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New Phase-Fitted and Amplification-Fitted Modified Runge-Kutta Method for Solving Oscillatory Problems

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Abstract

In this paper, a new phase-fitted and amplification-fitted modified Runge-Kutta (MRK) method is constructed to solve first-order ordinary differential equations with oscillatory solutions. This new method is based on the Runge-Kutta Zonneveld method with fourth algebraic order. The numerical results for the new method have been compared with other existing methods. Findings have shown that the new method is more efficient than the other existing methods.

AMS subject classification:
Keywords: Runge-Kutta Methods, Phase-Fitted, Amplification-Fitted and Periodic Solutions.

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1. Introduction

This study deals with the initial value problems (IVPs) of the form:

$$y'(x) = f(x, y), \; y(x_0) = y_0,$$
$$y'(x_0) = y'_0, \; x \in [a, b] \tag{1}$$

where

$$y(x) = [y_1(x), y_2(x), \ldots, y_s(x)]^T$$
$$f(x, y) = [f_1(x, y), f_2(x, y), \ldots, f_s(x, y)]^T$$

y_0 is a given vector of initial conditions and their solution is oscillatory. This problem often exists in a number of applied science area, such as astronomy, quantum mechanics, mechanics, and electronics.

Lately, there have been a number of numerical methods derived by several authors based on different approaches like minimal phase-lag, phase-fitted, and exponential-fitted for solving first-order oscillatory IVPs. See [1-3] they proposed application of phase-lag and dissipation error for solving oscillatory problems. Other than phase-lag, a lot of research is also focused on methods having high dissipative order, which is the distance of the computed solution from the standard cyclic solution. Van der Houwen and Sommeijer [3] constructed diagonally implicit Runge-Kutta Nyström method which have relatively low algebraic order and high order of dispersion for solving oscillatory problems. Van de Vyver [2] suggested a symplectic Runge-Kutta Nyström method with minimal phase-lag for solving oscillatory problems. Senu et al. [4] derived a zero dissipative Runge-Kutta Nyström method with minimal phase-lag. In 1993, Simos [5] derived a Runge-Kutta-Fehlberg method based on the idea of phase-lag of order infinity. Recently, the idea of phase-lag of order infinity has been used to develop new numerical methods. Anastassi et al. [6] suggested a family of Runge-kutta methods with zero phase-lag and derivatives for the numerical solution of the Schrödinger equation and related problems. Simos et al. [7-8] proposed a modified Runge-Kutta method with phase-lag of order infinity for solving the Schrödinger equation and a modified phase-fitted Runge-Kutta method for numerical solution of the Schrödinger equation.

In this work, the aim is to combine the idea of the phase-lag of order infinity together with zero amplification error. A five-stage phase-fitted and amplification-fitted modified Runge-Kutta method is constructed based on the coefficients of the Zonneveld method of algebraic order four.

2. Analysis Phase-Lag of the Method

An explicit m-stage MRK formula is given by

$$y_{n+1} = y_n + h \sum_{i=1}^{m} b_i f(x_n + c_i h, Y_i) \tag{2}$$

where

$$Y_i = g_i y_n + h \sum_{j=1}^{m} a_{ij} f(x_n + c_j h, Y_j) \tag{3}$$

The method is said to be explicit when $a_{ij} = 0$ for $i \leq j$ and implicit otherwise. The method in Eq. (2) and Eq. (3) can be reduced into Butcher tableau form (see Table 1).

Table 1:
m-stage modified explicit Runge-Kutta method

$$\begin{array}{c|c|ccccc}
0 & & & & & & \\
c_2 & g_2 & a_{21} & & & & \\
c_3 & g_3 & a_{31} & a_{32} & & & \\
. & . & . & . & & & \\
. & . & . & . & & & \\
. & . & . & . & & & \\
c_m & g_m & a_{m1} & a_{m2} & ... & a_{m,m-1} & \\
\hline
 & & b_1 & b_2 & ... & b_{m-1} & b_m
\end{array}$$

To develop the new method, we utilize the test equation based on [3]

$$y' = ivy \tag{4}$$

where v is real. Then we compare the theoretical solution and the numerical solution for this equation. By requiring that the solutions are in phase with maximal order in the step-size h, we derive the so-called dispersion relation.

