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Prediction of generalized order statistics from two independent sequences

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Abstract

The paper deals with various exact distribution-free prediction intervals for the future generalized order statistics (GOS) from a X-sequence of independent and identically (iid) continuous random variables, based on observed GOS from another independent Y -sequence of iid variables from the same distribution. The coverage probabilities of these intervals are exact expressions and are also free of the parent distribution F. Finally, a real life data set is used to illustrate the proposed procedures.

AMS subject classification: 11B39, 33C05, 11N13.
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1. Introduction

Let $X_{1,n,m,k}$, $X_{2,n,m,k}$, ..., $X_{n,n,m,k}$ be n GOS from an absolutely continuous cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. The idea of GOS has been introduced, see Kamps [8], as a unified approach to a variety of models of ordered random variables (r.v.'s) with different interpretations, such as ordinary order statistics, sequential order statistics, progressive type II censoring, record values, kth record values, Pfeifer's records. Review articles on GOS are found in Al-Hussaini [6], Cramer [7], Kamps and Cramer [9] and among others.

Prediction of future events on the basis of the past knowledge are of natural interest in this context. There are different types of predictions of future observation such as one-sample prediction, two-sample prediction and multi-sample prediction. We focus our attention here on the two-sample prediction. In this type, we use the first sequence to predict future observations from another independent sequence. In recent years, distribution-free confidence and prediction intervals (PIs) for order statistics (record values) from a future sequence Y of iid r.v.'s have been discussed extensively by many researches. In the context of records, Ahmadi and Balakrishnan [2, 3] proposed distribution-free confidence intervals for the quantiles of a distribution based on record ranges. Ahmadi and Balakrishnan [4] derived PIs for order statistics (or record) from the Y-sequence based on the record values (or order statistics) from the X-sequence. Raqab and Balakrishnan [14] derived PIs for records from the Y-sequence based on the record values from the X-sequence as well as outer and inner prediction intervals are derived based on X-records. Recently, Ahmadi and Balakrishnan [5] derived distribution-free PIs for order statistics based on record coverage, as well as PIs for future records based on observed order statistics are also obtained.

In this paper, we will construct PIs for future GOS from an independent Y-sequence based on observed GOS from an independent X-sequence. The rest of this paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we derive distribution-free PIs for GOs from a future sample based on observed GOS from the X-sequence. In Section 4, we show how the GOS spacings of the observed X-sequence can be used to construct upper and lower prediction limits for GOS spacings of the future Y-sequence. In Section 6, we make some remarks. A numerical example from an accelerated life-testing given by Nelson [11] is used for illustration in Section 7.

2. Preliminaries

Suppose we observed n GOS from X -sequence and we wish to predict the future u GOS from an independent Y-sequence. Then, the interval $(X_{i,n,m,k}, X_{j,n,m,k})$, $1 \le i \le j \le n$ is logical (termed the upper GOS coverage) to predict $Y_{r,u,\mu,q}$, $1 \le r \le u$. Now, let us denote the lth GOS from the X -sequence by $X_{l,n,m,k}$. When $m_1 = m_2 = \cdots = m_{l-1} = m$, the marginal pdf of the lth GOS $X_{l,n,m,k}$ is given by (see Kamps [8], p. 64)

$$f_{l,n,m,k}(x) = \frac{c_{l-1}}{(l-1)!} \left(\bar{F}(x)\right)^{\gamma_l - 1} g_m^{l-1}\left(F(x)\right) f(x), \tag{2.1}$$

where $c_{l-1} = \prod_{i=1}^{l} \gamma_i$, $\gamma_i = k + (n-i)(m+1)$, $\bar{F}(x) = 1 - F(x)$ and

$$g_m(x) = \begin{cases} \frac{1}{m+1}\left(1-(1-x)^{m+1}\right), & m \neq -1, \\ -\log(1-x), & m = -1, \end{cases} \quad x \in [0, 1).$$

Kamps and Cramer [9] have introduced the marginal pdf of the lth GOS $X_{l,n,\tilde{m},k}$, for $\gamma_i \neq \gamma_j$, $i \neq j$, $\forall\, i, j \in \{1, \ldots, n\}$, in the following form

$$f_{l,n,\tilde{m},k}(x) = c_{l-1} f(x) \sum_{i=1}^{l} a_i(l) \left(\bar{F}(x)\right)^{\gamma_i - 1}, \tag{2.2}$$

where $\tilde{m} = (m_1, \ldots, m_{n-1}) \in \mathbf{R}^{n-1}$, $\gamma_l = k + n - l + M_l \geq 1, \forall\, l \in \{1, \ldots, n-1\}$,

$$M_l = \sum_{i=l}^{n-1} m_i, \quad \gamma_n = k \quad \text{and} \quad a_i(l) = \prod_{j=1, j\neq i}^{l} \frac{1}{\gamma_j - \gamma_i}. \tag{2.3}$$

