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Direct numerical method for solving a class of fourth-order partial differential equation

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Abstract

In this paper, we classified a class of fourth-order partial differential equations (PDEs) to be fourth-order PDE of type I, II, III and IV. The PDE of type IV is solved by using an efficient numerical method. The PDE is first transformed to a system of fourth-order ordinary differential equations (ODEs) using the method of lines, then the resulting system of fourth-order ODEs is solved using direct Runge-Kutta method (RKFD). The RKFD method is constructed purposely for solving special fourth-order ODEs. Numerical results demonstrated that the RKFD method is in good agreement with the exact solutions.

Keywords: RKFD method, System of fourth-order ODEs, Fourth-order PDEs, Method of lines.

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1. Introduction

Fourth-order partial differential equations (PDEs) are widely used to describe the mathematical models in a variety of physical phenomenon and dynamic processes in physics, engineering and geological science such as fluids in lungs, physical flows include ice formation, episodic vibration of a uniform elastic beams and plate deviation theory (see [1, 2, 3, 4, 5, 6]). Several methods have been used to find the analytic solution of linear and non-linear fourth-order PDEs, for instance variation iteration method [7], homotopy perturbation method [8], Adomian Decomposition Method [9, 10]) and homotopy analysis method [11]. For most fourth-order PDEs the analytic solutions are not available, hence, there is a need to develop an efficient numerical methods to solve these PDEs because the solutions of fourth-order PDEs are of great importance to scientists and engineers.

In the last years, wide classes of PDEs problems in various fields of mathematics, engineering and physics have been solved by using numerical methods. For instance, the method of lines (MOL) is one such efficient technique for solving PDEs. This computational technique involves approximation of the derivatives by converting the partial differential equations to a system of ordinary differential equations and then the resulting ODE system is solved using numerical method. The MOL has been applied in [12, 13, 14] to solve second-order elliptic PDEs. Also, MOL has been used in [15] to solve wave equation while Khaliq and Twizell [16] solved fourth-order parabolic PDEs using MOL. Furthermore, PDE of type I and II have been solved by using finite difference method in [17, 18]. Caglar and Caglar [19] proposed B-spline method to solve PDE of type II. Soltanalizadeh [20] obtained the solution of PDE of type II by applying differential transformation method (DTM). Recently, Mechee et al. [31] categorized three types of third-order PDEs to be third-order PDE of type I, II and III.

Motivated and inspired by the ongoing research in this field, we have classified fourth-order PDEs to be of type I, II, III and IV. We proposed a new numerical method to solve fourth-order PDE of type IV together with MOL. First MOL is applied to convert the fourth-order PDE of type IV to a system of fourth-order ODEs and then used the RKFD method to solve the resulting system directly.

The outline of this paper is as follows: In section 1 we present the preliminaries. In section 2 the proposed numerical method is given in details. In section 3 we implement numerical experiments to show the accuracy of the proposed method. The concluding remarks are given in section 5.

2. Preliminaries

Definition 2.1. The general fourth-order PDEs in two variables can be presented as follows:

$$f\Bigg(x, t, u(x,t), \frac{\partial u(x,t)}{\partial x}, \frac{\partial u(x,t)}{\partial t}, \frac{\partial^2 u(x,t)}{\partial x^2}, \frac{\partial^2 u(x,t)}{\partial x \partial t}, \frac{\partial^2 u(x,t)}{\partial t^2}, \frac{\partial^3 u(x,t)}{\partial x^3},$$
$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}, \frac{\partial^3 u(x,t)}{\partial x \partial t^2}, \frac{\partial^3 u(x,t)}{\partial t^3}, \frac{\partial^4 u(x,t)}{\partial x^4}, \frac{\partial^4 u(x,t)}{\partial x^3 \partial t},$$
$$\frac{\partial^4 u(x,t)}{\partial x \partial t^3}, \frac{\partial^4 u(x,t)}{\partial x^2 \partial t^2}, \frac{\partial^4 u(x,t)}{\partial t^4}\Bigg) = 0. \tag{2.1}$$

