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An Investigation on Some theorems on K-Path Vertex Cover

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Abstract

In This Paper we examine upper limits on the estimation of δ_k (G) the base cardinality of a vertex K-Path Cover in G and give a few estimation and precise estimations of δ_k (G). We additionally demonstrate that δ_k (G) $\geq \frac{kn}{k+2} + \frac{m}{(k+1)(k+2)}$ for every graph G with n vertices and m edges. The base cardinality of vertex K-Path cover concepts widely used in secure communication in wireless sensor networks (WSNs).

Keywords: Graph, Base cardinality of a vertex K-Path Cover, WSN.

Introduction and motivation

In this paper, A subset S of vertices of a graph G is called a K-path vertex cover if every path of order kin G contains at least one vertex from S. Denote by δ_k (G) the minimum cardinality of a K-path vertex cover in G. It is shown that the problem of determining δ_k(G) is NP-hard for each k ≥ 2, while for trees the problem can be solved in linear time. We investigate upper bounds on the value of δ_k(G) and provide several estimations and exact values of δ_k(G). We also prove that δ_3(G) ≤ (2n + m)/6, for every graph G with n vertices and m edges we consider finite graphs without loops and multiple edges and use standard graph theory notations [3]. In particular, by the order of a path P we understand the number of vertices on P while the length of a path is the number of edges of P.

We introduce a new graph invariant that generalizes the intensively studied concept of vertex cover. Our research is motivated by the following problem [2] related to secure communication in wireless sensor networks (WSNs). The topology of WSN can be represented by a graph, in which vertices represent sensor devices and edges represent communication channels between pairs of sensor devices. Traditional security techniques cannot be applied directly to WSN, because sensor devices are limited in their computation, energy, communication capabilities. Moreover, they are often deployed inaccessible areas, where they can be captured by an attacker. In general, a standard sensor device is not considered as tamper resistant and it is undesirable to make all devices of a sensor network tamper-proof due to increasing cost. Therefore, the design of WSN security protocols has become a challenge in security research.

We focus on the Canvas scheme [1, 3] which should provide data integrity in a sensor network. The k-generalized Canvas scheme [5] guarantees data integrity under the assumption that at least one node which is not captured exists on each path of the length k−1 in the communication graph. The scheme combines the properties of cryptographic primitives and the network topology. The model distinguishes between two kinds of sensor devices—protected and unprotected. The attacker is unable to copy secrets from a protected device. This property can be realized by making the protected device tamper-resistant or placing the protected device at a safe location, where capture is problematic. On the other hand, an unprotected device can be captured by the attacker, who can also copy secrets from the device and gain control over it. During the deployment and initialization of a sensor network, it should be ensured, that at least one protected node exists on each path of the length k − 1 in the communication graph [15]. The problem to minimize the cost of the network by minimizing the number of protected vertices is formulated in [15].

Formally, let G be a graph and let k be a positive integer. A subset of vertices S ⊆ V(G) is called a t-path vertex cover if every path of order k in G contains at least one vertex from S. We denote by δ_k (G) the minimum cardinality of a t-path vertex cover in G.

Clearly, 2-path vertex cover corresponds to the well-known vertex cover (a subset of vertices such that each edge of the graph is incident to at least one vertex of the set). Therefore δ_2(G) is equal to the size of the minimum vertex cover of a graph G. Moreover, the value of δ_3(G) corresponds to the so-called dissociation number of a graph [8, 14] defined as follows.

A subset of vertices in a graph G is called a dissociation set if it induces a sub graph with maximum degree 1. The number of vertices in a maximum cardinality dissociation set in G is called the dissociation number of G and is denoted by diss (G). Clearly, δ_3 (G) = |V(G)| − diss(G).

A related coloring is the t-path chromatic number, which is the minimum number of colors that are necessary for coloring the vertices of G in such a way that each color class forms a p_k-free set [1, 12] and it is easy to see that

$$\delta_k(G) \le \frac{\chi_k(G)-1}{\chi_k(G)} \ |V(G)|$$

Since the minimum vertex cover problem is NP-hard [5], it is not surprising that so is the problem of determining δ_k for each k$\ge$ 3. We provide details in Section 2—actually; we reduce the minimum vertex cover problem to the minimum t-path vertex cover problem.