Applying the above method (2) and (3) to the test equation (4) we obtain

$$y_n = a_*^n y_0$$

with

$$a_* = A_m(H^2) + iHB_m(H^2),\ H = vh \tag{5}$$

The amplification factor is $a_* = a_*(H)$, and y_n denotes the approximation to $y(x_n)$. A comparison of Eq. (5) with the solution of Eq. (4) leads to the following definition of the dispersion or phase error or phase-lag and the amplification error.

Definition 2.1. An explicit m-stage MRK, presented in Table 1 the quantities:

$$t(H) = H - arg[a_*(H)],\ a(H) = 1- \mid a_*(H) \mid \tag{6}$$

are called the dispersion or phase error or phase-lag and the amplification error respectively. If $t(H) = O(H^{r+1})$, and $a(H) = O(H^{s+1})$ then the method is said to be phase-lag order r and dissipative order s.

From the Eq. (6) it follows that,

$$a(H) = 1 - \sqrt{[A_m^2(H^2) + H^2 B_m^2(H^2)]} \tag{7}$$

Meanwhile, for the Runge-Kutta method given in Table 1, the following formula is used for the direct calculation of the phase-lag order r and the phase-lag constant q

$$\tan(H) - H\left[\frac{B_m(H^2)}{A_m(H^2)}\right] = qH^{r+1} + O(H^{s+3}). \tag{8}$$

The analysis of phase-fitted (dispersion of order infinity) and amplification-fitted (dissipation of order infinity) are based on dispersion and dissipation quantities that have discussed above. The modified RK method is phase-fitted and amplification-fitted if the following conditions hold:

$$t(H) = 0 \qquad \text{and} \qquad a(H) = 0.$$

3. Construction of the New Method

In this section, we will persent the construction of a method with phase-lag of order infinity and zero amplification error which is based on Zonneveld fourth order method with five-stages as follows [11].

Table 2:
Butcher Tableau for five-stage fourth order RK method

$$\begin{array}{c|ccccc}
0 & & & & & \\
\frac{1}{2} & \frac{1}{2} & & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & & \\
1 & 0 & 0 & 1 & & \\
\frac{3}{4} & \frac{5}{32} & \frac{7}{32} & \frac{13}{32} & -\frac{1}{32} & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 0
\end{array}$$

subsituting the cofficents in Table 2 into equation (9) and chosing a_{21} and g_2 as free parameters for the optimized value of the maximum global error we have:

$$A_5(H^2) = 1 + \frac{1}{12}H^4 a_{21} + \left(-\frac{1}{3}g_2 - \frac{1}{6} - \frac{1}{6}a_{21}\right)H^2,$$
$$B_5(H^2) = \left(-\frac{1}{6}a_{21} - \frac{1}{12}g_2\right) + \frac{2}{3} + \frac{1}{3}g_2. \tag{9}$$

Applying equations (10) to equations (7) and (8), we obtain:

$$a(H) = \left(1 + \frac{1}{12}H^4 a_{21} + \left(\frac{-1}{3}a_{21} - \frac{1}{6} - \frac{1}{6}g_2\right)H^2\right)^2$$
$$+H^2\left(\left(-\frac{1}{6}a_{21} - \frac{1}{12}g_2\right)H^2 + \frac{2}{3} + \frac{1}{3}g_2\right)^2 - 1 = 0 \tag{10}$$

$$\tan(H) - H\left[\frac{\left(-\frac{1}{6}a_{21} - \frac{1}{12}g_2\right) + \frac{2}{3} + \frac{1}{3}g_2}{1 + \frac{1}{12}H^4 a_{21} + \left(-\frac{1}{3}g_2 - \frac{1}{6} - \frac{1}{6}a_{21}\right)H^2}\right] = 0 \tag{11}$$

Solving Eq. (11) and Eq. (12) simultaneously using maple package, obtaining the solution as given below:

$$\begin{aligned}
a_{21} = {} & \frac{1}{(16 + h^4 - 4^2)h^2}(RootOf(786h^2 + 64h^2 \\
& + 2304^4 - 2304\tan(h)h - 960\tan(h)^2)h^2 \\
& + 208(h)^2h^4 - 16\tan(h)^2h^6 + 4h^8\tan(h)^2 + 576h^3\tan(h) - 16h^6 + 4h^8 \\
& + (-4h^4 - 4\tan(h)^2h^4 + 8\tan(h)^2h^2 + 8h^2 \\
& - 96\tan(h)^2 - 96)Z + (\tan(h)^2 + 1)Z^2)),
\end{aligned}$$