In general the linear fourth-order PDE in n variables can be written as follows:

$$\begin{aligned}
&\sum_{j=1}^{n} f_j(x_1, x_2, \ldots, x_n) \frac{\partial u(x_1, x_2, \ldots, x_n)}{\partial x_j} \\
&\quad + \sum_{j_1 \le j_2 = 1}^{n} g_{j_1, j_2}(x_1, x_2, \ldots, x_n) \frac{\partial^2 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2}} \\
&\quad + \sum_{j_1 \le j_2 \le j_3 = 1}^{n} h_{j_1, j_2, j_3}(x_1, x_2, \ldots, x_n) \frac{\partial^3 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \\
&\quad + \sum_{j_1 \le j_2 \le j_3 \le j_4 = 1}^{n} z_{j_1, j_2, j_3, j_4}(x_1, x_2, \ldots, x_n) \frac{\partial^4 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \partial x_{j_4}} \\
&\quad = f(x_1, x_2, \ldots, x_n).
\end{aligned} \tag{2.2}$$

Generally, quasi linear fourth-order PDE in n variables will be in the following form

$$\begin{aligned}
&\sum_{j=1}^{n} f_j(x_1, x_2, \ldots, x_n, u) \frac{\partial u(x_1, x_2, \ldots, x_n)}{\partial x_j} \\
&\quad + \sum_{j_1 \le j_2 = 1}^{n} g_{j_1, j_2}(x_1, x_2, \ldots, x_n, u) \frac{\partial^2 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2}} \\
&\quad + \sum_{j_1 \le j_2 \le j_3 = 1}^{n} h_{j_1, j_2, j_3}(x_1, x_2, \ldots, x_n, u) \frac{\partial^3 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \\
&\quad + \sum_{j_1 \le j_2 \le j_3 \le j_4 = 1}^{n} z_{j_1, j_2, j_3, j_4}(x_1, x_2, \ldots, x_n, u) \frac{\partial^4 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \partial x_{j_4}} \\
&\quad = f(x_1, x_2, \ldots, x_n, u).
\end{aligned} \tag{2.3}$$

Here, Some new definitions for a class of quasi linear fourth-order PDE are presented.

Definition 2.2. The quasi linear fourth-order PDE of type I which has n independent variables can be defined as follows:

$$
\begin{aligned}
u_{x_{j_k}} = & \sum_{j=1\&\neq j_k}^{n} f_j(x_1, x_2, \ldots, x_n, u)\frac{\partial u(x_1, x_2, \ldots, x_n)}{\partial x_j} \\
& + \sum_{j_1\leq j_2=1\&\neq j_k}^{n} g_{j_1,j_2}(x_1, x_2, \ldots, x_n, u)\frac{\partial^2 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}} \\
& + \sum_{j_1\leq j_2\leq j_3=1\&\neq j_k}^{n} h_{j_1,j_2,j_3}(x_1, x_2, \ldots, x_n, u)\frac{\partial^3 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}} \\
& + \sum_{j_1\leq j_2\leq j_3\leq j_4=1\&\neq j_k}^{n} z_{j_1,j_2,j_3,j_4}(x_1, x_2, \ldots, x_n, u)\frac{\partial^4 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}\partial x_{j_4}} \\
& - f(x_1, x_2, \ldots, x_n, u), \qquad (2.4)
\end{aligned}
$$

for $j_k = 1, 2, \ldots, n$. Hence the general form of fourth-order PDE of type I in two variables is defined as

$$u_t = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \qquad (2.5)$$

or

$$u_x = f(x, t, u, u_t, u_{tt}, u_{ttt}, u_{tttt}). \qquad (2.6)$$

Definition 2.3. The quasi linear fourth-order PDE of type II which has n independent variables is given as

