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On the generalized non-commutative sphere and their K-theory

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Abstract

In the work we follow the work of J.Cuntz, he associate a universal C^*-algebra to every locally finite flag simplicial complex. In this article we define a universal C^*-algebra S_3^{nc}, associated to the 3-dimensional noncommutative sphere. We analyze the topological information of the noncommutative sphere by using it's skeleton filtration (I_k). We examine the K-theory of the quotient S_3^{nc}/I_k, and I_k for such $k, k \leq 7$.

Keywords: Universal C^*-algebra , Noncommutative sphere, K-theory of C^*-algebra.

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1. Introduction

Noncommutative 3-sphere play an important role in the non commutative geometry which introduced by Allain Connes [4] .In noncommutative geometry, the set of points in the space are replaced by the set of continuous functions on the space. In fact noncommutative geometry change our point of view from the topological space itself to the functions on the space (The algebra of functions on the space). Indeed, the main idea was noticed first by Gelfand-Niemark [8] they were give an important theorem which state that: any commutative C^*-algebra is isomorphic of the continuous functions of a compact space, namely the space of characters of the algebra. More generally any C^*-algebra (commutative or not) is isomorphic to a closed subalgebra of the algebra of bounded operators on a Hilbert space. Our motivation is the problem of classification of noncommutative spheres by using the

K-theory of C^*-algebras associated these spheres.

2. Universal C^*-algebras

In this section we give some basic definitions and facts on C^*-algebras and universal C^*-algebras and their K-theory which we will use in this article for basic theory of C^*-algebras, we refer to [15], [10]and [7].

Definition 2.1 *A C^*-algebra A is a complex Banach algebra with a conjugate-linear involution $*: A \to A$, such that* $(x^*)^* = x, (xy)^* = y^* x^*, \|x^* x\| = \|x\|^2$
for all x, y in A. The C^*-condition $\|x^* x\| = \|x\|^2$ implies that the involution is an isometry in the sense that $\|x^*\| = \|x\|$ for all x in A.

A C^*-algebra is called unital if it possesses a unit. It follows easily that $\|1\| = 1$.

Many C^*-algebras can be constructed in the form of universal C^*-algebras. In the following, we state the main definition of the universal C^*-algebras.

Definition 2.2 *Let A be a complex $*$-algebra, such that for each $x \in A$. Then we define the enveloping C^*-algebra $C^*(A)$ to be the closure of the quotient $A/\{x \mid \|x\|_u = 0\}$ with the induced norm.*

$C^*(A)$ has the following universal property: if $\varphi: A \to B$ is a $*$-homomorphism between C^*-algebras A and B, then there exists a unique $*$-homomorphism $\varphi': C^*(A) \to B$ such that $\varphi' \circ \alpha = \varphi$, where α is the canonical $*$-homomorphism from A to $C^*(A)$. Also, $\|\varphi(x)\|_u \leq \|x\|_u$. In particular, if $\|x\|_u = 0$ this means that x belongs to the kernel of φ.

Definition 2.3 *Let Λ and I be two index sets and let P be the $*$-algebra of all noncommutative polynomials in variables $x_i, x_i^*, i \in I$, with involution $(a_1 \ldots a_n)^* = a_n^* \ldots a_1^*, a_k \in \{x_i, x_{i^*} \mid i \in I\}$. Let $R = \{P_\lambda | \lambda \in \Lambda\}$ be a subset of P, and J_R be the ideal in P generated by the relations R. Then $A := P/J_R$ is called the universal$*$-algebra with generators and relations $x_i, x_i^*, i \in I$ relations R. The enveloping C^*-algebra of A is called the universal C^*-algebra with generators $x_i, x_i^*, i \in I$ and relations R.*

3. K-theory groups

The K-theory of C^*-algebras generalizes and extends the classical topological K-theory which was introduced by Atiyah [1] to the noncommutative case. More

precisely, topological K-theory of a compact space X is just the K-theory of the unital commutative C^*-algebra $C(X)$.
K-theory has found many applications in representation theory of groups, topology, geometry, index theory and many other subjects of mathematics and physics. The K-theory of C^*-algebras plays a central role in what is called noncommutative geometry, which was pioneered by A. Connes. K-theory of C^*-algebras is one of two powerful tools of this theory, the other one being cyclic homology. In noncommutative geometry the place of K–theory of spaces , [1], [9], is taken by the K-theory of C^*-algebras and Banach algebras.
By now, the K-groups of many important classes of C^*-algebras are known. So, the next important question is what K-theory can tell us about various kinds of C^*-algebras respectively about noncommutative spaces.
The K-theory of C^*-algebras is defined by means of projections and unitaries. Here we give the basic definitions of $K_0(A)$ and $K_1(A)$ groups of such unital C^*-algebra A, for more details we refer [2], [6], [7], [16] and [17].