However, since the question whether there is a t-path vertex cover of size at most t can be expressed in the monadic second order logic, by famous Courcelle s' theorem [8], the minimum t-path vertex cover problem can be solved in linear time on graphs with bounded tree width, e. g. trees, series–parallel graphs, outer planar graphs, etc. In Section 3, we determine the exact value of δ_k for trees and present a linear time algorithm which returns an optimal solution for trees. Then Section 4is devoted to outer planar graphs. We present a tight upper bound for δ_k (G).

In Section 5, we provide several estimations for the size of minimum k-path vertex cover on degree of its vertices. Finally, we prove that δ_3 (G) $\le$ (2n + m)/6, for every graph G with n vertices and m edges.

Theorem 1. For every graph G without isolated vertices:

$$\delta_k(G) \le |V(G)| - \frac{k-1}{k} \sum_{u \in v(G)} \frac{1}{1+d(U)}$$

Proof. Let us arbitrarily order the vertices of G, and let us start with S being the empty set. One by one let us add vertex v_i to S unless at least two of its neigh bounds are already there. The probability that v_i will eventually land in S, is $\frac{2}{(1+d(v_i)}$, because it is the probability that in random ordering of vertices of G, v_i precedes each of its neigh bounds except for one. From this one can deduce that the expected size of the set S is at least $\sum_{v_i \in v(G)} \ {}_i \frac{2}{(1+d(v_i))}$ Since S is a 1degenerated graph, S is a forest. Using Theorem 2 we get $\delta_k(s) \le \frac{1}{k}|V(s)|$. Finally, to construct a t-path vertex cover for G we put into solution $|V(G)|\setminus S$ and all the vertices forming the minimum t-path vertex cover of S.

Theorem 2. For every positive rational number $\frac{a}{b}>1$, a, b$\in$G, and the smallest positive integer k, such that $\frac{a}{b}\leq 2k+2$ there exists a graph G with average degree d(G)= $\frac{a}{b}$ and

$$\delta_k\ (G)\geq \frac{kn}{k+2}+\frac{m}{(k+1)(k+2)}$$

Proof. Denote by H_n a complete graph on n vertices without edges of one perfect matching (assume n is even). Clearly $|V(H_n)| = n$ and

$$|E(H_n)|=\frac{n(n-1)}{2}-\frac{n}{2}=\frac{n(n-2)}{2}$$

Clearly $diss\,(H_n))$ = 2, because any arbitrary three vertices of H_n form a path of order three. Therefore $\psi_3\ ((H_n)\ n-2$, for $n \geq 2$.

We construct the graph G as the disjoint union of

- x components H_{2k+2}
- y components H_{2k+4}.

We let $x = (2b - a + 2kb)\ (k + 2)$ and $y = (a - 2kb)(k + 1)$.

First, we verify the average degree of the graph G. There are $x(2k + 2)$ vertices of degree $2k$ and $y(2k + 4)$ vertices of degree $2k + 2$; hence we get

$$\text{d (G) } =\frac{x(2k+2)(2k)+y(2k+4)(2k+2)}{x(2k+2)+y(2k+4)}$$

$$=\frac{(2b-a+2kb)(k+2)(2k+2)(2k)+(a-2kb)(k+1)(2k+4)(2k+2)}{(2b-a+2kb)(k+2)(2k+2)+(a-2kb)(k+1)(2k+4)}$$

$$=\frac{(2b-a+2kb)(2k)+(a-2kb)+(2k+2)}{(2b-a+2kb)+(a-2kb)}$$

$$=\frac{4kb-(a+2kb)(2k)+(a-2kb)(2k)+2(a-2kb)}{2b}$$

$$=\frac{a}{b}.$$

Regular graphs

In the main theorem of this section we shall use the following result of Erdős and Gallai [5].