$$
\begin{aligned}
u_{x_{j_k},x_{j_k}} = & \sum_{j=1\&\neq j_k}^{n} f_j(x_1, x_2, \ldots, x_n, u)\frac{\partial u(x_1, x_2, \ldots, x_n)}{\partial x_j} \\
& + \sum_{j_1\leq j_2=1\&\neq j_k}^{n} g_{j_1,j_2}(x_1, x_2, \ldots, x_n, u)\frac{\partial^2 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}} \\
& + \sum_{j_1\leq j_2\leq j_3=1\&\neq j_k}^{n} h_{j_1,j_2,j_3}(x_1, x_2, \ldots, x_n, u)\frac{\partial^3 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}} \\
& + \sum_{j_1\leq j_2\leq j_3\leq j_4=1\&\neq j_k}^{n} z_{j_1,j_2,j_3,j_4}(x_1, x_2, \ldots, x_n, u)\frac{\partial^4 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}\partial x_{j_4}} \\
& - f(x_1, x_2, \ldots, x_n, u), \qquad (2.7)
\end{aligned}
$$

for $j_k = 1, 2, \dots, n$. Hence the general form of fourth-order PDE of type II in two variables is defined as

$$u_{tt} = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \tag{2.8}$$

or

$$u_{xx} = f(x, t, u, u_t, u_{tt}, u_{ttt}, u_{tttt}). \tag{2.9}$$

Definition 2.4. The quasi linear fourth-order PDE of type III which has n independent variables can be defined as follows:

$$\begin{aligned} u_{x_{j_k}, x_{j_k}, x_{j_k}} = & \sum_{j=1 \& \neq j_k}^{n} f_j(x_1, x_2, \dots, x_n, u) \frac{\partial u(x_1, x_2, \dots, x_n)}{\partial x_j} \\ & + \sum_{j_1 \le j_2 = 1 \& \neq j_k}^{n} g_{j_1, j_2}(x_1, x_2, \dots, x_n, u) \frac{\partial^2 u(x_1, x_2, \dots, x_n)}{\partial x_{j_1} \partial x_{j_2}} \\ & + \sum_{j_1 \le j_2 \le j_3 = 1 \& \neq j_k}^{n} h_{j_1, j_2, j_3}(x_1, x_2, \dots, x_n, u) \frac{\partial^3 u(x_1, x_2, \dots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}} \\ & + \sum_{j_1 \le j_2 \le j_3 \le j_4 = 1 \& \neq j_k}^{n} z_{j_1, j_2, j_3, j_4}(x_1, x_2, \dots, x_n, u) \frac{\partial^4 u(x_1, x_2, \dots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \partial x_{j_4}} \\ & - f(x_1, x_2, \dots, x_n, u), \end{aligned} \tag{2.10}$$

for $j_k = 1, 2, \dots, n$. Hence the general form of fourth-order PDE of type III in two variables is defined as follows:

$$u_{ttt} = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \tag{2.11}$$

or

$$u_{xxx} = f(x, t, u, u_t, u_{tt}, u_{ttt}, u_{tttt}). \tag{2.12}$$

Definition 2.5. The quasi linear fourth-order PDE of type IV which has n independent variables is defined as follows:

$$\begin{aligned} u_{x_{j_k}, x_{j_k}, x_{j_k}, x_{j_k}} = & \sum_{j=1 \& \neq j_k}^{n} f_j(x_1, x_2, \dots, x_n, u) \frac{\partial u(x_1, x_2, \dots, x_n)}{\partial x_j} \\ & + \sum_{j_1 \le j_2 = 1 \& \neq j_k}^{n} g_{j_1, j_2}(x_1, x_2, \dots, x_n, u) \frac{\partial^2 u(x_1, x_2, \dots, x_n)}{\partial x_{j_1} \partial x_{j_2}} \end{aligned}$$

$$+ \sum_{j_1 \le j_2 \le j_3 = 1 \& \neq j_k}^{n} h_{j_1, j_2, j_3}(x_1, x_2, \ldots, x_n, u) \frac{\partial^3 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3}}$$

$$+ \sum_{j_1 \le j_2 \le j_3 \le j_4 = 1 \& \neq j_k}^{n} z_{j_1, j_2, j_3, j_4}(x_1, x_2, \ldots, x_n, u) \frac{\partial^4 u(x_1, x_2, \ldots, x_n)}{\partial x_{j_1} \partial x_{j_2} \partial x_{j_3} \partial x_{j_4}}$$

