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On Regularity of Solution to Diffusion Approximation of GI/G/1 Queueing System

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Abstract.

Regularity of non-stationary and stationary solutions to the diffusion approximation of the GI/G/1 queueing system under the elementary return boundary condition are discussed in this paper. Some boundedness of the solutions are also verified by using maximum principle.

Introduction

We discuss explicit non-stationary and stationary solutions to an initial boundary value problem of a linear partial differential equation of parabolic type, used in the elementary return boundary formulation of diffusion approximation to the GI/G/1 queueing system. It has been one of open problems in the literature [18].

Diffusion approximation is one of the most useful methods for tracing the temporal behavior of queueing systems. It describes the probability distribution function of the customer number in the system or virtual waiting time of a customer at each time, which is formulated by an initial boundary value problem of a linear partial differential equation of parabolic type. It is especially efficient for the GI/G/1 queueing system, where the inter-arrival times are independent and identically distributed random variables, customers are served in order of arrival, the service times of customers are independent and identically distributed random variables, and the inter-arrival and service times form independent sequences. The justification of this approach was provided by Kleinrock [12].

Even though the queue length is assumed to be infinite, the customer number in the system and virtual waiting time take non-negative values. Therefore, we have to consider the problem on the interval $\boldsymbol{R}_+ \equiv (0, \infty)$. As a result, some boundary conditions at $x = 0$ and $x \to \infty$ are necessary. In general, there exist two formulations of the diffusion approximation of the GI/G/1 system according to the form of boundary conditions: *the reflecting barrier* and *elementary return* formulations. The former formulation models the sample path of the object to be reflected instantaneously at the point $x = 0$, which means that it does not rest at that point. In the latter formulation, however, the sample path remains at the point $x = 0$ with time subject to the exponential distribution, then jumps into the region $x > 0$. The distance of the jump is assumed to be subject to a probability distribution function provided in advance. The equation of diffusion approximation of the GI/G/1 queueing system under the elementary return boundary condition is described as follows [17]:

$$(\frac{\partial}{\partial t} - L)w = \Lambda R(t)\phi(x) \quad (x,t) \in \boldsymbol{R}_+^2, \qquad 1.1$$

where the operator $L \equiv -\beta \frac{\partial}{\partial x} + \frac{\alpha}{2}\frac{\partial^2}{\partial x^2}$, and $\boldsymbol{R}_+ \equiv (0,\infty)$. In the above formulation, $\phi(x)$ is a function denoting the location of a sample path right after resting at $x = 0$. Gelenve [7] formulated $\phi(x) = \delta(x-d)$, where $\delta(x)$ is the Dirac's delta function, and $d > 0$ is a constant. Takahashi [17] considered a model in which multiple N calls are present and formulated this term by $\phi(x) = \sum_{i=1}^{N} \delta(x - d_i)$ with $d_i > 0$ $(i = 1,2,\ldots,N)$.

The independent variable t stands for time, and x for the approximated number of customers in the queue or the virtual waiting time. By virtue of the renewal limit theorems, when it approximates the number of customers in the queue, α and β are defined as $(\alpha,\beta) = (\lambda^3 V_a + \mu^3 V_s, \lambda - \mu)$, where λ and μ are the arrival rate and the customers served in a unit of time, respectively, and V_a and V_s are the variances of the inter-arrival and service time, respectively. In case (1. 1) approximates the time evolution of the virtual waiting time, they are expressed as $(\alpha,\beta) = (\varrho\mu^{-1}(\lambda^2 V_a + \mu^2 V_s), \varrho - 1)$, with $\varrho = \lambda/\mu$. They are positive and negative constants. Let us introduce other notations in (1. 1). Λ represents the mean interval of the events in which the system becomes empty; $R(t)$, the probability that the systems becomes empty at time t, and $w(x,t) \in \boldsymbol{R}$ is the probability density function of the customer number or the virtual waiting time at time t. Note that both $w(x,t)$ and $R(t)$ are unknown variables in this formulation. Due to the definitions of $R(t)$ and $w(x,t)$, the following relationship should be satisfied:

$$R(t) + \int_0^\infty w(x,t)\,\mathrm{d}x = 1 \quad \forall t \in (0,\infty). \qquad 1.2$$

Condition (1. 2) is rewritten as the following boundary condition:

$$Bw(t) \equiv (\frac{\alpha}{2}\frac{\partial}{\partial x} - \beta w)|_{x=0} = \frac{d}{dt}R(t) + \Lambda R(t). \qquad 1.3$$