Definition 3.1 *Let A be a unital C^*-algebra. Two projections p,q in A are called*

- Murray-von Neumann equivalent, denoted $p \sim q$, if there exist some element $v \in A$ such that $p = vv^*$ and $q = v^*v$.
- unitarily equivalent, denoted $p \sim_u q$, if there exist a unitary $u \in A$ such that $q = upu^*$.
- homotopy equivalent, denoted $p \sim_h q$ if there exist a continuous path of projections in A, $p(t):[0,1] \to A$, such that $p(0) = p, p(1) = q, t \in [0,1]$.

It is not hard to prove that the relations above are equivalence relations.
Now, let us denote by $M_\infty(A) := \bigcup_{n\geq 1} M_n(A)$, the union of the $M_n(A)$,

$$A \to M_n(A) \to M_{n+1}(A) \to ... \to M_\infty(A),\ a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

and denote by $P_\infty(A)$ the set of projections in $M_\infty(A)$.
We have the following theorem.

Theorem 3.2 *The equivalence relations, $\sim, \sim_u, \sim_h$ coincide on the set of projections $P_\infty(A)$ in $M_\infty(A)$.*
Now, we describe the $K_0(A)$-group.

Definition 3.3 *Let A be a unital C^*-algebra. For any $p \in P_\infty(A)$ denote by $[p]$ the equivalence class of p with respect to the equivalence relation $\sim$. The set $V(A) := \{[p] | p \in P_\infty(A)\}$ with the addition given by $[p]+[q]=[p\oplus q]$, where*

$p \oplus q := \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$, *is a semi-group. Actually* $V(A)$ *is an abelian semi-group, since if* $p, q \in P_\infty(A)$, *such that* $p \in P_n(A)$ *and* $q \in P_m(A)$, *put* $v = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \in P_{n+m}(A)$, *this gives* $vv^* = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} = p \oplus q$ *and* $vv^* = \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} = q \oplus p$, *so* $p \oplus q \sim q \oplus p$.

We denote by $K_0(A)$ the Grothendieck group of $V(A)$. Recall that the Grothendieck group $G(S)$ for some semi-group S is the universal abelian group generated by S.

The K_0 group satisfies a limited set of properties(which characterize this groups on the category of all C^*-algebras. For a C^*-algebra, K_0 satisfies the following short list of properties) homotopy invariance, half-exactness, continuity, stability, and $K_0(\mathbb{C}) = 0$.

On the other hand, $K_1(A)$, it is described by unitaries in $M_\infty(A)$.

Definition 3.4 *Let* A *be a unital* C^**-algebra. Denote by* $U(M_\infty(A))$ *is the group of unitaries in* $M_\infty(A)$. *Define* $K_1(A) := U(M_\infty(A))/U_0(M_\infty(A))$, *where* $U_0(M_\infty(A))$ *is the path component of the unit in* $M_\infty(A)$.

Moreover, $K_1 : A \to K_1(A)$ defines a functor from the category of $C^* -$algebras to the category of abelian groups, with the same properties as the functor K_0 but normalized in the sense that $K_1(\mathbb{C}) = 0$ and $K_1(C_0(0,1)) = \mathbb{Z}$.

The following theorem states a deep relation between $K_0(A)$ and $K_1(A)$ for a C^*-algebra A, since the algebraic equivalence relations of projections in C^*-algebras can be described in terms of homotopy equivalence, so one can find the following natural isomorphism .

Theorem 3.5 *For any* C^**-algebra* A *and the suspension* SA *of* A, *there is a natural isomorphism* $K_1(A) \to K_0(SA)$.

There is a long exact sequence and a connecting map between K_0 and K_1 for each short exact sequence as follows.