$$- f(x_1, x_2, \ldots, x_n, u), \tag{2.13}$$

for $j_k = 1, 2, \ldots, n$. Hence the general form of fourth-order PDE of type IV in two variables is given as

$$u_{tttt} = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \tag{2.14}$$

or

$$u_{xxxx} = f(x, t, u, u_t, u_{tt}, u_{ttt}, u_{tttt}). \tag{2.15}$$

Recently, several numerical techniques have been used to solve a vast classes of PDEs. However, these numerical methods give numerical solutions of some types of PDEs. The main objective of this paper is to solve fourth-order PDE of type IV numerically. First, the fourth-order PDE of type IV is transformed to a system of fourth-order ODEs using the method of lines. Then the resulting system of fourth-order ODEs is solved using RKFD method, which constructed purposely for directly solving fourth-order ODEs (see [21, 22]).

3. The description of the proposed numerical method

Special fourth-order ODEs in which the function does not depend explicitly on the first derivative ($y'(x)$), the second derivative ($y''(x)$) and the third derivative ($y'''(x)$). This type of ODEs commonly can be found in various fields of applied science and engineering such as beam theory [23], fluid dynamics [24], neural networks [25] and electric circuits [26]. Such equations can be written in the following form

$$y^{(iv)}(x) = f(x, y), \; x \ge x_0 \tag{3.1}$$

with initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y'_0, \quad y''(x_0) = y''_0, \quad y'''(x_0) = y'''_0,$$

where $f : \mathcal{R} \times \mathcal{R}^d \rightarrow \mathcal{R}^d$. Traditionally researchers and engineers used to solve fourth-order ODEs by transforming them into an equivalent first-order system of ODEs and then applying a suitable numerical method to this system (see [27, 28, 29]). However, this technique wasted a lot of computing time and human effort. Therefore, the direct numerical methods would be more efficient for solving fourth-order ODEs (3.1) (see [21, 22, 30]).

3.1. The direct numerical RKFD method

The general form of s-stage RKFD method for solving fourth-order ODEs (3.1) is given as follows:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2}\, y''_n + \frac{h^3}{6}\, y'''_n + h^4 \sum_{i=1}^{s} b_i k_i, \tag{3.2}$$

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{2}\, y'''_n + h^3 \sum_{i=1}^{s} b'_i k_i, \tag{3.3}$$

$$y''_{n+1} = y''_n + hy'''_n + h^2 \sum_{i=1}^{s} b''_i k_i, \tag{3.4}$$

$$y'''_{n+1} = y'''_n + h \sum_{i=1}^{s} b'''_i k_i, \tag{3.5}$$

where

$$k_1 = f(x_n, y_n), \tag{3.6}$$

$$k_i = f(x_n + c_i h,\, y_n + hc_i y'_n + \frac{h^2}{2} c_i^2 y''_n + \frac{h^3}{6} c_i^3 y'''_n + h^4 \sum_{j=1}^{s} a_{ij} k_j). \tag{3.7}$$

for $i = 2, 3, \ldots, s$.

The parameters $b_i, b'_i, b''_i, b'''_i, a_{ij}$ and c_i of the RKFD method are to be determined for $i = 1, 2, \ldots, s$; $j = 1, 2, \ldots, s$ and supposed to be real. The RKFD method is an explicit method if $a_{ij} = 0$ for $i \leq j$ and is an implicit method if $a_{ij} \neq 0$ for some i such that $i \leq j$. The RKFD method can be represented in Butcher tableau as follows (see Table 1):

Table 1: The Butcher tableau for RKFD method.

c	A
	b^T
	b'^T
	b''^T
	b'''^T

In [21], direct RKFD methods of orders four and five for solving special fourth-order ODEs are constructed. while the variable step size technique of RKFD pairs of orders 6(5) and 5(4) are developed in [22]. The RKFD methods of orders four and five can be written in Butcher tableau and given in Tables 2 and 3 respectively.