$$\begin{aligned}
g_2 = {} & \frac{1}{(16 + h^4 - 4h^2)h(-2\tan(h)h + h^2 - 4}(192\tan(h) + 20\tan(h)h^4 - 80\tan(h)h^2 \\
& + \tan(h)h^2 RootOf(768h^2 + 64h^4 + 4h^8\tan(h)^2 + 576h^3\tan(h) - 16h^6 \\
& + 2304\tan(h)^2 - 2304\tan(h)h + 4h^8 \\
& - 4\tan(h)^4 - 960\tan(h)^2h^2 + 208\tan(h)^2h^4 - 16\tan(h)^2h^6 + (-4h^4 - 4\tan(h)^2h^4 \\
& + 8h^2 + 8\tan(h)^2h^2 - 96 - 96\tan(h)^2)Z - 2304\tan(h)h - 960\tan(h)^2h^2 - 16h^6 \\
& + 2304\tan(h)^2 + 4h^8\tan(h)^2 + 576h^3\tan(h) + 208\tan(h)^2h^4 + 4h^8 \\
& + (-4h^4 - 4\tan(h)^2h^4 + 8\tan(h)^2h^2 \\
& + 8h^2 - 96\tan(h)^2 - 96)Z + (\tan(h)^2 + 1)Z^2) - 2h^6\tan(h) \\
& + 2h RootOf(768h^2 + 64h^4 - 2304\tan(h)h - 960\tan(h)^2h^2 - 16h^6 \\
& + 2304\tan(h)^2 + 4h^8(h)^2 + 576h^3\tan(h) + 208\tan(h)^2h^4 - 16\tan(h)^2h^6 + 4h^8 \\
& + (-4h^4 - 4\tan(h)^2h^4 + 8\tan(h)^{[2}h^2 + 8h^2 - 96\tan(h)^2 - 96)Z \\
& + (\tan(h)^2 + 1)Z^2) - 128h - 8h^5 + 32h^3).
\end{aligned}$$

In corresponding taylor series expansion of the solution is given in equation bellow:

$$a_{21} = \frac{1}{2} - \frac{1}{120}H^4 - \frac{13}{4480}H^6 - \frac{229}{1209600}H^8 + \frac{8549}{63866880}H^{10}$$

$$+\frac{1755109}{38745907200}H^{12} + \frac{2947561}{996323328000}H^{14} + \cdots \tag{12}$$

$$g_2 = 1 + \frac{1}{40}H^4 + \frac{1}{672}H^6 - \frac{37}{34560}H^8 - \frac{2143}{5913600}H^{10}$$

$$-\frac{319}{13478400}H^{12} + \frac{4488229}{268240896000}H^{14}$$

$$+\frac{44755279}{7904165068800}H^{16} + \cdots \tag{13}$$

This new method is denoted as PHAFRK4.

3.1. Analysis of Stability

An m-stage modified Runge-Kutta method (2) and (3) is applied to equation (4), we obtain

$$y_{n+1} = y_n + \hat{h}BY, \tag{14}$$

$$Y = y_n G + \hat{h}AY \tag{15}$$

where

$$Y = [Y_1, Y_2, \dots, Y_s], G = [g_1, g_2, \dots, g_s]$$

and

$$B = [b_1\, b_2 \cdots b_s]^T,\ \hat{h} = hv$$

From (11), we have

$$Y = (I - \hat{h}A)^{-1} y_n G \tag{16}$$

substituting equation (16) into equation (14), we obtan:

$$y_{n+1} = R(\hat{h})y_n,$$

where

$$R(\hat{h}) = 1 + \hat{h}B(1 - \hat{h}A)^{-1}G \tag{17}$$

is the stability function of the method.