For each exact sequence $0 \to J \xrightarrow{i} A \xrightarrow{\pi} B \to 0$ of C^*-algebras, there is an exact sequence

$$K_1(J) \to K_1(A) \to K_1(B) \xrightarrow{\delta} K_0(J) \to K_0(A) \to K_0(B).$$

The connecting map δ is called the index map in K-theory.

On the other hand Bott gives a very deep result in K-theory, Bott periodicity, it is very important to to determine the K-theory of C^*-algebras related in an extension. The following theorem is given by Bott see [3].

Theorem 3.6 *The map* $\beta_A : K_0(A) \to K_1(SA)$ *is an isomorphism for any* C^**-algebra* A.

By using the connecting map and Bott periodicity, one can give a definition of the higher $K-$group, K_n, for $n \geq 1$, as follows.

Definition 3.7 *For each integer* $n \geq 2$ *and* C^**-algebra* A *we set* $K_n(A) := K_0(S^n A)$.

By using the above theorems and definitions in this section we get the following very useful result

Proposition 3.8 *For each short exact sequence*

$$0 \to I \to A \to B \to 0$$

there is a six-term exact sequence

$$\begin{array}{ccccc} K_0(I) & \to & K_0(A) & \to & K_0(B) \\ \delta\uparrow & & & & \downarrow\partial \\ K_1(B) & \leftarrow & K_1(A) & \leftarrow & K_1(I). \end{array}$$

Here δ is the index map and ∂ is given by $\partial([p]) = [exp(2\pi ix)]$, where $[p]$ is a class of projections in $K_0(B)$ and x is a selfadjoint lift of p to A.

4. Commutative 3-sphere and their K-theory

Define the commutative 3-sphere by

$$S^3 = \{(x_0, x_1, x_2, x_3) \in \Re^4 \mid x_0^{\ 2} + x_1^{\ 2} + x_1^{\ 2} + x_3^{\ 2} = 1\}.$$

We denote by $C(S^3)$ the algebra of all complex-valued continuous functions on S^3 with pointwise addition and multiplication. The algebra $C(S^3)$ with involution defined by $f^*(x) = \overline{f(x)}$ for each $f \in (S^3), x \in X$, and with the norm $\|f\| = sup\{|f(x)|, x \in S^3\}$ is a commutative C^*-algebra. In the language of Universal C^*-algebra , this algebra is isomorphic to the universal C^*-algebra generated by self-adjoint elements x_0, x_1, x_3, x_3 with relations $x_i x_j = x_j x_i, \sum_i x_i^{\ 2} = 1$ where $i, j \in \{0,1,2,3\}$.

S^3 is homeomorphic to the one-point compactification of R^3, so we have an isomorphism $C(S^3) \cong C_0(\tilde{R^3})$. Since the real line is homeomorphic to the open interval, $C_0(R)$ isomorphic to $S\mathbb{C}$ and consequently S^3C is isomorphic to $C_0(R^3)$. By Bott periodicity, we obtain $K_0(C_0(R^3)) = 0$ and $K_1(C_0(R^3)) = Z$ Now, consider the split exact sequence

$$0 \to C_0(R^3) \to \tilde{C_0(R^3)} \to C \to 0,$$

and apply the split exactness of $K-$groups. Then we obtain

$$K_0(C(S^3)) \cong K_0(\tilde{C_0(R^3)}) \cong K_0(C_0(R^3)) \oplus Z = Z$$

and

$$K_1(C(S^3)) \cong K_1(C_0(R^3)) = Z.$$

$K_0(C(S^3))$ is generated by the unit of $C(S^3)$ and $K_1(C(S^3))$ is generated by the 2×2 unitary matrix with entries viewed as the coordinate functions on S^3.

5. Noncommutative sphere and their K-theory

Noncommutative circle and their K-theory was introduced in [11]. The author in [13], and [14] analyze the topological information of the noncommutative circle and 2-dimensional sphere by using it's skeleton filtration.

Definition 5.1 *A simplicial complex Σ consists of a set of vertices V_Σ and a set of non-empty subsets of V_Σ, the simplexes in Σ, such that:*

- If $s \in V_\Sigma$, then $\{s\} \in \Sigma.$
- If $F \in \Sigma$ and $\emptyset \neq E \subset F$ then $E \in \Sigma$.