Theorem 2. 1 (*[5]*). *If G is a graph on n vertices that does not contain a path of order k, then it cannot have more than* $\frac{n(k+2)}{2}$ *edges. Moreover, the bound is achieved when the graph consists of disjoint cliques on k −1 vertices.*

Theorem 2. 2. Let k ≥ 2 and d ≥ k −1 be positive integers. Then, for any d-regular graph G, the following holds:

$$\delta_k\ (G) \geq +\frac{d-k+2}{2d-k+2}\,|V(G)|$$

Proof. Let S$\subseteq V(G)$ be a vertex K-path cover and T= $V(G)\setminus$ S. Let E_s , E_T be the set of edges with both end vertices is S and T, respectively. Let E_{ST} be the set of edges with one end vertex in S and the second vertex in T. Then obviously

$$|E(G)| = \frac{1}{2}d|V(G)| = |E_S| + |E_{ST}| + |E_T|.$$

Since G is d-regular, $d|S| = 2|E_S| + |E_{ST}|$. therefore $|S| \geq \frac{1}{d}|E_{ST}|$. similarly $|E_{ST}|\ 2|E_T| = d|T|$. since the graph induced on the set E_T does not contain a path of order k. according to theorem 3. 1 we have $|E_T| \leq \frac{|T|(k+2)}{2}$. Combining all the previous formulas, we immediately have

$$|S| \geq \frac{1}{d}|E_{ST}| = \frac{1}{d}(d|T| - 2|E_T|) \geq (d|T| - |T|(k-2)) = \frac{d-k+2}{d}|T|$$

Then

$$\frac{|S| + (T)}{|S|} = 1 + \frac{|T|}{|S|} \leq 1 + \frac{d}{d-k+2} = \frac{2d-k+2}{d-k+2}$$

and

$$|S| \geq \frac{d-k+2}{2d-k+2}\,|V(G)|.$$

Path vertex cover for trees

We begin our investigation with the class of trees, that present an important underlying communication topology for WSN. Courcelle's theorem [8] guarantees the existence of a linear time algorithm, basically using dynamic programming. In this section, we describe such an algorithm in detail, and we use it then to derive a sharp upper bound δ_k (T) $\leq |v(T)/k|$ for an arbitrary tree T.

In order to simplify our consideration, consider that the input tree is rooted at a vertex u. By properly rooted sub tree we denote a sub tree T_v, induced by a vertex v and its descendants (with respect to u as the root) and satisfies the following properties:

1. T_v contains a path on k vertices;
2. $T_v \setminus v$ does not contain a path on k vertices.

The algorithm PVCP Tree systematically searches for a properly rooted tree T_v, puts v into a solution and removes T_v from the input tree T.

Function PVCP Tree (T, k)
Input: A tree T on n vertices and a positive integer k;
Output: A k-path vertex cover S of T ;
Form an arbitrary vertex u $\in$ T, make T rooted in u;
S := ϕ;
while T contains a properly rooted sub tree T_v do
S := $S \cup \{v\}$
T := $T \setminus T_v$;
return S;

Theorem 3. 1.
Let T be a tree and k be a positive integer. The algorithm PVCP Tree (T, k) returns an optimal k-path vertex cover of T of size at most $\frac{|v(T)|}{k}$. Therefore, $\delta_k(T) \leq \frac{|v(T)|}{k}$

Proof.
First, we shall argue that PVCP Tree returns an optimal solution. We prove this by induction on the number of vertices in the tree T. If T does not contain any path on k vertices, the empty set is the optimal solution. Suppose T contains a path on k vertices. Let T_v be a properly rooted sub tree of T. Since any k-path cover of T contains a vertex of T_v it follows that $\delta_k(T) = \delta_k(T \setminus T_V) + 1$ and hence the result follows by induction.

To prove that the returning set S contains at most $\frac{|v(T)|}{k}$ vertices, we argue that each loop of the algorithm inserts one vertex into S and removes from T one properly rooted sub tree having at least k vertices. _
Concerning the time complexity of the algorithm, it is straightforward to implement the algorithm PVCP Tree such that the returning set S in computed in linear time.