3. PIs for a future generalized order statistics

In the following theorem, under the assumption that $m_1 = m_2 = \cdots = m_{n-1} = m$ and $\mu_1 = \mu_2 = \cdots = \mu_{u-1} = \mu$, we obtain two-sided distribution-free PIs for $Y_{r,u,\mu,q}$ with coverage probabilities being free of the parent distribution F. Then the survival function of $X_{l,n,m,k}$ is given by (see Kamps [8], p. 73)

$$\bar{F}_{l,n,m,k}(x) = c_{l-1} \left(\bar{F}(x)\right)^{\gamma_l} \sum_{\ell=0}^{l-1} \frac{g_m^\ell\,(F(x))}{\ell! c_{l-\ell-1}}. \tag{3.1}$$

Theorem 3.1. Let $X_{1,n,m,k} \leq X_{2,n,m,k} \leq \cdots \leq X_{n,n,m,k}$ be GOS sample with size n based on continuous distribution function $F(x)$. Moreover, let $Y_{1,u,\mu,q} \leq Y_{2,u,\mu,q} \leq \cdots \leq Y_{u,u,\mu,q}$ be GOS from a future random sample of size u from the same cdf $F(x)$. If $X_{j,n,m,k}$ is the jth GOS, then $(X_{i,n,m,k}, X_{j,n,m,k})$, $1 \leq i < j \leq n$, is a two-sided PI for $Y_{r,u,\mu,q}$, $1 \leq r \leq u$, $\mu > -1$, whose coverage probability is free of F and is given by

$$\pi_1\,(i, j; r) = \frac{d_{r-1}}{(\mu+1)^{r-1}} \sum_{\nu=0}^{r-1} \left(\sum_{\ell=i}^{j-1} \phi_{\ell,\nu}(j) + \sum_{\ell=0}^{i-1} \left(\phi_{\ell,\nu}(j) - \phi_{\ell,\nu}(i)\right) \right), \tag{3.2}$$

where

$$\phi_{\ell,\nu}(j) = \frac{\bar{b}_\nu(r)}{\gamma_{j+1}} \prod_{s=0}^{\ell} \frac{\gamma_{j-s+1}}{\gamma_{j-s} + \delta_{r-\nu}} \quad \text{and} \quad \bar{b}_\nu(r) = \frac{(-1)^\nu}{(r-\nu-1)!\nu!}. \tag{3.3}$$

Proof. Under the assumption that $X_{j,n,m,k}$, $1 \leq j \leq n$ are continuous r.v.'s, we can write

$$P\left(X_{i,n,m,k} \leq y \leq X_{j,n,m,k}\right) = P\left(X_{i,n,m,k} \leq y\right) - P\left(X_{j,n,m,k} \leq y\right). \tag{3.4}$$

From (3.1) and (3.4), we readily obtain

$$\begin{aligned} P\left(X_{i,n,m,k} \leq y \leq X_{j,n,m,k}\right) &= \left(\bar{F}(y)\right)^{\gamma_j} \sum_{\ell=i}^{j-1} \frac{\eta_{\ell,j}}{\ell!} g_m^\ell\,(F(y)) + \sum_{\ell=0}^{i-1} \frac{1}{\ell!} g_m^\ell\,(F(y)) \\ &\times \left(\eta_{\ell,j} \left(\bar{F}(y)\right)^{\gamma_j} - \eta_{\ell,i} \left(\bar{F}(y)\right)^{\gamma_i}\right), \end{aligned} \tag{3.5}$$

where $\eta_{\ell,j} = \dfrac{c_{j-1}}{c_{j-\ell-1}} = \prod_{v=1}^{\ell} \gamma_{j-v+1}$.

Using the conditioning argument, we then have

$$\begin{aligned} P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q} \le X_{j,n,m,k}\right) &= \int_{-\infty}^{\infty} P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q} \le X_{j,n,m,k} | Y_{r,u,\mu,q} = y\right) dF_{r,u,\mu,q}(y) \\ &= \int_{-\infty}^{\infty} P\left(X_{i,n,m,k} \le y \le X_{j,n,m,k}\right) dF_{r,u,\mu,q}(y). \end{aligned} \tag{3.6}$$

Upon using (2.1) and (3.5) in (3.6), we obtain

$$\begin{aligned} P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q} \le X_{j,n,m,k}\right) &= \frac{d_{r-1}}{(r-1)!}\left(\sum_{\ell=i}^{j-1} \frac{\eta_{\ell,j}}{\ell!} I_\ell(j) + \sum_{\ell=0}^{i-1} \frac{1}{\ell!}\left(\eta_{\ell,j} I_\ell(j) - \eta_{\ell,i} I_\ell(i)\right)\right), \end{aligned} \tag{3.7}$$

where

$$I_\ell(j) = \int_{-\infty}^{\infty} \left(\bar{F}(y)\right)^{\gamma_j+\delta_r-1} g_m^\ell\left(F(y)\right) g_\mu^{r-1}\left(F(y)\right) f(y)dy. \tag{3.8}$$