The closest contributions to ours is those by Czachórski [1], [3] for single and multi-server models with infinite length queue. They derived a solution not by solving the partial differential equation, but by considering the probabilistic behavior of a sample path. The obtained solution includes functions satisfying some integral equations, which are solvable by only numerical calculations. They also investigated a model with a bounded queue [2] by using a similar approach. For instance, their solution was provided in such a form

$$\tilde{w}(x,s) = \tilde{\phi}(x,s;\psi) + \tilde{g}_1(s)\tilde{\phi}(x,s,1) + \tilde{g}_{N-1}(s)\tilde{\phi}(x,s,N-1)$$

where $\tilde{f}(s)$ is the temporal Laplace transform of a function $f(t)$ in general. The functions $\tilde{g}_i(s)$ $(i = 1, N-1)$ are calculated from a lengthy algebraic equations.

Czachórski [1] also advocated that as t tends to infinity, the non-stationary solution converges to a solution to the problem

$$\begin{cases} -Lw_{\phi} = \Lambda R_{\infty}\phi(x) \quad \text{in } \boldsymbol{R}_{+}, \\ Bw_{\phi}|_{x=0} = w_{\phi}(0) = 0, \\ \lim_{x\to\infty} w_{\phi}(x) = 0, \end{cases}$$

which is obtained by replacing $R(t)$ in (1. 1) by $R_{\infty} \equiv \lim_{t\to\infty} R(t)$. In this paper, we theoretically prove this fact by making use of the theory of linear partial differential equations of parabolic type. The principal difficulty in obtaining an explicit non-stationary form of w lies in the calculation of the inverse Laplace transform with respect to time, which we have overcome with classical theories of the complex analysis and ordinary differential equation. For details, see Lemma 4. 4 in Section 4. In this paper, we provide a purely theoretical solution.

Let us refer to existing results concerning other formulations of diffusion approximation for the GI/G/1 system. The diffusion approximation of the GI/G/1 system under the reflecting barrier formulation is investigated by Heyman [11], Kobayashi [13] and Newell [16]. In this formulation, they considered the homogeneous version of (1. 1) with a homogeneous boundary condition:

$$Bw(t) \equiv \left(\frac{\alpha}{2}\frac{\partial}{\partial x} - \beta w\right)\Big|_{x=0} = 0. \qquad 1.4$$

To this problem, an explicit non-stationary and stationary solutions were provided [16]. Under this formulation, however, the sample path of the objects does not remain at $x = 0$, which results in the lower accuracy of the approximation, as Takahashi [17] pointed out. When the buffer length is finite, the domain of x is a bounded interval $(0, L)$ with $L > 0$. In this case, reflecting barrier boundary conditions are imposed on both boundaries, and an explicit solution was provided by Kobayashi [14] when the buffer length is finite.

In the rest of this section, we discuss the mathematical formulation. From the classical theory of partial differential equations, it is sufficient to impose initial and two more boundary conditions so that the problem is well-posed:

$$w(x,0) = \delta(x - x_0) \quad x \in \boldsymbol{R}_{+}, \qquad 1.5$$

$$\lim_{x\to+\infty} w(x,t) = 0 \quad t > 0, \qquad 1.6$$

$$w(0,t) = 0 \quad \forall t > 0. \qquad 1.7$$

As Czachórski did, Gelenbe [9] and Takahashi [17] also derived the stationary solution by letting t go to infinity in (1. 1) and (1. 3). The solution is consistent with our result as we will state in Theorem 5. 1.

We investigate problem (1. 1), (1. 3), (1. 5)–(1. 7) from the viewpoint of the theory of partial differential equations, and provide the explicit form of the non-stationary and stationary solutions. We also investigate the mathematically rigorous conditions for the well-posedness of the problems. Furthermore, we theoretically prove that the obtained solution $w(x,t)$ satisfies $w(x,t) \geq 0$.