For this new method, we obtained the stability polynomial are three different stages of the solutions. First, we take the value of a_{21} and g_2 up to h^6 from their series solution.

$$\begin{aligned} a_{21} &= \frac{1}{2} - \frac{1}{120}\,H^4 - \frac{13}{4480}\,H^6, \\ g_2 &= 1 + \frac{1}{40}\,H^4 + \frac{1}{672}\,H^6. \end{aligned}$$

$$\begin{aligned} R(\hat{h}) = 1 + h + \frac{1}{2}\hat{h}^2 + \frac{1}{6}\hat{h}^3 + \frac{1}{24}\hat{h}^4 \\ + \frac{1}{120}\hat{h}^5 + \frac{1}{720}\hat{h}^6 + \frac{1}{840}\hat{h}^7 \\ -\frac{19}{13440}\hat{h}^8 - \frac{29}{80640}\hat{h}^9 - \frac{13}{53760}\hat{h}^{10} + \cdots . \end{aligned} \tag{18}$$

Secondly, we take the values a_{21} and g_2 up to h^8 from their series solution.

$$\begin{aligned} a_{21} &= \frac{1}{2} - \frac{1}{120}\,H^4 - \frac{13}{4480}\,H^6 - \frac{229}{1209600}\,H^8, \\ g_2 &= 1 + \frac{1}{40}\,H^4 + \frac{1}{672}\,H^6 - \frac{37}{34560}\,H^8. \end{aligned}$$

$$\begin{aligned} R(\hat{h}) = 1 + h + \frac{1}{2}\hat{h}^2 + \frac{1}{6}\hat{h}^3 + \frac{1}{24}\hat{h}^4 \\ + \frac{1}{120}\hat{h}^5 + \frac{1}{720}\hat{h}^6 + \frac{1}{840}\hat{h}^7 \\ -\frac{19}{13440}\hat{h}^8 - \frac{29}{80640}\hat{h}^9 - \frac{13}{53760}\hat{h}^{10} - \frac{1753}{14515200}\hat{h}^{11} - \frac{229}{14515200}\hat{h}^{12} + \cdots . \end{aligned} \tag{19}$$

Lastly, we take the values a_{21} and g_2 up to h^{10} from their series solution.

$$\begin{aligned} a_{21} &= \frac{1}{2} - \frac{1}{120}\,H^4 - \frac{13}{4480}\,H^6 - \frac{229}{1209600}\,H^8 + \frac{8549}{63866880}\,H^{10} \\ g_2 &= 1 + \frac{1}{40}\,H^4 + \frac{1}{672}\,H^6 - \frac{37}{34560}\,H^8 - \frac{2143}{5913600}\,H^{10} \end{aligned}$$

$$\begin{aligned} R(\hat{h}) = 1 + \hat{h} + \frac{1}{2}\hat{h}^2 + \frac{1}{6}\hat{h}^3 + \frac{1}{24}\hat{h}^4 + \frac{1}{120}\hat{h}^5 \\ + \frac{1}{720}\hat{h}^6 + \frac{1}{840}\hat{h}^7 - \frac{19}{13440}\hat{h}^8 \\ - \frac{13}{18144}\hat{h}^9 - \frac{877}{1814400}\hat{h}^{10} - \frac{551}{2280960}\hat{h}^{11} - \frac{3023}{95800320}\hat{h}^{12} \\ - \frac{3779}{479001600}\hat{h}^{13} + \frac{8549}{766402560}\hat{h}^{14} + \cdots . \end{aligned} \tag{20}$$

we next obtained the stability region of the new method from the above three stability polynomials by equating each to the Euler formula and then solve for h using maple package, i.e.

$$R(\hat{h}) = e^{I\theta} = \cos(\theta) + I\sin(\theta).$$

The stability region for the new method is shown in Figure 1.

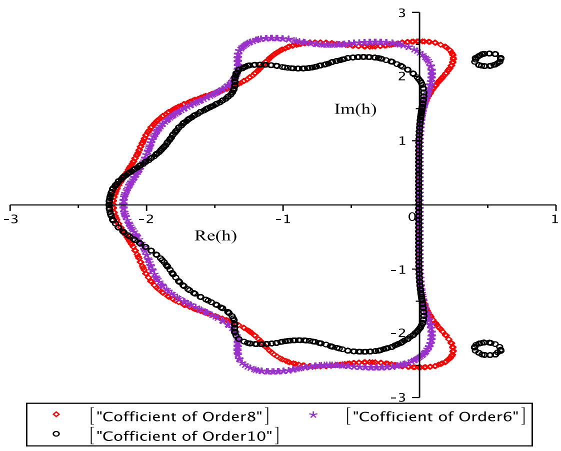

Figure 1: The stability region for the new method PHAFRK4 for different order

Definition 3.1. A Runge-kutta method is said to be absolutely stable if $\forall\, \hat{h}\, \epsilon(-h, 0)$, $| R(\hat{h}) | < 1$.