Making the transformation $t = \bar{F}(y)$, equation (3.8) reduces to

$$\begin{aligned} I_\ell(j) &= \int_0^1 t^{\gamma_j+\delta_r-1}\left(\frac{1-t^{m+1}}{m+1}\right)^\ell \left(\frac{1-t^{\mu+1}}{\mu+1}\right)^{r-1} dt \\ &= \frac{(\mu+1)^{-(r-1)}}{(m+1)^{\ell+1}} \sum_{\nu=0}^{r-1} b_\nu(r) B\left(\ell+1, \frac{\nu(\mu+1)+\gamma_j+\delta_r}{m+1}\right). \end{aligned} \tag{3.9}$$

Using $\nu(\mu+1)+\delta_r = \delta_{r-\nu}$, we have

$$\begin{aligned} I_\ell(j) &= \frac{(\mu+1)^{-(r-1)}}{(m+1)^{\ell+1}} \sum_{\nu=0}^{r-1} b_\nu(r) B\left(\ell+1, \frac{\gamma_j+\delta_{r-\nu}}{m+1}\right) \\ &= \frac{(\mu+1)^{-(r-1)}}{(m+1)^{\ell+1}} \sum_{\nu=0}^{r-1} b_\nu(r) \frac{\Gamma(\ell+1)\,\Gamma\left(\frac{\gamma_j+\delta_{r-\nu}}{m+1}\right)}{\Gamma\left(\frac{\gamma_j+\delta_{r-\nu}}{m+1}+\ell+1\right)} \\ &= \frac{(\mu+1)^{-(r-1)}}{(m+1)^{\ell+1}} \sum_{\nu=0}^{r-1} \frac{b_\nu(r)\Gamma(\ell+1)}{\left(\frac{\gamma_j+\delta_{r-\nu}}{m+1}+\ell\right)\left(\frac{\gamma_j+\delta_{r-\nu}}{m+1}+\ell-1\right)\cdots\left(\frac{\gamma_j+\delta_{r-\nu}}{m+1}\right)} \\ &= \frac{\Gamma(\ell+1)}{(\mu+1)^{r-1}} \sum_{\nu=0}^{r-1} b_\nu(r) \prod_{s=0}^{\ell} \frac{1}{\gamma_{j-s}+\delta_{r-\nu}}, \quad \mu > -1. \end{aligned} \tag{3.10}$$

It can be shown easily that

$$\begin{aligned}\frac{d_{r-1}}{(r-1)!}\eta_\ell(j)I_\ell(j) &= \frac{d_{r-1}c_{j-1}\Gamma\,(\ell+1)}{(r-1)!(\ell!c_{j-\ell-1})\,(\mu+1)^{r-1}}\sum_{\nu=0}^{r-1}b_\nu(r)\prod_{s=0}^{\ell}\frac{1}{\gamma_{j-s}+\delta_{r-\nu}}\\ &= \frac{d_{r-1}}{(\mu+1)^{r-1}}\sum_{\nu=0}^{r-1}\phi_{\ell,\nu}(j), \qquad (3.11)\end{aligned}$$

where

$$\phi_{\ell,\nu}(j)=\frac{\bar{b}_\nu(r)}{\gamma_{j+1}}\prod_{s=0}^{\ell}\frac{\gamma_{j-s+1}}{\gamma_{j-s}+\delta_{r-\nu}} \quad \text{and} \quad \bar{b}_\nu(r)=\frac{(-1)^\nu}{(r-\nu-1)!\nu!}. \qquad (3.12)$$

If we take $q\in\mathbf{N}$ for $\mu_1=\mu_2=\cdots=\mu_{u-1}=\mu=-1$, we obtain prediction of qth record based on GOS. With $q\in\mathbf{N}$ for $\mu=-1$, taking limit $\mu\to-1$, we obtain from (3.8),

$$\begin{aligned}I_\ell(j) &= \int_0^1 t^{\gamma_j+q-1}\left(\frac{1-t^{m+1}}{m+1}\right)^{\ell}(-\ln t)^{r-1}\,dt\\ &= \frac{1}{(m+1)^\ell}\sum_{\nu=0}^{\ell}b_\nu(\ell+1)\int_0^1 t^{\gamma_{j-\nu}+q-1}\,(-\ln t)^{r-1}\,dt\\ &= \frac{\Gamma\,(r)}{(m+1)^\ell}\sum_{\nu=0}^{\ell}\frac{b_\nu(\ell+1)}{\left(\gamma_{j-\nu}+q\right)^r}, \quad m>-1. \qquad (3.13)\end{aligned}$$

In this case, it is easy to verify that $d_{r-1}=q^r$ and

$$\frac{d_{r-1}}{(r-1)!}\eta_\ell(j)I_\ell(j)=\frac{q^r}{(m+1)^\ell}\sum_{\nu=0}^{\ell}\xi_{\ell,\nu}(j), \qquad (3.14)$$

where

$$\xi_{\ell,\nu}(j)=\frac{\bar{b}_\nu(\ell+1)}{\left(\gamma_{j-\nu}+q\right)^r}\prod_{s=1}^{\ell}\gamma_{j-s+1}.$$

Upon substituting (3.14) in (3.7), we get

$$\pi_1\,(i,j;r)=q^r\sum_{\ell=i}^{j-1}\sum_{\nu=0}^{\ell}\frac{\xi_{\ell,\nu}(j)}{(m+1)^\ell}+q^r\sum_{\ell=0}^{i-1}\sum_{\nu=0}^{\ell}\left(\frac{\xi_{\ell,\nu}(j)-\xi_{\ell,\nu}(i)}{(m+1)^\ell}\right). \qquad (3.15)$$