Mathematical formulation

In this section, we formulate again the problem to be solved. The unknown variables are $w(x,t)$ and $R(t)$ satisfying the following problem.

$$\begin{cases}(\frac{\partial}{\partial t}-L)w=\Lambda R(t)\phi(x) \quad (x,t)\in \boldsymbol{R}_{+}^{2}, (8)\\ w(x,0)=w_0(x) \quad x>0, (9)\\ Bw(t)=\frac{d}{dt}R(t)+\Lambda R(t) \quad t>0, (10)\\ \lim_{x\to\infty} w(x,t)=0 \quad t>0, (11)\\ w(0,t)=0 \quad t>0,\end{cases} \qquad 2.1$$

If we consider $R(t)$ be provided, the problem for w can be regarded to be linear, and $w(x,t)$ is represented by $R(t)$ explicitly. The main idea of our proof is as follows. First, we solve $w(x,t)$ explicitly by using (2. 1) $_1$–(2. 1) $_4$, which includes $R(t)$. Next, it is substituted into (2. 1) $_5$, which yields $R(t)$. As a result, $w(x,t)$ is presented explicitly including $R(t)$, while for $R(t)$, only the Laplace transformed representation is obtained.

Function spaces

In this section, we prepare function spaces used in the following arguments. Let G be a domain in $\boldsymbol{R}^n$ $(n=1,2)$. By $C(G)$ and $C^m(G)$ $(m\in\boldsymbol{N})$, we denote sets of continuous and m times continuously differentiable functions on G, respectively. We also introduce a notation $C^\infty(G)\equiv\bigcap_{m=0}^{\infty}C^m(G)$, and $L_p(G)$ $(0<p<\infty)$ stands for a space of integrable functions in the sense of Lebesgue measure of finite norm

$$\| f \|_{L_p(G)}^p \equiv \int_G |f(x)|^p \, dx.$$

For simplicity, we hereafter denote $\frac{\partial^\alpha u}{\partial x^\alpha}$ and $\frac{\partial^\alpha u}{\partial t^\alpha}$ (or sometimes $\frac{d^\alpha u}{dt^\alpha}$) by $D_x^\alpha u$ and $D_t^\alpha u$, respectively.

By $W_2^l(G)$ we mean a space of functions $u(x)$, $x\in G$ equipped with norm [21]

$$\| u \|_{W_2^l(G)}^2 = \sum_{|\alpha|<l} \|D^\alpha u\|_{L_2(G)}^2 + \| u \|_{\dot{W}_2^l(G)}^2,$$

where

$$\begin{cases}\| u \|_{\dot{W}_2^l(G)}^2=\sum_{|\alpha|=l}\| D_x^\alpha u \|_{L_2(G)}^2=\sum_{|\alpha|=l}\int_G | D_x^\alpha u(x)|^2 dx \text{if } l \text{ is an integer},\\ \| u \|_{\dot{W}_2^l(G)}^2=\sum_{|\alpha|=[l]}\int_G\int_G\frac{|D_x^\alpha u(x)-D_x^\alpha u(y)|^2}{|x-y|^{n+2\{l\}}}dxdy\\ \text{if } l \text{ is a non}-\text{integer}, l=[l]+\{l\},\ 0<\{l\}<1.\end{cases}$$

Next we introduce anisotropic Sobolev–Slobodetskiĭ spaces [21]

$$W_2^{l,\frac{l}{2}}(G_T)\equiv W_2^{l,0}(G_T)\cap W_2^{0,\frac{l}{2}}(G_T) \quad (G_T\equiv G\times(0,T)),$$

whose norms are defined by

$$\| u \|^2_{W_2^{l,\frac{l}{2}}(G_T)} = \int_0^T \| u(\cdot,t) \|^2_{W_2^l(G)} \,\mathrm{d}t + \int_G \| u(x,\cdot) \|^2_{W_2^{\frac{l}{2}}(0,T)} \,\mathrm{d}x$$
$$\equiv \| u \|^2_{W_2^{l,0}(G_T)} + \| u \|^2_{W_2^{0,\frac{l}{2}}(G_T)}.$$

For a function or a distribution, to which the spacial Laplace transform is applicable and has its support included in $\boldsymbol{R}_+$, we define the following function spaces [20] with $\gamma \in \boldsymbol{R}$:

$$H_\gamma^l(\boldsymbol{R}_+) \equiv \{f | \int_{\gamma-i\infty}^{\gamma+i\infty} |\eta|^{2l} |\tilde{\tilde{f}}(\eta)|^2 \mathrm{d}\eta < \infty\},$$

where $\tilde{\tilde{f}}(\eta)$ stands for the spacial Laplace transform of f:

$$\tilde{\tilde{f}}(\eta) \equiv L_x[f](\eta) \equiv \int_0^\infty f(x) \exp(-\eta x) \,\mathrm{d}x,$$