Now, our new method is absolutely stable since for all $\hat{h}\, \epsilon(-2.8, 0)$, $| R(\hat{h}) | < 1$ where we obtained using maple package.

3.2. Error Analysis

The local truncation error (LTE) of the new method is based on the Taylor series expansion of the differences y_{n+1} and $y(x_n + h)$

$$LTE = y_{n+1} - y(x_n + h) \tag{21}$$

$$
\begin{aligned}
LTE = h^5 \Bigg[& -\frac{1}{2880}(f_{xxxx} + f^4 f_{yyyy}) - \frac{1}{120}(f_y y(x) w^4 + f_y^2 f_{xy} y' + f_y^3 f_x + f_y^4 y') \\
& + f_{xxyy} y'^{(2)} + f_x f_{xyy} + f_y^2 f_{xx}) \\
& + \frac{1}{480}(f_y f_{xyy} y' - y'^{(2)} f_x f_{yyy} - f_{xx} f_{yy} y' - f_{xxyy} y'^{(2)} \\
& - f_x f_{xxy} + f_{xx} f_{xy} + f_{yy}^2 y'^{(3)} - f_y^2 f_{xx}) \\
& - \frac{1}{240}(f_x f_{xyy} y' + f_x f_y f_{xy} - f_{xy}^2 y') \\
& + \frac{1}{160}(f_{xy} f_{yy} y'^{(2)} - f_{xx}^2 f_{yy}) - \frac{1}{1440} f_y f_{yyy} y'^{(3)} \\
& - \frac{1}{80} f_y^2 f_{yy} y'^{(2)} - \frac{1}{60} f_x f_y f_{yy} y' \\
& - \frac{1}{720}(f_{xxxy} y' + f_{xyyy} y'^{(3)} - f_y f_{xxx}) \Bigg] + O(h^6).
\end{aligned} \tag{22}
$$

From equation (22), it is clear that the order of the new method is four because all the terms of h lower than h^5 are vanished.

4. Tested Problems and Numerical Results

In this section, we will apply the new method to solve different problems. The following explicit MRK method are selected for the numerical comparison.

- PHAFRK4: Combination between phase-fitted and amplification-fitted derived in this paper.
- RK5B: Fifth-order six-stage RK method given by Sakas and Simos [9].
- PLRK4: The modified RK method derived by Simos and Vigo-Aguiar [7].
- RK4M: Fourth-order five-stage RK method given in Butcher [10].
- RK4Z: Fourth-order five-stage RK method given in Hairer et al. [11].

Problem 1: (Homogeneous)

$$y_1' = y_2, \;\; y_1(x) = 1$$

$$y_2' = -64 y_1, \;\; y_2(x) = -2$$

Theoretical solution:

$$y_1(x) = -\frac{1}{4}\sin(8x) + \cos(8x)$$

$$y_2(x) = -2\cos(8x) - 8\sin(8x)$$

Source: Chawla and Rao [12].

Problem 2: (Inhomogeneous)

$$y_1' = y_2,\ \ y_1(x) = 1$$

$$y_2' = -v^2 y_1 + (v^2 - 1)\sin(x),\ \ y_2(x) = v + 1$$

Estimated frequency: $v = 10$

Theoretical solution:

$$y(x) = \cos(vx) + \sin(vx) + \sin(x)$$

$$y_2(x) = -v\sin(vx) + v\cos(vx) + \cos(x)$$

Source: Van der Howuen and Sommeijer [3].

Problem 3: (Periodic orbit system)

$$y_1' = y_3,\ \ y_1(x) = 1$$

$$y_3' = -y_1 + 0.001\cos(x),\ \ y_3(x) = 0$$

$$y_2' = y_4,\ \ y_2(x) = 0$$

$$y_4' = -y_2 + 0.001\sin(x),\ \ y_4(x) = 0.9995$$

Theoretical solution:

$$y_1(x) = \cos(x) + 0.0005x\sin(x)$$

$$y_2(x) = \sin(x) - 0.0005x\cos(x)$$

$$y_3(x) = -\sin(x) + 0.0005x\cos(x)$$

$$y_4(x) = \cos(x) + 0.0005x\sin(x)$$

Source: Stiefel and Bettis [1].