If we take $k\in\mathbf{N}$ and $q\in\mathbf{N}$ for $m_1=\cdots=m_{n-1}=m=-1$ and $\mu_1=\cdots=\mu_{u-1}=\mu=-1$, respectively; we obtain prediction of qth record based on kth record. With $k,q\in\mathbf{N}$, taking limit $(m,\mu)\to-1$, we obtain from (3.8),

$$I_\ell(j)=\int_0^1 t^{k+q-1}\,(-\ln t)^{\ell+r-1}\,dt=\frac{\Gamma(\ell+r)}{(k+q)^{\ell+r}}. \qquad (3.16)$$

From (3.16) and (3.7), we readily obtain

$$\pi_1\left(i, j ; r\right)=q^r \sum_{\ell=i}^{j-1} \frac{k^{\ell}}{(k+q)^{\ell+r}}\binom{\ell+r-1}{\ell}=\left(\frac{q}{k}\right)^r \frac{k}{k+q} \sum_{\ell=i}^{j-1} P\left(W_r=\ell\right), \quad (3.17)$$

where W_r is a binomial random variable with parameters $(\ell+r-1, k/(k+q))$. Hence, the theorem is proved. ■

In the following theorem, we assume that $\gamma_i \neq \gamma_j$ and $\delta_i \neq \delta_j$, $i \neq j$, $i, j = 1, 2, \cdots, n-1(u-1)$. Then the cdf of $X_{l,n,\tilde{m},k}$ is given by

$$\bar{F}_{l,n,\tilde{m},k}(x)=c_{l-1} \sum_{i=1}^{l} \frac{a_i(l)}{\gamma_i}\left(\bar{F}(x)\right)^{\gamma_i}, \quad (3.18)$$

where $1 \leq l \leq n$ and $\tilde{m}=(m_1, \ldots, m_{n-1})$, see Kamps and Cramer [9].

Theorem 3.2. Under the assumption of Theorem 3.1, $(X_{i,n,\tilde{m},k}, X_{j,n,\tilde{m},k})$, $1 \leq i \leq j \leq n$, is a two-sided PI for $Y_{r,u,\tilde{\mu},q}$, $1 \leq r \leq u$, whose coverage probability is free of F and is given by

$$\pi_2\left(i, j ; r\right)=\sum_{\nu=1}^{r} \delta_\nu a_\nu^{(\delta)}(r)\left(\sum_{\ell=i+1}^{j} \frac{a_\ell^{(\gamma)}(j)}{\gamma_\ell+\delta_\nu}+\sum_{\ell=1}^{i} \frac{a_\ell^{(\gamma)}(j)-a_\ell^{(\gamma)}(i)}{\gamma_\ell+\delta_\nu}\right), \quad (3.19)$$

where

$$a_\nu^{(\delta)}(r)=\prod_{s=1, s\neq \nu}^{r} \frac{\delta_s}{\delta_s-\delta_\nu}. \quad (3.20)$$

Proof. It is known that when Y is continuous, we have

$$P\left(X_{i,n,\tilde{m},k} \leq y \leq X_{j,n,\tilde{m},k}\right)=P\left(X_{i,n,\tilde{m},k} \leq y\right)-P\left(X_{j,n,\tilde{m},k} \leq y\right) \quad (3.21)$$

By Equations (3.18) and (3.21) can be expressed as

$$\begin{aligned} P\left(X_{i,n,\tilde{m},k} \leq y \leq X_{j,n,\tilde{m},k}\right) = \; & c_{j-1} \sum_{\ell=i+1}^{j} \frac{a_\ell(j)}{\gamma_\ell}\left(\bar{F}(y)\right)^{\gamma_\ell} \\ & +\sum_{\ell=1}^{i}\left(\frac{c_{j-1}}{\gamma_\ell} a_\ell(j)-\frac{c_{i-1}}{\gamma_\ell} a_\ell(i)\right)\left(\bar{F}(y)\right)^{\gamma_\ell}. \end{aligned} \quad (3.22)$$

Then we find

$$
\begin{aligned}
&P\left(X_{i,n,\tilde{m},k} \le Y_{r,u,\tilde{\mu},q} \le X_{j,n,\tilde{m},k}\right) \\
&= d_{r-1}c_{j-1} \sum_{\ell=i+1}^{j} \frac{a_\ell(j)}{\gamma_\ell} \sum_{\nu=1}^{r} \bar{a}_\nu(r) \int_{-\infty}^{\infty} \left(\bar{F}(y)\right)^{\gamma_\ell+\delta_\nu-1} f(y)dy \\
&\quad + d_{r-1} \sum_{\ell=1}^{i} \left(\frac{c_{j-1}}{\gamma_\ell} a_\ell(j) - \frac{c_{i-1}}{\gamma_\ell} a_\ell(i) \right) \sum_{\nu=1}^{r} \bar{a}_\nu(r) \int_{-\infty}^{\infty} \left(\bar{F}(y)\right)^{\gamma_\ell+\delta_\nu-1} f(y)dy \\
&= d_{r-1}c_{j-1} \sum_{\ell=i+1}^{j} \sum_{\nu=1}^{r} \frac{a_\ell(j)\bar{a}_\nu(r)}{\gamma_\ell\left(\gamma_\ell+\delta_\nu\right)} \\
&\quad + d_{r-1} \sum_{\ell=1}^{i} \sum_{\nu=1}^{r} \frac{c_{j-1}a_\ell(j) - c_{i-1}a_\ell(i)}{\gamma_\ell\left(\gamma_\ell+\delta_\nu\right)} \bar{a}_\nu(r), \qquad (3.23)
\end{aligned}
$$