Definition 5.2 *A simplicial complex Σ is called flag or full, if it is determined by its* 1*-simplexes in the sense that* $\{s_0,...,s_n\} \in \Sigma \Leftrightarrow \{s_i, s_j\} \in \Sigma$ for all $0 \le i < j \le n$.

Cuntz in [5] associate a universal C^*-algebra to every locally finite flag simplicial complex as follows.

Definition 5.3 *Let Σ be a locally finite flag complex. Denote by V the set of its vertices. Define C_Σ^f as the universal C^*-algebra with positive generators $h_s, s \in V$, satisfying the relations*

$$\sum_{s\in V} h_s h_t = h_t, \quad t \in V$$

and

$$h_s h_t = 0 \quad for \quad \{s,t\} \notin \Sigma.$$

Denote by I_k the ideal in C_Σ^f generated by products containing at least $n+1$-different generators. The filtration (I_k) of C_Σ^f is called the skeleton filtration.

Let we consider the flag complex Σ_{S^3} with vertices $\{V^{0^+}, V^{0^-}, V^{1^+}, V^{1^-}, ..., V^{3^+}, V^{3^-}\}$ and the condition that exactly the edges $\{V^{i^+}, V^{i^-}\}$ $\forall i \in \{0,1,2,3\}$ do not belong to Σ_{S^3}, the geometric realization of Σ_{S^3} is the $3-$sphere S^3. We consider the universal

C^*-algebra with 8 positive generators $h_{V^{i+}}, h_{V^{i-}}$ where

$V^{i+}, V^{i-} \in V_{\Sigma_{S^3}} := \{V^{0^+}, V^{0^-}, V^{1^+}, V^{1^-}, ..., V^{3^+}, V^{3^-}\}$

and satisfying the relations

$\sum_i h_{V^{i^+}} + \sum_i h_{V^{i^-}} = 1, h_{V^{i^+}} h_{V^{i^-}} = 0.$

We denote it by $C^f_{\Sigma_{S^3}}$..

Let $C^*(x_0, x_1, x_2, x_3)$ denote the unital universal C^*-algebra with self-adjoint generators x_0, x_1, x_2, x_3 satisfying the relations that $\sum_i x_i^2 = 1$, $x_i x_j \neq x_j x_i$ for all $i, j \in \{0,1,2,3\}$. The abelianization of this C^*-algebra is isomorphic to the algebra of continuous functions on the 3-sphere S^3 as shown in [5]. Consider the following the positive and negative parts $(x_i)_+$ and $(x_i)_-$ of (x_i).

We have

$(x_i)^2 = (x_i)_+^{\ 2} + (x_i)_-^{\ 2}$ *and* $(x_i)_+^{\ 2}(x_i)_-^{\ 2} = 0.$

and consider :

$x_i \mapsto \sqrt{h_{V^{i+}}} - \sqrt{h_{V^{i-}}}.$

and

$h_{V^{i+}} \mapsto (x_i)_+, and\, h_{V^{i-}} \mapsto (x_i)_+.$

Then we get a new positive generators for $C^*(x_0, x_1, x_2, x_3)$ and satisfy

$\sum_i h_{V^{i^+}} + \sum_i h_{V^{i^-}} = 1, and\, h_{V^{i^+}} h_{V^{i^-}} = 0, \forall i \in \{0,1,2,3\}.$

Therefore the algebra $C^*(x_0, x_1, x_2, x_3)$ is exactly isomorphism the algebra $C^f_{\Sigma_{S^3}}$.

For simplicity we denote $C^f_{\Sigma_{S^3}}$ by S_3^{nc}.

The $K-$theory of the algebra S_3^{nc} is given by the following proposition.

Proposition 5.4 *The evaluation map* $ev : S_3^{nc} \to C$ *at the vertex* V^{1^+}*, which maps the generator* $h_{V^{1^+}}$ *to* 1 *and all the other generators to* 0*, induces an isomorphism in* $K-$*theory.*

Proof.

Define $\kappa_{V^{i^+}}$ to be the homomorphism from S_3^{nc} to S_3^{nc} that maps the generator $h_{V^{i^+}}$ to 1 and the other generators to zero. It is clear that $\kappa_{V^{i^+}}$ is the composition of the evaluation map for the vertex V^{i^+} and the natural inclusion map $C \xrightarrow{j} S_3^{nc}$. Denote by id the identity homomorphism of S_3^{nc}. Our purpose is to show that

$K_*(\kappa_{V^{1^+}}) = K_*(id)$ for a fixed vertex V^{1^+}.