Problem 4: (Nonlinear system)

$$y_1' = y_3,\ \ y_1(0) = 1$$

$$y_3' = \frac{-y_1}{\left(\sqrt{y_1^2 + y_2^2}\right)^3},\ \ y_3(0) = 0$$

$$y_2' = y_4,\ \ y_2(0) = 0$$

$$y_4' = \frac{-y_2}{\left(\sqrt{y_1^2 + y_2^2}\right)^3},\ \ y_4(0) = 1$$

Theoretical solution:

$$y_1(x) = \cos(x)$$
$$y_2(x) = \sin(x)$$
$$y_3(x) = -\sin(x)$$
$$y_4(x) = \cos(x)$$

Source: Moo et al. [13].

Problem 5: (Inhomogeneous)

$$y_1' = y_2,\ \ y_1(0) = 1$$
$$y_2' = -y_1 + x,\ \ y_2(0) = 2$$

Theoretical solution:

$$y_1 = \sin(x) + \cos(x) + x$$
$$y_2 = \cos(x) - \sin(x) + 1$$

Source: Allen and Wing [14].

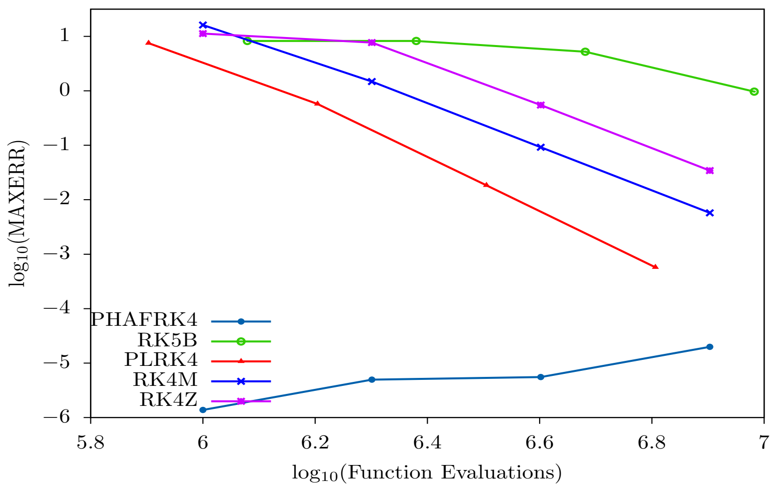

Figure 2: Comparison for PHAFRK4, RK5B, PLRK4, RK4M and RK4Z problem 1 with $b = 10000$.

4.1. Discussion and Conclusion

In this study, we have presented a new phase-fitted and amplification-fitted (PHAFRK4) method that can be used to solve first-order ordinary differential equations with periodic solutions. The numerical results are plotted in Figures 1, 2, 3, 4 and 5. Those Figures display the efficiency curves where the common logarithm of the maximum global error

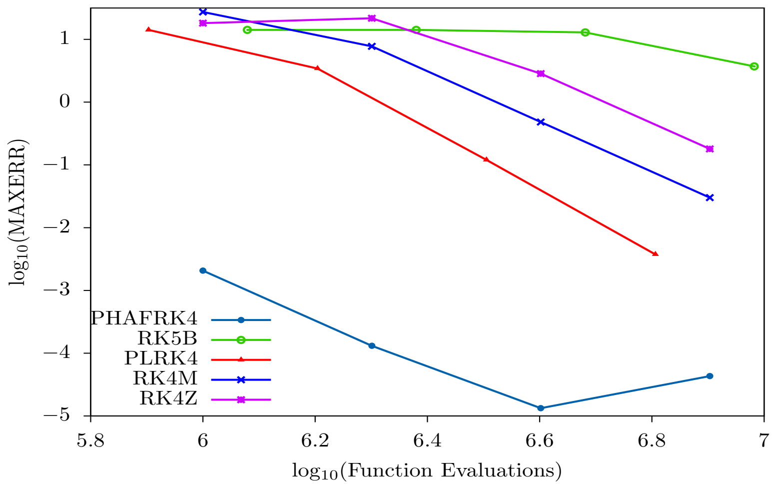

Figure 3: Comparison for PHAFRK4, RK5B, PLRK4, RK4M and RK4Z problem 2 with $b = 10000$.