where $\tilde{\mu} = (\mu_1, \ldots, \mu_{u-1})$,

$$
d_{r-1} = \prod_{s=1}^{r} \delta_s \quad \text{and} \quad \bar{a}_\nu(r) = \prod_{s=1, s\neq\nu}^{r} \frac{1}{\delta_s - \delta_\nu}.
$$

It is easy to verify that

$$
\begin{aligned}
d_{r-1}c_{j-1}a_\ell(j)\bar{a}_\nu(r) &= \left(\prod_{s=1}^{r}\delta_s\right)\left(\prod_{s=1}^{j}\gamma_s\right)\left(\prod_{s=1,s\neq\nu}^{r}\frac{1}{\gamma_s-\gamma_\nu}\right)\left(\prod_{s=1,s\neq\ell}^{j}\frac{1}{\delta_s-\delta_\ell}\right) \\
&= \gamma_\ell\delta_\nu\left(\prod_{s=1,s\neq\nu}^{r}\frac{\delta_s}{\delta_s-\delta_\nu}\right)\left(\prod_{s=1,s\neq\ell}^{j}\frac{\gamma_s}{\gamma_s-\gamma_\ell}\right) \\
&= \gamma_\ell\delta_\nu a_\nu^{(\delta)}(r)a_\ell^{(\gamma)}(j), \quad \text{say}, \qquad (3.24)
\end{aligned}
$$

where

$$
a_\nu^{(\delta)}(r) = \left(\prod_{s=1,s\neq\nu}^{r}\frac{\delta_s}{\delta_s-\delta_\nu}\right).
$$

Substituting (3.24) into (3.23), the result in (3.19) follows. ■

By choosing suitable parameters in the model of GOS, several models of ordered r.v.'s, such as ordinary order statistics, record values, sequential order statistics and progressively type II censoring are seen to be particular cases, see Table 2.

To illustrate the procedures in this section, we consider four simulated samples from a sequential 2-out-of-5 system based on components, using the algorithms of Aboeleneen [1]. The influence of failures on remaining components in the system is assumed to be described by the increasing sequence of parameters $\alpha_{i+1} = 1 + i/10$,

Table 1: The values of $\pi_2\,(i,\,j;r)$ for $m_i = (n-i+1)\alpha_i - (n-i)\alpha_{i+1} - 1$, $k = \alpha_n$, $\mu_1 = \cdots = \mu_{u-1} = 0$, $q = 1$ and some selected choices of n, u, i, j and r.