Table 2: The Butcher tableau for RKFD4 method of order four

$$
\begin{array}{c|ccc}
\frac{4}{11} & -\frac{1}{5} & & \\
\frac{17}{20} & \frac{19}{125} & \frac{19}{125} & \\
\hline
 & \frac{17}{200} & -\frac{7}{75} & \frac{1}{20} \\
 & \frac{1}{18} & \frac{209}{1926} & \frac{5}{1926} \\
 & \frac{47}{408} & \frac{847}{2568} & \frac{100}{1819} \\
 & \frac{47}{408} & \frac{1331}{2568} & \frac{2000}{5457}
\end{array}
$$

Table 3: The Butcher tableau for RKFD5 method of order five

$$
\begin{array}{c|ccc}
\frac{3}{5}+\frac{\sqrt{6}}{10} & \frac{4059}{187793} & & \\
\frac{3}{5}-\frac{\sqrt{6}}{10} & -\frac{1502}{532215} & \frac{1826}{569317} & \\
\hline
 & \frac{19}{1080} & \frac{13}{1080}-\frac{11\sqrt{6}}{2160} & \frac{13}{1080}+\frac{11\sqrt{6}}{2160} \\
 & \frac{1}{18} & \frac{1}{18}-\frac{\sqrt{6}}{48} & \frac{1}{18}+\frac{\sqrt{6}}{48} \\
 & \frac{1}{9} & \frac{7}{36}-\frac{\sqrt{6}}{18} & \frac{7}{36}+\frac{\sqrt{6}}{18} \\
 & \frac{1}{9} & \frac{4}{9}-\frac{\sqrt{6}}{36} & \frac{4}{9}+\frac{\sqrt{6}}{36}
\end{array}
$$

3.2. The Proposed Numerical Method

Here, we present a numerical technique to solve fourth-order PDEs of type IV based on the combination of the method of lines with RKFD method, the approach is as follows:

First, we consider the fourth-order PDE of type IV of the form

$$u_{tttt} = f(x, t, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}), \quad a \leq x \leq b \quad 0 < t < T, \tag{3.8}$$

with initial conditions

$$u(x, 0) = f_1(x), \quad u_x(x, 0) = f_2(x), \quad u_{xx}(x, 0) = f_3(x), \quad u_{xxx} = f_4(x), \tag{3.9}$$

and the boundary conditions,

$$u(a,t) = g_1(t), \quad u(b,t) = g_2(t). \tag{3.10}$$

We assume that the interval of the numerical solution in the directions of x and t be $[a, b]$ and $[0, T]$ respectively, with $h = \dfrac{b-a}{n}$ and $k = \dfrac{T}{m}$. Where n is the number of points in the direction of x on the interval $[a, b]$ and m is the number of points in the direction of t on the interval $[0, T]$. Problem (3.8) with initial conditions (3.9) and the boundary conditions (3.10) will be solved by combining the method of lines (MOL) and RKFD method by following these steps:

1. while $1 \le i \le m$ do the steps 2-6.

2. fix $x = x_i$ at the point (x, t) in PDE (3.8), then transform (3.8) to the following equation

$$u_i''''(t) = f\left(x, t, u(x,t), \frac{\partial u(x,y)}{\partial x}, \frac{\partial^2 u(x,y)}{\partial x^2}, \frac{\partial^3 u(x,y)}{\partial x^3}, \frac{\partial^4 u(x,y)}{\partial x^4}\right), \tag{3.11}$$

   where

$$u_i''''(t) = \frac{d^4 u(x,t)}{dx^4}, \quad i = 1, 2, \ldots, n-1. \tag{3.12}$$

3. Substituting finite difference formulas of the orders one, two, three and four into the derivatives on the right hand side of ODE (3.11), then a system of fourth-order ODEs is obtained

$$u_i''''(t) = f\left(x_i, t, u_{i-2}(t), u_{i-1}(t), u_i(t), u_{i+1}(t), u_{i+2}(t)\right), \tag{3.13}$$

   for $i = 1, 2, \ldots, n-1$, The central finite differences of orders one, two, three and four are presented as follows:

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i+1,j} - u_{i-1,j}}{h},$$

$$\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2},$$

$$\left.\frac{\partial^3 u(x,t)}{\partial x^3}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i+2,j} - 2u_{i+1,j} + 2u_{i-1,j} + u_{i-2,j}}{3h^3},$$

$$\left.\frac{\partial^3 u(x,t)}{\partial t^3}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i,j+2} - 2u_{i,j+1} + 2u_{i,j-1} + u_{i,j-2}}{3k^3},$$

$$\left.\frac{\partial^4 u(x,t)}{\partial x^4}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i+2,j} - 4u_{i+1,j} + 6u_{i,j} - 4u_{i-1,j} + u_{i-2,j}}{h^4},$$

$$\left.\frac{\partial^4 u(x,t)}{\partial t^4}\right|_{(x,t)=(x_i,t_j)} = \frac{u_{i,j+2} - 4u_{i,j+1} + 6u_{i,j} - 4u_{i,j-1} + u_{i,j-2}}{k^4}.$$

4. if $j = 1$, thus the initial conditions are

$$u_i(0) = f_1(x_i), \quad u_i'(0) = f_2(x_i), \quad u_i''(0) = f_3(x_i), \quad u_i'''(0) = f_4(x_i). \tag{3.14}$$

   if $2 \le j \le m$, thus the initial conditions are

$$\begin{aligned} u_i(t_{j-1}) &= u(x_i, t_{j-1}), \\ u_i'(t_{j-1}) &= \left.\frac{du(x, t_{j-1})}{dx}\right|_{x=x_i}, \\ u_i''(t_{j-1}) &= \left.\frac{d^2u(x, t_{j-1})}{dx^2}\right|_{x=x_i}, \\ u_i'''(t_{j-1}) &= \left.\frac{d^3u(x, t_{j-1})}{dx^3}\right|_{x=x_i}, \end{aligned} \tag{3.15}$$

5. set the boundary conditions as follows:

$$\begin{aligned} u_{0,j} &= u(a, t_j) = g_1(t_j), \\ u_{n,j} &= u(b, t_j) = g_2(t_j), \end{aligned} \tag{3.16}$$

6. solve the system of fourth-order ODEs (3.13) at $t = t_j$ with initial conditions (3.14) and (3.15) and boundary conditions (3.16) using the RKFD method.

4. Numerical Results

We applied our proposed method in the following study cases, which included fourth-order PDE.

Problem 4.1.

$$u_{tttt} = u, \quad 0 \le x \le 1, \quad t > 0,$$

with initial conditions,

$$u(x,0) = \mathrm{e}^{-x}, \quad u_x(x,0) = -\mathrm{e}^{-x}, \quad u_{xx}(x,0) = \mathrm{e}^{-x},$$
$$u_{xxx}(x,0) = -\mathrm{e}^{-x},$$

and boundary conditions,

$$u(a,t) = \mathrm{e}^{-a}\mathrm{e}^{-t}, \quad u(b,t) = \mathrm{e}^{-b}\mathrm{e}^{-t},$$

The exact solution is: $u(x,t) = \mathrm{e}^{-t}\mathrm{e}^{-x}$; (see Table 4 and Figure 1).

Table 4: The comparison between numerical and exact solutions for RKFD5 for Problem 1

Times (t_j)	x_i	Numerical solution	Exact solution	Absolute error
10^{-7}	0.1	9.048374180359595e-01	9.048373275522222e-01	9.048373728060000e-08
10^{-7}	0.2	8.187307530779818e-01	8.187306712049106e-01	8.187307121154674e-08
10^{-7}	0.3	7.408182206817179e-01	7.408181465998995e-01	7.408181834644978e-08
$50(10^{-7})$	0.4	6.703200460356393e-01	6.703166944437882e-01	3.351591851163960e-06
$50(10^{-7})$	0.5	6.065306597126334e-01	6.065276270669164e-01	3.032645716993798e-06
$50(10^{-7})$	0.6	5.488116360940265e-01	5.488088920427061e-01	2.744051320391350e-06
$100(10^{-7})$	0.7	4.965853037914095e-01	4.965803379632008e-01	4.965828208747247e-06
$100(10^{-7})$	0.8	4.493289641172216e-01	4.493244708500468e-01	4.493267174832116e-06
$100(10^{-7})$	0.9	4.065696597405991e-01	4.065655940643301e-01	4.065676268982799e-06