which is holomorphic in the half-plane $\mathrm{Re}\eta > \gamma$. It has been shown that $H_0^l(\boldsymbol{R}_+)$ is equivalent to a set of functions $f \in W_2^l(\boldsymbol{R}_+)$ satisfying $D_x^i f|_{x=0} = 0$ $(i = 1,2,\ldots,[l])$ [20]. If a mapping $t \in \boldsymbol{R}_+ \longmapsto \| f(t) \|_{W_2^l(G)}$ is m times continuously differentiable for each $t \in \boldsymbol{R}_+$, we represent

$$f \in C^m((\boldsymbol{R}_+; W_2^l(G)).$$

A function f defined on $\boldsymbol{R}$ is denoted by $f|_G$ when it is restricted on a domain in $G \subset \boldsymbol{R}$. Finally, we introduce notations concerning the distribution. We denote a function space of rapidly decreasing functions by S:

$$S \equiv \{f \in C^\infty(\boldsymbol{R}) | \; x^m D_x^n f(x) < \infty \;\forall m,n \in \{0\} \bigcup \boldsymbol{N}\}.$$

Then we denote a set of linear continuous mappings on S by S', which is known as the space of tempered distributions. We now define the Fourier and Laplace transforms of a distribution $\Phi \in S$ in accordance with the usual definitions of those of tempered distributions [23] [24]. We also say $\Phi \geq 0$ in the distribution sense if it takes non-negative values on any test function $f \in S$, that satisfies $f(x) \geq 0 \;\forall x \in \boldsymbol{R}$. We define the support of a function f defined on $\boldsymbol{R}$:

$$supp(f) \equiv \overline{\{x \in \boldsymbol{R} | \; f(x) \neq 0\}},$$

where $\overline{G}$ stands for the closure of a domain G in general. For a tempered distribution Φ, we say $\Phi = 0$ on G if it vanishes on any function $f \in S$ defined on G. Now, let us denote the union of any continuous domain G_λ $(\lambda \in \boldsymbol{N})$ by $\bigcup_\lambda G_\lambda$ on which Φ vanishes. We define the support of Φ, denoted by $supp(\Phi)$, as the complement of $\bigcup_\lambda G_\lambda$: $supp(\Phi) \equiv (\bigcup_\lambda G_\lambda)^c$ [24].

Explicit non-stationary solution

In this section, we provide an explicit form of a non-stationary solution, which is one of the main results of this paper.

Compatibility condition

First, let us introduce a new concept of the compatibility condition. Note that since the right-hand side of the equation and initial data are formulated as distributions in (2. 1), compatibility conditions in the ordinary sense do not make sense. Nevertheless, the solution in the region $t > 0$ has some regularity, as discuss later. Therefore, we introduce the following concept of compatibility conditions. For $k \in \{0\} \cup \boldsymbol{N}$, let us define

$$\tilde{\tilde{w}}^{(k)}(\eta) = D_t^k \tilde{\tilde{w}}|_{t=0}(\eta).$$

Then, the compatibility condition in the weak sense of order m for (2. 1) means that

$$\lim_{\eta\to\infty} \eta\,\{(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}^{(k)}(\eta) - \frac{\alpha}{2}\lim_{\eta'\to\infty} \eta'\, \tilde{\tilde{w}}^{(k)}(\eta')\} = D_t^k \psi(0)$$

hold for $k = 1,2, \ldots, m$. For the usual definition of the compatibility condition, which does not make sense here, refer to the appendix of this paper, [15], and [19].

Explicit non-stationary solution

In this subsection, we state the main result of this paper. We state sufficient conditions for the well-posedness and the explicit form of the solution under these conditions. Let $l > 0$ be arbitrarily provided, and assume following items:

(i) $supp(\phi),\ supp(w_0) \subset \boldsymbol{R}_+$;

(ii) $\lim_{\eta\to 0} \tilde{\tilde{\phi}}\,(\eta) = \lim_{\eta\to 0} \tilde{\tilde{w}}_0\,(\eta) = 1$;

(iii) $\phi \geq 0$. If ϕ is a distribution, it holds in a distribution sense;

(iv) $\phi \in H_0^l(\boldsymbol{R}),\ w_0 \in H_0^{1+l}(\boldsymbol{R})$;

(v) $(H *_x \phi - H) \in H_0^{1+l}(\boldsymbol{R}_+)$, where $H(x)$ stands for the Heaviside step function, and $f *_x g = \int_{\boldsymbol{R}} f\,(x - y, t) g(y, t)\ \mathrm{d}y$

(vi) for functions or distributions f and g in general;

(vii) the compatibility conditions in the weak sense up to order 1 are satisfied.