				j						
n	u	r	i	4	5	6	7	8	9	10
10	5	1	1	0.4787	0.5488	0.5960	0.6270	0.6466	0.6583	0.6644
			2	0.2550	0.3251	0.3723	0.4033	0.4229	0.4346	0.4406
			3	0.1033	0.1734	0.2206	0.2516	0.2712	0.2829	0.2890
		3	1	0.2094	0.3133	0.4266	0.5450	0.6647	0.7815	0.8897
			2	0.1614	0.2654	0.3787	0.4971	0.6168	0.7335	0.8418
			3	0.0901	0.1941	0.3073	0.4258	0.5454	0.6622	0.7704
		5	1	0.0112	0.0230	0.0428	0.0748	0.1273	0.2171	0.3942
			2	0.0099	0.0217	0.0414	0.0735	0.1259	0.2158	0.3929
			3	0.0066	0.0184	0.0381	0.0702	0.1226	0.2125	0.3896
	10	1	1	0.4420	0.4735	0.4886	0.4956	0.4985	0.4996	0.4999
			2	0.1907	0.2222	0.2374	0.2443	0.2473	0.2484	0.2487
			3	0.0638	0.0953	0.1105	0.1174	0.1203	0.1214	0.1218
		2	1	0.5702	0.6563	0.7084	0.7378	0.7530	0.7600	0.7626
			2	0.3252	0.4113	0.4634	0.4929	0.5081	0.5150	0.5176
			3	0.1334	0.2196	0.2716	0.3011	0.3163	0.3232	0.3258
		3	1	0.5146	0.6479	0.7470	0.8150	0.8575	0.8810	0.8916
			2	0.3464	0.4796	0.5788	0.6467	0.6862	0.7127	0.7233
			3	0.1645	0.2978	0.3969	0.4649	0.5074	0.5309	0.5415
		7	1	0.0562	0.1109	0.1930	0.3064	0.4527	0.6292	0.8238
			2	0.0496	0.1043	0.1864	0.2998	0.4461	0.6226	0.8172
			3	0.0328	0.0875	0.1696	0.2830	0.4293	0.6058	0.8004
30	10	1	1	0.4075	0.4813	0.5379	0.5817	0.6160	0.6430	0.6645
			2	0.2285	0.3023	0.3589	0.4027	0.4370	0.4640	0.4855
			3	0.0976	0.1714	0.2279	0.2718	0.3060	0.3331	0.3545
		3	1	0.1070	0.1573	0.2108	0.2657	0.3208	0.3752	0.4282
			2	0.0815	0.1318	0.1853	0.2402	0.2953	0.3496	0.4027
			3	0.0448	0.0951	0.1486	0.2036	0.2587	0.3130	0.3660
		5	1	0.0088	0.0157	0.0250	0.0370	0.0516	0.0689	0.0902
			2	0.0074	0.0144	0.0237	0.0357	0.0502	0.0676	0.0895
			3	0.0047	0.0116	0.0209	0.0329	0.0475	0.0648	0.0860
	20	1	1	0.4527	0.5050	0.5380	0.5592	0.5730	0.5821	0.5881
			2	0.2215	0.2737	0.3068	0.3280	0.3418	0.3509	0.3569
			3	0.0839	0.1361	0.1692	0.1904	0.2042	0.2133	0.2193
		3	1	0.3319	0.4398	0.5348	0.6157	0.6832	0.7385	0.7839
			2	0.2329	0.3407	0.4357	0.5166	0.5841	0.6394	0.6844
			3	0.1167	0.2246	0.3195	0.4004	0.4679	0.5233	0.5683
		5	1	0.1048	0.1647	0.2320	0.3034	0.3761	0.4519	0.5724
			2	0.0844	0.1443	0.2116	0.2830	0.3558	0.4307	0.5399
			3	0.0491	0.1090	0.1763	0.2477	0.3204	0.3977	0.5060

Table 2: Models of ordered r.v.'s and their correspondence, see Cramer and Kamps [9].

Models	$\gamma_n = k$	γ_r	$m_r \ (1 \le r < n)$
ordinary order statistics	1	$n - r + 1$	0
record values	1	1	-1
sequential order statistics	α_n	$(n - r + 1)\alpha_r$	$(n - r + 1)\alpha_r - (n - r)\alpha_{r+1} - 1$
progressive type II censoring	$R_n + 1$	$N - r + 1 - \sum_{i=1}^{r-1} R_i$	R_r

$i = 0, 1, \ldots, n-1$. Furthermore, suppose we have a future sample of u order statistics from an independent Y-sequence based on observed sequential order statistics. Table 1 gives the values of $\pi_2\,(i,\, j;\, r)$ for $u = 5$, 10 and some selected values of i and j.

4. PIs for generalized order statistics spacings

In this section, we use GOS spacings $D_{i,j,n,m,k} = X_{j,n,m,k} - X_{i,n,m,k}$, $i < j$ from the X-sequence to construct distribution-free upper and lower prediction limits for GOS spacing $D^*_{r,s,u,m\mu,q} = Y_{s,u,m\mu,q} - Y_{r,u,m\mu,q}$, $r < s$ from the Y-sequence. It is easy to show that

$$\begin{aligned} P\left(D_{i,j,n,m,k} \ge D^*_{r,s,u,\mu,q}\right) &= P\left(X_{j,n,m,k} - X_{i,n,m,k} \ge Y_{s,u,\mu,q} - Y_{r,u,\mu,q}\right) \\ &\ge P\left(X_{j,n,m,k} \ge Y_{s,u,\mu,q},\, X_{i,n,m,k} \le Y_{r,u,\mu,q}\right) \\ &\ge P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q}\right) - P\left(X_{j,n,m,k} \le Y_{s,u,\mu,q}\right). \end{aligned} \tag{4.1}$$

From equations (2.1) and (3.1), using the conditioning argument, we have

$$\begin{aligned} P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q}\right) &= \int_{-\infty}^{\infty} P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q} | Y_{r,u,\mu,q} = y_r\right) dF_{r,u,\mu,q}(y) \\ &= \frac{d_{r-1}}{(r-1)!} \int_{-\infty}^{\infty} \left\{1 - c_{i-1}\,(1 - F(y))^{\gamma_i} \sum_{\ell=0}^{i-1} \frac{g_m^{\ell}(F(y))}{\ell! c_{i-\ell-1}}\right\} \\ &\qquad \times (1 - F(y))^{\delta_r - 1}\, g_\mu^{r-1}(F(y)) f(y) dy \\ &= 1 - \frac{d_{r-1}}{(r-1)!} \sum_{\ell=0}^{i-1} \frac{\eta_{\ell,i}}{\ell!} I_\ell(i), \end{aligned} \tag{4.2}$$

