, 2\}$. Define the complex fuzzy binary operation $\underline{F} = (F, f_{xy})$ over the complex fuzzy space $(\mathbb{Z}_3, E^2)$ as follows: $F(x, y) = x +_3 y$, where $+_3$ refers to addition modulo 3 and $f_{xy}(re^{i\theta_r}, se^{i\theta_s}) = (r.s)e^{i\wedge(\theta_r.\theta_s)}$. Then $\left((\mathbb{Z}_3, E^2), \underline{F}\right)$ is a complex fuzzy group.

Definition 7.4 Let $\left((X, E^2), \underline{F}\right)$ be a complex fuzzy group and let $U = \left\{(x, r_x e^{i\theta_x}) : x \in U_0\right\}$ be a complex fuzzy subspace of (X, E^2). $(U, \underline{F})$ is called a complex *fuzzy subgroup* of the complex fuzzy group $\left((X, E^2), \underline{F}\right)$ if:

(i) $\underline{F}$ is closed on the complex fuzzy subspace *U*, *i.e.,*

$$(x, r_x e^{i\theta_x})\,\underline{F}\,(y, r_y e^{i\theta_y}) = (xFy, r_{xFy}e^{i\theta_{xFy}}) = \left(xFy, (r_x f_{xy} r_y)e^{i(\theta_x f_{xy} \theta_y)}\right)$$

(ii) $(U, \underline{F})$ satisfies the conditions of an ordinary group.

Theorem 7.2 *The subspace* $(U\,;\,\underline{F})$ *is a complex fuzzy subgroup of the complex fuzzy group* $\big((X\,,E^2),\ \underline{F}\big)$ *if and only if:*

(i) (U_0, F) *is an ordinary subgroup of* (X, F);

(ii) $f_{xy}(r_x e^{i\theta_x},\ r_y e^{i\theta_y}) = r_{xFy} e^{i\theta_{xFy}} = (r_x f_{xy} r_y)\ e^{i(\theta_x f_{xy} \theta_y)}$.

Proof. Suuppose conditions (i) and (ii) are satisfied. Then

(a) $\underline{F}$ is closed on the complex fuzzy subspace U. Let $(x\,,\,r_x e^{i\theta_x}),\ (y\,,\,r_y e^{i\theta_y}) \in U$. Then

$$(x\,,\,r_x e^{i\theta_x})\,\underline{F}\,(y\,,r_y e^{i\theta_y}) = \big(x\,F\,y\,,\ f_{xy}\,(r_x e^{i\theta_x}\,,\ r_y e^{i\theta_y})\big)$$
$$= \big(x\,F\,y\,,\ r_{xFy} e^{i\theta_{xFy}}\big) \in U\,.$$

(b) $(U,\ \underline{F})$ satisfies the conditions of an ordinary group. Let $(x\,,\ r_x e^{i\theta_x}),$ $(y\,, r_y e^{i\theta_y}),\ (z\,,\,r_z e^{i\theta_z}) \in U$. Then

(b1)
$$\big((x, r_x e^{i\theta_x})\underline{F}(y, r_y e^{i\theta_y})\big)\underline{F}(z, r_z e^{i\theta_z}) = \big(xFy,\, r_{xFy}\, e^{i\theta_{xFy}}\big)\underline{F}(z,\, r_z e^{i\theta_z})$$
$$= \big((xFy)Fz,\, r_{(xFy)Fz}\, e^{i\theta_{(xFy)Fz}}\big)$$
$$= \big(xF(yFz),\, r_{xF(yFz)} e^{i\theta_{xF(yFz)}}\big)$$
$$= (x,\, r_x e^{i\theta_x})\underline{F}\big(yFz,\, r_{yFz} e^{i\theta_{yFz}}\big)$$
$$= (x,\, r_x e^{i\theta_x})\underline{F}\big((y,\, r_y e^{i\theta_y})\underline{F}(z,\, r_z e^{i\theta_z})\big).$$

(b2)
$$(x\,,\,r_x e^{i\theta_x})\,\underline{F}\,(e,\,r_e e^{i\theta_e}) = (xFe,\ r_{xFe} e^{i\theta_{xFe}})$$
$$= (x\,,\ r_x e^{i\theta_x})$$
$$= (eFx\,,\ r_{eFx} e^{i\theta_{eFx}})$$
$$= (e,\, r_e e^{i\theta_e})\,\underline{F}\,(x\,,\, r_e e^{i\theta_e}).$$

(b3) Each $(x\,, r_x\, e^{i\theta_x})$ has an inverse $(x^{-1}, r_{x^{-1}} e^{i\theta_{x^{-1}}}),$ since

$$(x\,,\,r_x e^{i\theta_x})\,\underline{F}\,(x^{-1},\ r_{x^{-1}} e^{i\theta_{x^{-1}}}) = (x\,F\,x^{-1},\ r_{xFx^{-1}} e^{i\theta_{xFx^{-1}}})$$
$$= (x^{-1}Fx\,,\ r_{x^{-1}Fx}\, e^{i\theta_{x^{-1}Fx}})$$
$$= (e\ ,\ r_e e^{i\theta_e})$$
$$= (x^{-1},\ r_{x^{-1}} e^{i\theta_{x^{-1}}})\,\underline{F}(x\,,\ r_x e^{i\theta_x})\,.$$

From (a) and (b) we conclude that $(U\,;\,\underline{F})$ is a complex fuzzy subgroup of $\big((X\,,E^2),\ \underline{F}\big)$.

Conversely if $(U\ ;\ \underline{F})$ is a complex fuzzy subgroup of $\left((X\ ,E^{2}),\ \underline{F}\right)$ then (i) holds by the associativity. Alsothe following holds $(r_x e^{i\theta_x} f_{xy} r_y e^{i\theta_y}) = f_{xy}(r_x e^{i\theta_x}, r_y e^{i\theta_y}) = r_{xFy} e^{i\theta_{xFy}}$.

Conclusion

In this study, we have generalised the study initiated by Dib (1994) about fuzzy groups to complex fuzzy groups. The present work generalises a new algebraic structure by combining three branches of mathematics to get a huge structure carrying several properties which could be gained from the properties of complex numbers, algebra and fuzzy theory.

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