Then problem (2.1) has a unique solution $(w(x,t), R(t))$ satisfying

$$w \in C^\infty(\boldsymbol{R}_+; W_2^{2+l}(\boldsymbol{R}_+)),\ R \in C^\infty(\boldsymbol{R}_+)$$

of the form $w(x,t) = u(x,t) + v(x,t) + w_\phi(x)$, where

$$u(x,t) = [\Lambda\Gamma_1 * (\overline{R}\phi^*) + \Gamma_1 *_x (w_0 - w_\phi)^*]|_{\boldsymbol{R}_+},$$

$$v(x,t) = -2\exp(\frac{\beta x}{\alpha}) G *_t (\bar{\psi} - Bu|_{x=0}),$$

$$w_\phi(x) = -\frac{2\Lambda R_\infty}{\alpha}\exp(\frac{2\beta x}{\alpha}) *_x (H *_x \phi - H),$$

$$\tilde{\tilde{R}}(s) = \frac{R_\infty - \alpha r C_2(s)}{\alpha r \Lambda C_1(s) - (s+\Lambda)}. \qquad 4.1$$

Here, L_x^{-1} stands for the inverse Laplace transform with respect to x, f^* stands for an extension of a function $f(x)$ defined on $\boldsymbol{R}_+$ into $\boldsymbol{R}$ by zero, $\overline{R}(t) \equiv R(t) - R_\infty \equiv R(t) - \lim_{t\to\infty} R(t), \bar{\psi}(t) \equiv \frac{d}{dt}R(t) + \Lambda\overline{R}(t)$, and

$$\Gamma_1(x,t) \equiv \frac{1}{\sqrt{2\pi\alpha t}}\exp(\frac{-(x-\beta t)^2}{2\alpha t}),\ J(x,t) \equiv \sqrt{\frac{2\pi}{\alpha t}}\exp(-\frac{x^2+\beta^2 t^2}{2\alpha t}),$$

$$K(x,t) = -\frac{\beta}{\alpha}\int_x^\infty J(z,t)\exp(\frac{\beta}{\alpha}(x-z))\ \mathrm{d}z,\ G(x,t) \equiv \frac{1}{2\pi}(J(x,t)+K(x,t)),$$

$$f *_t g \equiv \int_0^t f(x,t-\tau)g(x,\tau)\ \mathrm{d}\tau,\ f * g = \int_0^t \int_{\boldsymbol{R}} f(x-y,t-\tau)g(y,\tau)\ \mathrm{d}y\mathrm{d}\tau,$$

$$C_1(s) \equiv \frac{1}{\alpha r}\int_0^\infty \exp\left(-(\frac{\beta}{\alpha}+r)y\right)\phi^*(y)\ \mathrm{d}y,$$

$$C_2(s) \equiv \frac{1}{\alpha r}\int_0^\infty \exp\left(-(\frac{\beta}{\alpha}+r)y\right)(w_0-w_\phi)^*(y)\ \mathrm{d}y,\ r = \sqrt{\frac{2}{\alpha}(s+\frac{\beta^2}{2\alpha})} \in \boldsymbol{C}.$$

Condition (ii) equals $\int_0^\infty \phi(x)\ \mathrm{d}x = 1$ and $\int_0^\infty w_0(x)\ \mathrm{d}x = 1$ if ϕ and w_0 are functions in an ordinary sense, respectively. For the problem formulated by Gelenbe [7] and Czachórski [1], usually $\phi = \delta(x-d)$ with $d>0$ and $w_0(x) = \delta(x-x_0)$ with $x_0>0$. In this case, the solution is provided in the following corollary. Let us assume $x_0>0$ and $d>0$. Then, problem (2.1) with ϕ and w_0 replaced with $\delta(x-d)$ and $\delta(x-x_0)$, respectively, has a unique solution $w(x,t)$ satisfying

$$w \in \bigcap_{m=0}^{\infty} C^\infty(\boldsymbol{R}_+; W_2^m(\boldsymbol{R}_+))$$

of the form

$$w(x,t) = u'(x,t) + v'(x,t) + w_d(x), \qquad 4.2$$

where

$$u'(x,t) = \Lambda\Gamma_1(x-d,t) *_t \overline{R} + \Gamma_1(x-x_0,t)|_{\boldsymbol{R}_+} - \Gamma_1 *_x w_d^*|_{\boldsymbol{R}_+},$$

$$v'(x,t) = -2\exp(\frac{\beta x}{\alpha})G *_t (\bar{\psi} - Bu'|_{x=0}),$$

$$w_d(x) = \begin{cases} \dfrac{\Lambda R_\infty}{\beta}(\exp(\dfrac{2\beta x}{\alpha})-1) & \text{on } (0,d), \\ \dfrac{\Lambda R_\infty}{\beta}(1-\exp(-\dfrac{2\beta d}{\alpha}))\exp(\dfrac{2\beta x}{\alpha}) & \text{on} (d,\infty). \end{cases}$$

***Preliminaries for proof of* Theorem 4. 1**

In this subsection, we prepare lemmas that prove the sufficiency of conditions in the statement in Theorem 4. 1. First, we show (1. 2) is derived from condition (ii).