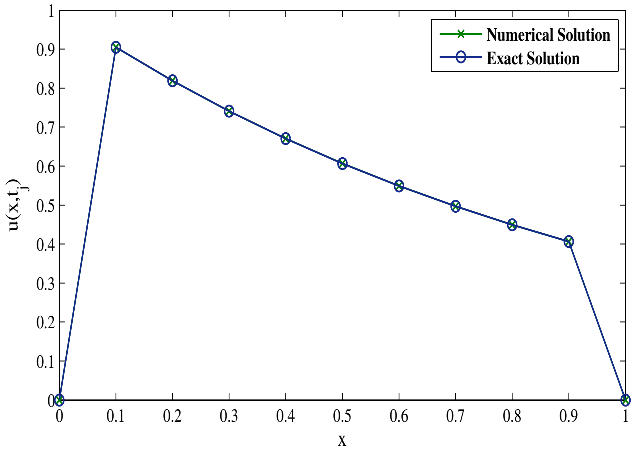

Figure 1: The comparison between numerical and exact solutions for Problem 1

Problem 4.2.

$$u_{tttt} = 8(u - u_{xx}), \quad 0 \le x \le 1, \quad t > 0,$$

with initial conditions,

$$u(x,0) = \cos(x), \quad u_x(x,0) = -\sin(x), \quad u_{xx}(x,0) = -\cos(x),$$
$$u_{xxx}(x,0) = \sin(x),$$

and boundary conditions,

$$u(a,t) = e^{-2t}\cos(a), \quad u(b,t) = e^{-2t}\cos(b),$$

The exact solution is: $u(x,t) = e^{-2t}\cos(x)$; (see Table 5 and Figure 2).

Table 5: The comparison between numerical and exact solutions for RKFD5 for Problem 2

Times (t_j)	x_i	Numerical solution	Exact solution	Absolute error
10^{-7}	0.1	9.950041652780257e-01	9.950039662772126e-01	1.990008131613763e-07
10^{-7}	0.2	9.800665778412416e-01	9.800663818279456e-01	1.960132960387995e-07
10^{-7}	0.3	9.553364891256060e-01	9.553362980583272e-01	1.910672787763801e-07
$50(10^{-7})$	0.4	9.210609940028851e-01	9.210517834389980e-01	9.210563887140921e-06
$50(10^{-7})$	0.5	8.775825618903728e-01	8.775737861086328e-01	8.775781739966959e-06
$50(10^{-7})$	0.6	8.253356149096783e-01	8.253273615947959e-01	8.253314882411544e-06
$100(10^{-7})$	0.7	7.648421872844885e-01	7.648268905937102e-01	1.529669077826590e-05
$100(10^{-7})$	0.8	6.967067093471654e-01	6.966927753523189e-01	1.393399484650448e-05
$100(10^{-7})$	0.9	6.216099682706644e-01	6.215975361956202e-01	1.243207504419974e-05

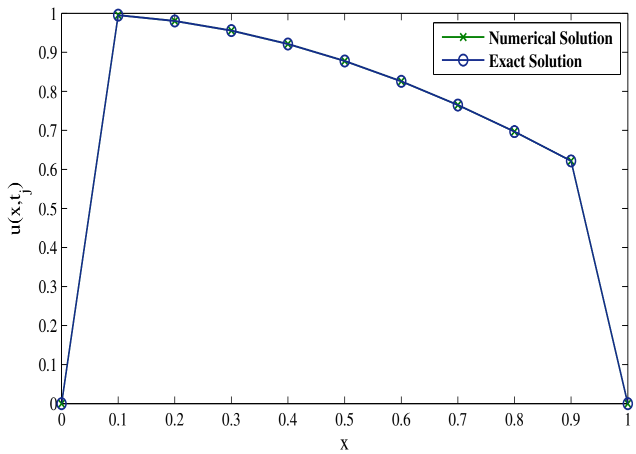

Figure 2: The comparison between numerical and exact solutions for Problem 2

Problem 4.3.