Outerplanar graphs
In this section we study the 3-path vertex cover of outerplanar graphs.

Theorem 3. Let G be an outerplanar graph of order n. Then $\delta_3(G) \leq \frac{n}{2}$

Proof. We prove the statement by induction on the number of vertices of G. Let H be a maximal outerplanar graph such that G is its sub graph satisfying V(H) = V(G). (Recall that H is maximal outerplanar if H is outerplanar, but adding any edge destroys that property.) It is easy to see that H is 2-connected and all its inner faces are triangles. Observe that every t-path vertex cover in H is a k-path vertex cover in G, since G is a sub graph of H. Therefore, it suffices to find a 3-path vertex cover in H of size at most $\frac{n}{2}$ Obviously, if H consists of a single triangle, then $\delta_3(H) = 1 \le \frac{3}{2} = \frac{n}{2}$
Assume H has at least four vertices. Since H is a 2-connected outerplanar graph, the closed trail bounding the outer face contains each vertex of H precisely once, hence H is Hamiltonian. Let $v_1, v_2, ...v_n$ be the cyclic ordering of vertices of H along the Hamiltonian cycle. Colour a vertex vi white, if the degree of vi in H is 2, otherwise colour it black. Since all the
inner faces of H are triangles, there are no two consecutive white vertices, unless H consists of a single triangle, which is excluded. Hence, the white vertices induce an independent set. The edge $v_i v_{i+1}$ is called good, if it has a white end vertex, otherwise it is bad. If all the edges incident with the outer faces are good, it implies that n is even, and half of the vertices are white. Then the set of all black vertices is a 3-path vertex cover of size $\frac{n}{2}$ since the white vertices form an independent set.
Assume there is a bad edge, i. e. there is at least one pair of consecutive black vertices, say v_i, v_{i+1} We claim that there is an edge $v_i v_i$ of H such that v_i, v_{i+1} and v_i form a triangular face adjacent to the outer face, all the three vertices v_i, v_{i+1}, and v_i are black and all the edges $v_{i+1} v_{i+2}, v_{i+3}, ..., v_{i+1}, v_i$ (indices taken modulo n) are good
For the sake of contradiction, suppose that this is not the case. Let e = v_i, v_{i+1} be a bad edge. There is a unique triangular face of H incident both with v_i and v_{i+1}; let v_i be the third vertex incident with the face. It is easy to see that v_i must be black. The vertices v_i, v_{i+1} and v_i, v_{i+1} cut the Hamiltonian cycle into the edge v_i, v_{i+1} and two paths; let σ (e) be the smaller of their lengths.
Let $e_0 = v_{i0} v_{i0+1}$ be the bad edge with σ (e) minimal; let v_{i0} be the corresponding black vertex such that together with v_{i0} and v_{i0+1} they form a triangular face of H. Without loss of generality assume that P = v_{i0+1},..., v_{i0} is the path of length σ (e). Observe that P contains at least one white vertex, thus, it contains good edges. If all the edges $v_{i0+1} v_{i0+2}, ..., v_{i0-1} v_{i0}$ are good, we are done. Otherwise, we find a bad edge e_1 = $v_i v_{i1+1}$ of P. Again, there is a unique triangular face of H incident both with vi1 and v_{i1+1}; let v_{i1} be the third (black) vertex incident with the face. Since $\{ v_{i0+1}, v_{i0} \}$ is a 2-cut of H separating the vertices $v_{i0+2}, ...v_{i0-1}$ from the rest of the graph, we have $v_{i1} \in \{ v_{i0+1}, v_{i0} \}$ But then $\sigma(e_1) < \sigma(e_0)$, a contradiction with the choice of e_0

Therefore, there is an edge e = $v_i v_i$ of H such that v_i, v_{i+1}, and v_i form a triangular face adjacent to the outer face, all the three vertices v_i, v_{i+1} and v_i are black, and all the edges $v_{i+1}v_{i+2},....,v_{i-1}v_i$ are good. It means that on the $p = v_{i+1},....,v_i$ black and white vertices alternate, thus, σ (e) = 2s for some integer S