It is substantial for the formulation of the diffusion approximation to convert the global condition (1. 2) into the local boundary condition (2. 1) $_2$. The following lemma refers to the mathematical sufficiency of condition (ii). Since the right-hand side of (1. 1) and the initial condition (1. 5) include the Dirac's delta function, which mathematically makes sense as a distribution on the space of test functions, we deal with the problem by applying the spacial Laplace transform. Under condition (ii) in the statement of Theorem 4. 1, a continuous solution $w(x,t)$ to (2.1), if it exists, satisfies (1.2).

Proof. Applying the spacial Laplace transform to (2. 1) yields

$$\begin{cases} \frac{\partial \tilde{\tilde{w}}}{\partial t}(\eta,t) - \eta(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}(\eta,t) = \frac{\alpha}{2}\eta w(0,t) + \frac{d}{dt}R(t) + \Lambda R(t)(\tilde{\tilde{\phi}}(\eta) - 1)(15) \\ \quad \eta \in \boldsymbol{C},\ t \in \boldsymbol{R}_+, (16) \\ \lim_{\eta\to 0} \eta\,\tilde{\tilde{w}}(\eta,t) = 0 \quad t \in \boldsymbol{R}_+, (17) \\ \tilde{\tilde{w}}(\eta,0) = \tilde{\tilde{w}}_0(\eta) \quad \eta \in \boldsymbol{C}. \end{cases} \qquad 4.3$$

Then, under condition (ii) in the statement of Theorem 4. 1,

$$\frac{d}{dt}\int_0^\infty w\,(x,t)\ \mathrm{d}x = \lim_{\eta\to 0}\frac{\partial \tilde{\tilde{w}}}{\partial t} = \frac{d}{dt}R(t).$$

Condition (ii) also implies $\lim_{\eta\to 0}\tilde{\tilde{w}}\,(\eta,0) = 1$, which means

$$\lim_{t\to 0}\int_0^\infty w\,(x,t) = 1,$$

and thus (1. 2) holds.

Next, we refer to the relationship between the derivative of $R(t)$ at $t = 0$ and the order of the compatibility condition in the weak sense. $R(t)$ provided in Theorem 4. 1 satisfies

$$D_t^k R(0) = 0\ (k = 0,1). \qquad 4.4$$

Under condition (i) in Theorem 4. 1, this is equivalent to the compatibility condition in the weak sense of order 1.

Proof. Introducing notations

$$\tilde{\tilde{w}}^{(0)}(\eta) = \tilde{\tilde{w}}_0(\eta), \quad \tilde{\tilde{w}}^{(k)}(\eta) \equiv \frac{\partial^k \tilde{\tilde{w}}}{\partial t^k}|_{t=0}(\eta)\ (k \geq 1),$$

it is sufficient to show the fact

$$\tilde{\tilde{w}}^{(k)}(\eta) = \eta^k(\frac{\alpha\eta}{2} + \beta)^k \tilde{\tilde{w}}_0(\eta). \qquad 4.5$$

In fact, the compatibility condition of order 0 in the weak sense becomes

$$\lim_{\eta\to\infty}\eta\,\{(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}_0(\eta) - \frac{\alpha}{2}\lim_{\eta'\to\infty}\eta'\,\tilde{\tilde{w}}_0(\eta',t)\}|_{t=0}$$
$$= \frac{dR}{dt}(0) + \Lambda R(0).$$

It is obvious that

$$\lim_{\eta\to\infty}\eta^m\,|\tilde{\tilde{w}}_0(\eta)| = 0(m = 0,1,2,\ldots,[2+l]),$$

since $w_0 \in H_0^{2+l}(\boldsymbol{R}_+)$. Noting that condition (i) implies $R(0) = 0$, these result in

$$\frac{dR}{dt}(0) = 0. \qquad 4.6$$

Next, by using (4. 3),

$$\tilde{\tilde{w}}^{(1)}(\eta) = \eta(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}^{(0)}(\eta) + \frac{\alpha}{2}\eta\lim_{\eta'\to\infty}\eta'\,\tilde{\tilde{w}}_0(\eta',0)$$
$$+ \frac{dR}{dt}(0) + \Lambda R(0)(\tilde{\tilde{\phi}}(\eta) - 1)$$