Consider the homomorphisms α and β from S_3^{nc} to $M_2(S_3^{nc})$ in the form

$$\alpha = \begin{pmatrix} id & 0 \\ 0 & \alpha_1 \end{pmatrix} \; and \; \beta = \begin{pmatrix} \beta_0 & 0 \\ 0 & \beta_1 \end{pmatrix}.$$

Here α_1, β_0 and β_1 are defined as follows :

$\alpha_1(h_{V^{i^+}}) = h_{V^{i^+}} + h_{V^{i^-}}, \alpha_1(h_{V^{i^-}}) = 0 \; for \; i = 0,1,$

$\alpha_1(h_s) = h_s$, for all other generators h_s.

$\beta_0(h_{V^{0^+}}) = h_{V^{0^+}} + h_{V^{0^-}}, \beta_0(h_{V^{0^-}}) = 0 \;\; \beta_0(h_s) = h_s$, for all other generators h_s

and

$\beta_1(h_{V^{1^+}}) = h_{V^{1^+}} + h_{V^{1^-}}, \beta_1(h_{V^{1^-}}) = 0 \;\; \beta_1(h_s) = h_s$ for all other generators h_s.

One has the following homotopies

- α is homotopic to β, by using the homomorphisms φ_t from S_3^{nc} to $M_2(S_3^{nc})$ mapping $h_{V^{0^+}}, h_{V^{0^-}}$ to $R_t\beta(h_{V^{0^+}})R_t^*, R_t\beta(h_{V^{0^-}})R_t^*$ and all other generators h_s to $\beta(h_s)$ where R_t are rotation matrices, $t \in [0, \pi/2]$. Clearly, we have $\varphi_0 = \beta$ and $\varphi_{\pi/2} = \alpha$.
- α_1 is homotopic to $\kappa_{V^{0^+}}$, using the homomorphism which maps $h_{V^{0^+}}$ to $t(h_{V^{0^+}} + h_{V^{0^-}}) + (1-t)1$, $h_{V^{0^-}}$ to 0, $h_{V^{1^+}}$ to $t(h_{V^{1^+}} + h_{V^{1^-}})$, $h_{V^{1^-}}$ to 0 and all the other generators h_s to th_s, $t \in [0,1]$.
- β_0 is homotopic to $\kappa_{V^{0^+}}$, using the homomorphism, which maps $h_{V^{0^+}}$ to $t(h_{V^{0^+}} + h_{V^{0^-}}) + (1-t)1$, $h_{V^{0^-}}$ to 0 and all the other generators h_s to th_s.
- β_1 is homotopic to $\kappa_{V^{1^+}}$, using the homomorphism, which maps $h_{V^{1^+}}$ to $t(h_{V^{1^+}} + h_{V^{1^-}}) + (1-t)1$, $h_{V^{1^-}}$ to 0 and all the other generators h_s to th_s.

From the above homotopies

$K_*(\alpha) = K_*(id \oplus \alpha_1) = K_*(id \oplus \kappa_{V^{0^+}})$

and

$K_*(\kappa_{V^{0^+}} \oplus \kappa_{V^{1^+}}) = K_*(\beta).$

Since $K_*(\alpha) = K_*(\beta)$, this implies that $K_*(\kappa_{V^{1^+}}) = K_*(id)$, and in the other side $K_*(\kappa_{V^{1^+}}) = K_*(ev) \circ K_*(j)$. This means that $K_*(ev)$ is the inverse of $K_*(j)$ and this prove that $K_*(S_3^{nc})$ and $K_*(C)$ are isomorphic.

Next we study the K-theory of the skeleton filtration for S_3^{nc} . We study the K-

theory of the quotient S_3^{nc}/I_k for all k. We find that the K-theory of the quotients S_3^{nc}/I_k is related to the K-theory of I_k as follows.