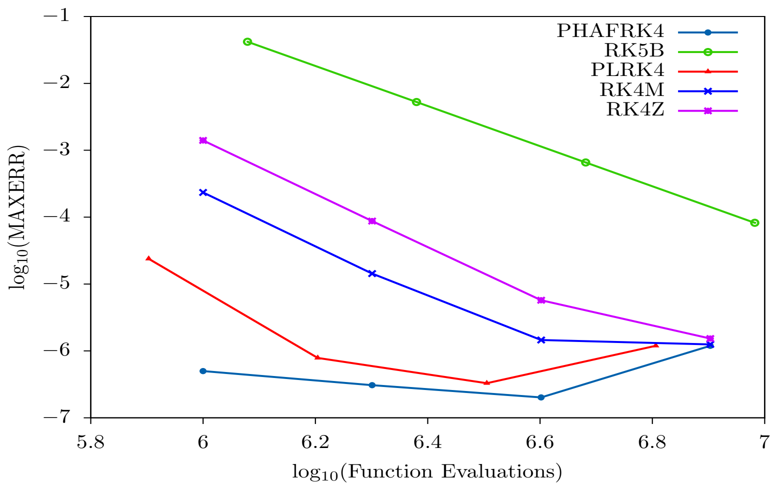

Figure 4: Comparison for PHAFRK4, RK5B, PLRK4, RK4M and RK4Z problem 3 with $b = 10000$.

throughout the integration versus computational cost measured by the number of function evaluations.

This new method is based on Zonneveld's five-stage fourth algebraic order. From Figures 1 to 5, numerical results have shown that the new method is more accurate and efficient when solving first-order differential equations with oscillatory solutions than the existing methods.

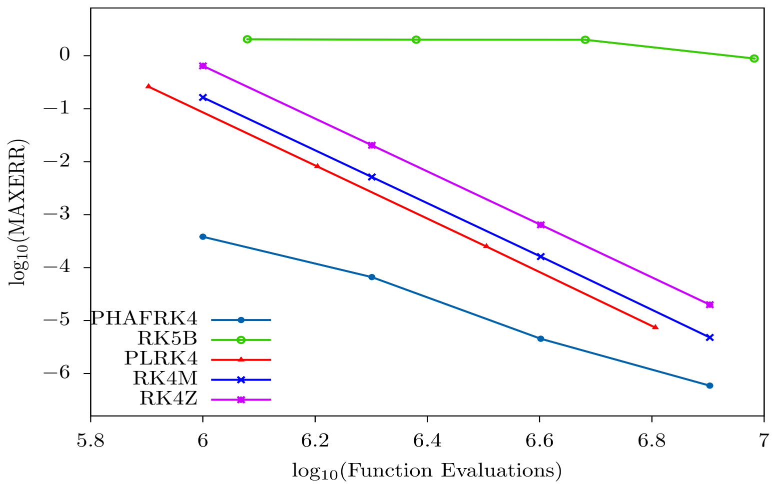

Figure 5: Comparison for PHAFRK4, RK5B, PLRK4, RK4M and RK4Z problem 4 with $b = 10000$.

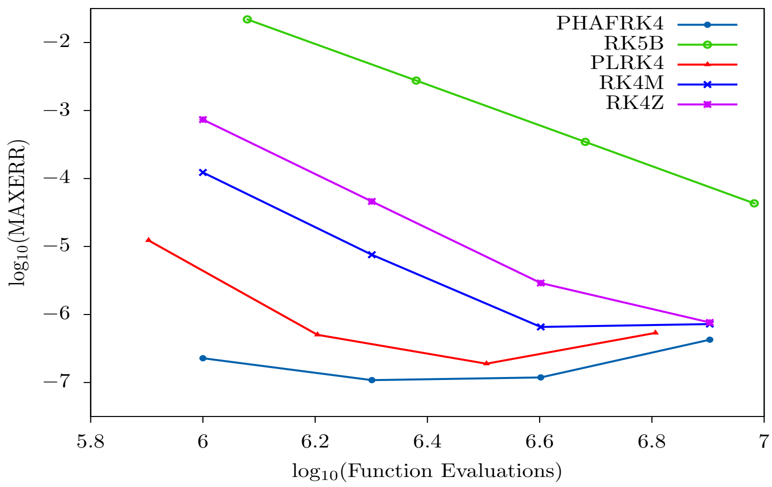

Figure 6: Comparison for PHAFRK4, RK5B, PLRK4, RK4M and RK4Z problem 5 with $b = 10000$.

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