$= \eta(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}_0(\eta)$, 4.7

and the compatibility condition of order 1 in a weak sense becomes

$$\lim_{\eta\to\infty} \eta \{(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}^{(1)}(\eta) - \frac{\alpha}{2}\lim_{\eta'\to\infty} \eta' \, \tilde{\tilde{w}}^{(1)}(\eta')\}$$

$= D_t^2 R(0) + \Lambda \frac{dR}{dt}(0)$. 4.8

The left-hand side of (4. 8) vanishes due to (4. 7), and applying (4. 6) to the right-hand side of (4. 8) leads to

$$D_t^2 R(0) = 0.$$

In a similar manner, we have

$$\tilde{\tilde{w}}^{(2)}(\eta) = \eta(\frac{\alpha}{2}\eta - \beta)\tilde{\tilde{w}}^{(1)}(\eta) + \frac{\alpha}{2}\eta \lim_{\eta'\to\infty} \eta' \, \tilde{\tilde{w}}^{(1)}(\eta')$$
$$= \eta^2(\frac{\alpha}{2} - \beta)^2 \tilde{\tilde{w}}_0(\eta).$$

Iterating this process recursively, we have (4. 5) and consequently

$D_t^k R(0) = 0 \quad (k = 0,1,2, \dots, [2 + l])$.

Thanks to (4. 4) in Lemma 4. 2, it is not necessary to impose any initial condition on $R(t)$.

Now let us consider some problems as a fundamentals of the arguments held later. Next two lemmas provide explicit solutions to the Cauchy problem and the initial boundary value problems with sufficient regularities of data. Let us assume the following conditions with $l > 0$ arbitrarily provided:

(i) $R_1(t) \in C^\infty(\boldsymbol{R}_+)$;

(ii) $\phi_1(x) \in H_0^l(\boldsymbol{R})$

(iii) $u_{10}(x) \in H_0^{2+l}(\boldsymbol{R})$.

Then, to the Cauchy problem

$$\begin{cases} (\frac{\partial}{\partial t} - L)u_1 = R_1(t)\phi_1(x) \text{ in } \boldsymbol{R} \times \boldsymbol{R}_+, (24) \\ u_1(x,0) = u_{10}(x) \text{ on } \boldsymbol{R}, \end{cases} \quad 4.9$$

a solution $u_1 \in C^\infty(\boldsymbol{R}_+; W_2^{2+l}(\boldsymbol{R}_+))$ uniquely exists, which is represented as follows:

$$u(x,t) = \Gamma_1 * (R_1\phi_1) + \Gamma_1 *_x u_{10}.$$

Proof. Applying the Fourier transform with respect to x to (4. 9), we have

$$\begin{cases} \frac{\partial \hat{u}_1}{\partial t} + \hat{L}(\xi)\hat{u}_1 = R(t)\hat{\phi}(\xi) \text{ in } \boldsymbol{R} \times \boldsymbol{R}_+, (26) \\ \hat{u}_1(\xi,0) = \hat{u}_{10}(\xi) \text{ on } \boldsymbol{R}, \end{cases} \quad 4.10$$

where $\hat{L}(\xi) \equiv i\beta\xi + \frac{\alpha}{2}\xi^2$and

$$\hat{f}(\xi,t) \equiv F_x[f](\xi,t) \equiv \int_0^\infty \exp(-ix\xi) f(x,t) \, \mathrm{d}x$$

is the Fourier transform of a function f with respect to x. The solution to (4. 10) is represented as follows:

$$\hat{u}(\xi,t) = \hat{\phi}(\xi) \int_0^t \exp(-\hat{L}(\xi)(t-\tau)) R(\tau) \, \mathrm{d}\tau + \hat{u}_{10}(\xi) \exp(\hat{L}(\xi)t).$$

Thus, applying the inverse Fourier transform and the well known Parseval's theorem, we directly obtain the desired result.