Theorem 5.5 *For each* $k \le 7$ *we have isomorphisms*
$K_0(S_3^{nc}/I_k) = K_1(I_k) \oplus Z$
and
$K_1(S_3^{nc}/I_k) = K_0(I_k).$

Proof.
Consider the skeleton filtration
$S_3^{nc} = I_0 \supset I_1 \supset I_2 \supset ... \supset I_7.$
The following short sequence

$$0 \to I_k \xrightarrow{i} S_3^{nc} \xrightarrow{\pi} S_3^{nc}/I_k \to 0$$

is exact. $\pi : S_3^{nc} \to S_3^{nc}/I_k$ is the quotient map. Hence, by the six term exact sequence in $K-$theory, we get

$$\begin{array}{ccccc} K_0(I_k) & \xrightarrow{i_*} & K_0(S_3^{nc}) & \xrightarrow{\pi_*} & K_0(S_3^{nc}/I_k) \\ \uparrow & & & & \downarrow \\ K_1(S_3^{nc}/I_k) & \leftarrow & K_1(S_3^{nc}) & \leftarrow & K_1(I_k). \end{array}$$

Since we have $K_*(S_3^{nc}) \cong K_*(\mathbb{C})$ by 5.4, the above exact cyclic sequence becomes

$$\begin{array}{ccccc} K_0(I_k) & \xrightarrow{i_*} & Z & \xrightarrow{\pi_*} & K_0(S_3^{nc}/I_k) \\ \uparrow & & & & \downarrow \\ K_1(S_3^{nc}/I_k) & \leftarrow & 0 & \leftarrow & K_1(I_k). \end{array}$$

Where π_* maps the class [1] in $K_0(S_3^{nc})$ to the class [1] in $K_0(S_3^{nc}/I_k)$. So $K_0(S_3^{nc}/I_k)$ contains a copy of Z and $Z \xrightarrow{\pi_*} K_0(S_3^{nc}/I_k)$ is an embedding.
Consequently $i_* = 0$ and we have an isomorphism
$K_1(S_3{}^{nc}/I_k) = K_0(I_k)$
and a short exact sequence

$$0 \to Z \to K_0(S_3^{nc}/I_k) \to K_1(I_k) \to 0.$$

In the remaining part of the proof we show that this short exact sequence splits. Let $\alpha : S_3^{nc}/I_k \to C$ be the evaluation map.
Consider the natural map $ev : S_3^{nc} \to C$ which sends $h_{V^{1+}}$ to 1 and all other generators to 0. Then we get the commutative diagram

$$\begin{array}{ccc} S_3^{nc} & \xrightarrow{\pi} & S_3^{nc}/I_k \\ & ev & \downarrow \alpha \\ & & C \end{array}$$

Since K_* is a functor, we obtain the following corresponding commutative diagram in K-theory,

$$\begin{array}{ccc} K_*(S_3^{nc}) & \overset{\pi_*}{\rightarrow} & K_*(S_3^{nc}/I_k) \\ & ev_* & \downarrow \alpha_* \\ & & K_*(C), \end{array}$$

By proposition 5.4 the $K-$groups of S_3^{nc} and C coincide, and the map

$ev_* : K_*(S_3^{nc}) \rightarrow K_*(C)$

is equivalent to the identity. So the composition $\alpha_* \circ \pi_*$ is equivalent to the identity and thus the sequence (5) splits. Hence we obtain the desired isomorphism

$K_0(S_3^{nc}/I_k) = K_1(I_k) \oplus Z.$

Now, we apply the above theorem $k=1$, to obtain $K_0(I_1)$ and $K_0(I_1)$.

Lemma 5.6 *Let* S_3^{nc}/I_k *as in the above, then* $K_0(S_3^{nc}/I_1) = Z^8$ *and* $K_1(S_3^{nc}/I_1) = 0$.

Proof.

Let $\dot{h}_i$ denote the image of a generator h_i for S_3^{nc}/I_0. One has the following relations :

$$\sum_i \dot{h}_i = 1, \quad \dot{h}_i \dot{h}_j = 0, \quad i \neq j.$$

For every $\dot{h}_i$ in S_3^{nc}/I_1 we have

$$\dot{h}_i = \dot{h}_i (\sum_i \dot{h}_i) = \dot{h}_i^2.$$

Hence S_3^{nc}/I_1 is generated by 8 different orthogonal projections and therefore $S_3^{nc}/I_1 \cong C^8$. And therefore $K_0(S_3^{nc}/I_1) = Z^8$, $K_1(S_3^{nc}/I_1) = 0$. Apply theorem 2.5 and the lemma 2.6 above we get. $K_1(I_1) = Z^7$ and $K_0(I_1) = 0$.

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