Let W be the set of white vertices on P, let B be the set of black vertices on P, including v_{i+1} and v_i. Obviously |W| = Sand |B| = s + 1; vertices from W induce an independent set in H and v_{i+1} is adjacent to precisely one of them. Let S' be a 3-path vertex cover in the graph $H' = H \setminus \{ \; v_i, v_{i+1},....,v_i \; \}$ of size at most $\frac{|V(G')|}{2} = \frac{n-2s-2}{2}$ 2given by induction. Then $S = s' \cup \{v_i \; \} \cup (B \setminus \{v_{i+1} \; \})$ is a 3-path vertex cover in H of size at most $\frac{n-2s-2}{2} + 1 + S = \frac{n}{2}$

The bound $\frac{n}{2}$ in Theorem 3 is the best possible, since it is easy to find outerplanar graphs with. $\delta_3 \geq \frac{n}{2}$ Consider an arbitrary 2-connected outerplanar graph G. Let $v_1, v_2,.....,v_n$ be the ordering of its vertices along the Hamiltonian cycle (the boundary of the outer face). For each edge v v_i, v_{i+1} incident with the outer face $(v_{n+1} = v_1)$, add a new vertex u_i and two new edges $u_i v_i$ and v_i, v_{i+1} Let the resulting graph be H. It is easy to see that H is a 2-connected outerplanar graph on 2nvertices. We claim that $\delta_3(H) \geq n$. Let S be a 3-path vertex cover in H. Divide the vertices of H into n pairs of the form $v_i u_i$ Observe that if $u_i \notin S$ and, $v_i \notin S$ then $u_{i-1} \in S$ and $v_{i-1} \setminus \in S$ (otherwise there is a path on 3 vertices in H not covered by S). Therefore, the average number of vertices in S is either at least $\frac{1}{2}$ (if at least one vertex of a pair is in S), or $\frac{2}{4}$ (if no vertex is . in S, then in the previous pair both are). Altogether, $|S| \geq n$

Upper bounds on degree of vertices

In this section we provide several upper bounds on path vertex cover based on degrees of a graph. First, recall theorem of Caro [6] and Wei [29], which states

$$\delta_2(G) \leq |V(G)| - \sum_{u \in v(G)} \frac{1}{1 + d(U)}$$

for every graph G. Since the only graphs for which this is best possible are the disjoint unions of cliques, additional structural assumptions allow improvements (see e. g. [15–18, 23]).

The problem of dissociation number of graph was also studied in [14], where Göring et al. proved

$$\delta_2(G) \le |V(G)| - \sum_{u \in V(G)} \frac{1}{1+d(U)} - \sum_{uv \in E(G)} \frac{2}{|N(U) \cup N(V)(|N(U) \cup N(V|-1|}$$

Theorem 5. 1. Let G be a graph on n vertices and m edges. Then $\delta_3(G) \le \frac{2n+m}{6}$

Theorem 5. 2. Let a, b are integers such that $b \le a \le 2b$ There is a graph on n vertices and m edges such that $\frac{m}{n} = \frac{a}{b}$ and $\delta_3(G) \ge \frac{2n+m}{6}$

Proof. Let x = 3(2b − a) and y = 2(a − b). We construct the graph G having two types of components:
x components c_4 (a cycle on 4 vertices) with 4 edges, $\delta_3(C_4) = 2$.
y components H_6 (k_6 with a perfect matching removed) with 12 edges, $\delta_3(C_6) = 4$
It is easy to see that n = 4x + 6y and m = 4x + 12y.
Then.

$$\frac{2n+m}{6} = \frac{12x+24y}{6} = 2x+4y = \delta_3(G)$$

Moreover,

$$\frac{m}{n}\frac{2x+6y}{2x+3y} = \frac{6(2b-a)+12(a-b)}{6(2b-a)+6(a-b)} = \frac{6a}{6b} = \frac{a}{b}$$

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