where $I_\ell(i)$ is given by (3.8). Hence, from (3.11), we get

$$P\left(X_{i,n,m,k} \le Y_{r,u,\mu,q}\right) = 1 - \frac{d_{r-1}}{(\mu+1)^r}\sum_{\ell=0}^{i-1}\sum_{\nu=0}^{r-1}\phi_{\ell,\nu}(i) = 1 - \frac{\psi_i(r)}{(\mu+1)^r}, \quad \text{say,} \tag{4.3}$$

where $\phi_{\ell,\nu}(i)$ is given by (3.12) and

$$\psi_i(r) = d_{r-1}\sum_{\ell=0}^{i-1}\sum_{\nu=0}^{r-1}\phi_{\ell,\nu}(i).$$

Thus, from (4.3) into (4.1), with $\mu > -1$, it is easy to verify that

$$P\left(D_{i,j,n,m,k} \ge D^*_{r,s,u,\mu,q}\right) \ge \frac{\psi_j(s)}{(\mu+1)^s} - \frac{\psi_i(r)}{(\mu+1)^r} = \pi_3\left(i,j;r,s\right). \tag{4.4}$$

With $q = 1$ and $\mu_1 = \cdots = \mu_{u-1} = \mu = -1$, we obtain

$$P\left(D_{i,j,n,m,k} \ge Y_{U(s)} - Y_{U(r)}\right) \ge \sum_{\ell=0}^{j-1}\sum_{\nu=0}^{\ell}\frac{\phi_{\ell,\nu}(j,s)}{(m+1)^\ell} - \sum_{\ell=0}^{i-1}\sum_{\nu=0}^{\ell}\frac{\phi_{\ell,\nu}(i,r)}{(m+1)^\ell}, \tag{4.5}$$

where $\phi_{\ell,\nu}(j,r) \equiv \phi_{\ell,\nu}(j)$ and $\phi_{\ell,\nu}(j)$ is given by (3.12). Proceeding similarly, if we take $m = \mu = -1$ and $k, q \in \mathbf{N}$, then from (3.8),

$$P\left(X_{U_j^{(k)}} - X_{U_i^{(k)}} \ge Y_{U_s^{(k)}} - Y_{U_r^{(k)}}\right)$$
$$\ge \left(\frac{q}{k}\right)^s \frac{k}{k+q}\sum_{\ell=0}^{j-1}P\left(W_s = \ell\right) - \left(\frac{q}{k}\right)^r \frac{k}{k+q}\sum_{\ell=0}^{i-1}P\left(W_r = \ell\right).$$

Hence, the lower and upper prediction limits for $D^*_{r,s,u,\mu,q}$ with prediction coefficient $\ge 1-\alpha$ are, respectively, $D_{i,j,n,m,k}$ and $D_{k_1,k_2,n,m,k}$.

Analogous to the results presented in this section, when $\gamma_i \ne \gamma_j$ and $\delta_i \ne \delta_j$, $i \ne j$, we obtain the following result

$$\begin{aligned} P\left(D_{i,j,n,\tilde{m},k} \ge D^*_{r,s,u,\tilde{\mu},q}\right) &\ge \sum_{\ell=1}^{j}\sum_{\nu=1}^{s}\frac{\delta_\nu a_\ell^{(\gamma)}(j)a_\nu^{(\delta)}(s)}{\gamma_\ell+\delta_\nu} - \sum_{\ell=1}^{i}\sum_{\nu=1}^{r}\frac{\delta_\nu a_\ell^{(\gamma)}(i)a_\nu^{(\delta)}(r)}{\gamma_\ell+\delta_\nu} \\ &= \pi_4\left(i,j;r,s\right). \end{aligned}$$

5. Some remarks

By suitably choosing $m_1,\ldots,m_{n-1}$, k, $\mu_1,\ldots,\mu_{u-1}$ and q, we can easily get the following results:

1. If we take $m_1 = \cdots = m_{n-1} = m = -1$ and $k = 1$, then we obtain from (3.2),

$$\pi_1(i, j; r) = \frac{\bar{c}_{r-1}}{\Gamma(r)(\mu+1)^{r-1}} \sum_{\ell=i}^{j-1} \sum_{\nu=0}^{r-1} \binom{r-1}{\nu} \frac{(-1)^\nu}{(1+\delta_{r-\nu})^{\ell+1}}, \tag{5.1}$$

   as well as if we put in above equation ($\mu = 0$ and $q = 1$) we get the result of Theorem 3.1 in Ahmadi and Balakrishnan [4].

2. In particular, with $k = q = 1$, equation (3.17) reduces to

$$\pi_1(i, j; r) = \sum_{\ell=i}^{j-1} \frac{1}{2^{\ell+r}} \binom{\ell+r-1}{\ell} = \frac{1}{2} \sum_{\ell=i}^{j-1} P(W = \ell), \tag{5.2}$$

   which is the same result of Theorem 1 in Raqab and Balakrishnan [14], where W is a binomial random variable with parameters $(\ell + r - 1, 1/2)$.