$$u_{tttt} = 4u + u_{xx} + 12\sin(2t), \quad -\pi \le x \le \pi, \quad t > 0,$$

with initial conditions,

$$u(x,0) = \cos(2x), \quad u_x(x,0) = -2\sin(2x),$$
$$u_{xx}(x,0) = -4\cos(2x), \quad u_{xxx}(x,0) = 8\cos(2x),$$

and boundary conditions,

$$u(a,t) = \cos(-2\pi) + \sin(2t), \quad u(b,t) = \cos(2\pi) + \sin(2t),$$

The exact solution is: $u(x,t) = \cos(2x) + \sin(2t)$; (see Table 6 and Figure 3).

Table 6: The comparison between numerical and exact solutions for RKFD5 for Problem 3

Times (t_j)	x_i	Numerical solution	Exact solution	Absolute error
10^{-7}	$-\frac{4}{5}\pi$	3.090169943749472e-01	3.090171943749472e-01	2.000000000057511e-07
10^{-7}	$-\frac{3}{5}\pi$	-8.090169943749475e-01	-8.090167943749475e-01	2.000000000057511e-07
10^{-7}	$-\frac{2}{5}\pi$	-8.090169943749473e-01	-8.090167943749473e-01	2.000000000057511e-07
$50(10^{-7})$	$-\frac{1}{5}\pi$	3.090169943749475e-01	3.090269943749473e-01	9.999999999787956e-06
$50(10^{-7})$	0	1.000000000000000e+00	1.000010000000000e+00	1.000000000006551e-05
$50(10^{-7})$	$\frac{1}{5}\pi$	3.090169943749475e-01	3.090269943749473e-01	9.999999999787956e-06
$100(10^{-7})$	$\frac{2}{5}\pi$	-8.090169943749473e-01	-8.089969943749487e-01	1.999999999868773e-05
$100(10^{-7})$	$\frac{3}{5}\pi$	-8.090169943749475e-01	-8.089969943749488e-01	1.999999999868773e-05
$100(10^{-7})$	$\frac{4}{5}\pi$	3.090169943749472e-01	3.090369943749459e-01	1.999999999874325e-05

5. Conclusion

In this paper, four types of fourth-order PDEs are categorized as types I, II, III and IV. These classes of fourth-order PDEs often arise in various fields of physics and engineering. Finding solutions for PDEs directly using classical methods can be complicated. Hence, we established a new numerical technique for solving fourth-order PDEs of type IV. The PDE of type IV is first converted to a system of fourth-order ODEs using the method of lines. The system of ODEs is then solved using direct Runge-Kutta type method, which we derived purposely to solve special fourth-order ODEs of the form $y^{(iv)} = f(x,y)$. The proposed direct technique requires less computational work; also, it has a good accuracy. To show the efficiency of the new technique, we solved various problems of fourth-order PDEs of type IV. From the numerical results, we observed that the method is viable for a class of PDEs and has a good agreement with the exact

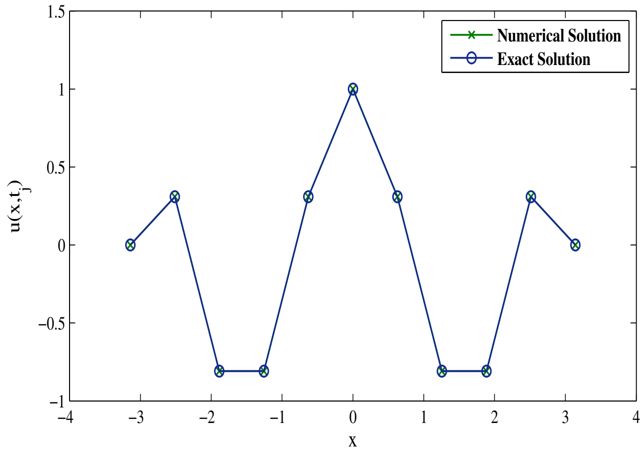

Figure 3: The comparison between numerical and exact solutions for Problem 3

solutions. The new method is efficient and provides encouraging results. Hence, we can conclude that RKFD method can be used as an alternative efficient method to solve PDE of type IV.

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