3. If we put $k = 1$ and $m_1 = \cdots = m_{n-1} = m = -1$, then from (4.5),

$$P\left(X_{U(j)} - X_{U(i)} \geq Y_{U(s)} - Y_{U(r)}\right) \geq \sum_{\ell=0}^{j-1} \frac{1}{2^{s+\ell}} \binom{s+\ell-1}{\ell} - \sum_{\ell=0}^{i-1} \frac{1}{2^{r+\ell}} \binom{r+\ell-1}{\ell}, \tag{5.3}$$

   which is same relation of Raqab and Balakrishnan [14] [see $\alpha_5(r, s; m, n)$, p. 1960].

4. GOS spacings $D_{i,j,n,m,k}$ represent the width of PIs. So we can be considered as an optimality criterion while comparing different intervals, evaluation of $E\left(D_{i,j,n,m,k}\right)$ is of natural interest, see Raqab [12, 13].

6. Illustrative example (real life data)

In this section, we consider the real life data set which given in Nelson [11] to illustrate the methods proposed in the previous sections. These data which was also used in Lawless ([10], p. 185), concerning the data on time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are contained in the sample (∗)

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Also, from the data (∗), we observe the following seven upper record values (∗∗):

0.96, 4.15, 8.01, 31.75, 33.91, 36.71, 72.89

Table 3: PIs for future order statistics based on observed records.

n	r	(i, j)	(U_i, U_j)	π_1	n	r	(i, j)	(U_i, U_j)	π_1
10	9	(1, 7)	(0.96, 72.89)	0.8068	100	86	(1, 5)	(0.96, 33.91)	0.8018
	10	(1, 7)	(0.96, 72.89)	0.8462		88	(1, 5)	(0.96, 33.91)	0.8076
20	17	(1, 7)	(0.96, 72.89)	0.8048		90	(1, 5)	(0.96, 33.91)	0.8085
	18	(1, 6)	(0.96, 36.71)	0.8269		92	(1, 5)	(0.96, 33.91)	0.8019
		(1, 7)	(0.96, 72.89)	0.8460		82	(1, 6)	(0.96, 36.71)	0.8031
	19	(1, 6)	(0.96, 36.71)	0.8377		84	(1, 6)	(0.96, 36.71)	0.8199
		(1, 7)	(0.96, 72.89)	0.8750		86	(1, 6)	(0.96, 36.71)	0.8356
	20	(1, 7)	(0.96, 72.89)	0.8460		88	(1, 6)	(0.96, 36.71)	0.8495
50	41	(1, 7)	(0.96, 72.89)	0.8015		90	(1, 6)	(0.96, 36.71)	0.8607
	43	(1, 6)	(0.96, 36.71)	0.8268		92	(1, 6)	(0.96, 36.71)	0.8674
		(1, 7)	(0.96, 72.89)	0.8383		94	(1, 6)	(0.96, 36.71)	0.8661
	45	(1, 5)	(0.96, 33.91)	0.8014		96	(1, 6)	(0.96, 36.71)	0.8491
		(1, 6)	(0.96, 36.71)	0.8518		81	(1, 7)	(0.96, 72.89)	0.8002
		(1, 7)	(0.96, 72.89)	0.8720		85	(1, 7)	(0.96, 72.89)	0.8379
	47	(1, 6)	(0.96, 36.71)	0.8589		89	(1, 7)	(0.96, 72.89)	0.8731
		(1, 7)	(0.96, 72.89)	0.8964		93	(1, 7)	(0.96, 72.89)	0.9012
	49	(1, 6)	(0.96, 36.71)	0.8038		97	(1, 7)	(0.96, 72.89)	0.8986
		(1, 7)	(0.96, 72.89)	0.8800		99	(1, 7)	(0.96, 72.89)	0.8357

Table 4: PIs for upper records based on observed future order statistics.

n	r	(i, j)	$(X_{i:n}, X_{j:n})$	π_1
19	1	(1, 19)	(0.19, 72.89)	0.9000
	2	(1, 19)	(0.19, 72.89)	0.8176
	3	(1, 19)	(0.19, 72.89)	0.6364
	1	(3, 19)	(0.96, 72.89)	0.8000
	2	(3, 19)	(0.96, 72.89)	0.8046
	3	(3, 19)	(0.96, 72.89)	0.6352
	1	(1, 17)	(0.19, 33.91)	0.8000
	2	(1, 17)	(0.19, 33.91)	0.5828
	3	(1, 17)	(0.19, 33.91)	0.3341
	1	(3, 17)	(0.96, 33.91)	0.7000
	2	(3, 17)	(0.96, 33.91)	0.5698
	3	(3, 17)	(0.96, 33.91)	0.3329

Based on the above seven upper records, PIs for future order statistics were obtained with prediction coefficient of at least 0.80 and the results are displayed in Table 3. By using the sample ($**$), PIs for future upper records, these intervals are presented in

Table 4. For more numerical examples for some special cases of GOS see Ahmadi and Balakrishnan [4] and Raqab and Balakrishnan [14].

7. Conclusion

In this paper, we have derived distribution-free PIs for future GOS from the Y -sequence based on GOS from the X-sequence with coverage probabilities of these intervals being free of the parent distribution. Also, we have described how GOS spacings $D_{i,j,n,m,k}$ can be used to construct distribution-free upper and lower prediction limits for GOS spacing $D^*_{r,s,u,\mu,q}$.

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