Next, we consider a boundary value problem as follows:

$$\begin{cases}(\frac{\partial}{\partial t}-L)v_1 = 0 \text{ in } \boldsymbol{R}_+^2, (28)\\ Bv_1(t) = \psi_1(t) \text{ on } \boldsymbol{R}_+, (29)\\ v_1(x,0) = 0 \text{ on } \boldsymbol{R}_+, (30)\\ \lim_{x\to\infty} v_1(x,t) = 0 \text{ on } \boldsymbol{R}_+.\end{cases} \qquad 4.11$$

For (4. 11), we introduce the following lemma. Let $\psi_1(t) \in L_2(\boldsymbol{R}_+) \cap C^\infty(\boldsymbol{R}_+)$, and assume that the compatibility condition of order 1 holds. Then, problem (4.11) has a unique solution $v_1 \in C^\infty(\boldsymbol{R}_+^2)$ of the form:

$$v_1(x,t) = -2\exp(\frac{\beta x}{\alpha})G *_t \psi_1,$$

where $G(x,t)$ is a function defined in the statement of Theorem 4. 1.

Proof. Noting the vanishing initial value of v_1, the temporal Laplace transform to (4. 11) with respect to t yields the following equation with $s \in \boldsymbol{C}$:

$$s\tilde{v}_1(x,s) - L\tilde{v}_1(x,s) = 0.$$

Here $\tilde{f}$ is the temporal Laplace transform of a function f with respect to time:

$$\tilde{f}(x,s) = L_t[f](x,s) \equiv \int_0^\infty \exp(-st)f(x,t)\,\mathrm{d}t.$$

For simplicity of calculation, we introduce a new function $q(x,s)$ and a constant d', which is determined later, satisfying

$$\tilde{v}_1(x,s) = q(x,s)\exp(-d'x).$$

Then $q(x,s)$ satisfies

$$(s-\beta d' - \frac{\alpha}{2}d'^2)q + (\beta+\alpha d')\frac{\partial q}{\partial x} - \frac{\alpha}{2}\frac{\partial^2 q}{\partial x^2} = 0. \qquad 4.12$$

Take $d' = -\frac{\beta}{\alpha}$, and (4. 12) is rewritten as follows:

$$(s+\frac{\beta^2}{2\alpha})q - \frac{\alpha}{2}\frac{\partial^2 q}{\partial x^2} = 0. \qquad 4.13$$

From (4. 13) and the requirement $\lim_{x\to\infty}\tilde{v}_1(x,s) = 0$ due to (4. 11) $_4$, $q(x,s)$ takes form

$$q(x,s) = A(s)\exp(-rx)$$

with a function $A(s)$. Then the boundary condition (4. 11) $_2$ yields

$$A(s) = -\frac{2\widetilde{\psi_1}(s)}{\alpha r+\beta},$$

and we have

$$\tilde{v}_1(x,s) = -\frac{2\tilde{\psi}_1(s)}{\alpha r+\beta}\exp((\frac{\beta}{\alpha}-r)x).$$

By virtue of the well known equality of the Laplace transform

$$L_t[f](s)L_t[g](s) = \int_0^t f(t-\tau)g(\tau)\,\mathrm{d}\tau,$$

we have

$$v_1(x,t) = -2\exp(\frac{\beta x}{\alpha})L_t^{-1}[\frac{exp\ (-rx)}{\alpha r+\beta}] *_t \psi_1,$$

where $L_t^{-1}[f]$ stands for the inverse Laplace transform of a function f with respect to time. In the following, we calculate

$$L_t^{-1}[\frac{exp\ (-rx)}{\alpha r+\beta}] = \frac{1}{2\pi i}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{exp\ (-rx)}{\alpha r+\beta}\exp(st)\ \mathrm{d}s.$$

Let us denote $s=\sigma_1+i\sigma_2$. Then, we choose σ_1 in such a way that the path of integrand lies in the domain of analyticity of the integrand. This means that σ_1 takes the form such as $\sigma_1 = -\frac{\beta^2}{2\alpha}+\frac{\alpha}{2}a^2$ with a constant a. From the analyticity of the integrand with respect to s in the half plane $\sigma_1 > -\frac{\beta^2}{2\alpha}$, it is not difficult to see that it vanishes in the region $t<0$, and we execute following calculations for $t>0$. We have

$$\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{exp\ (-rx)}{\alpha r+\beta}\exp(st)\ \mathrm{d}s = \int_{H(a)}\frac{r\ exp\ (-rx+(\frac{\alpha}{2}r^2-\frac{\beta^2}{2\alpha})t)}{r+\beta\alpha^{-1}}\ \mathrm{d}r$$

$$= \int_{a-i\infty}^{a+i\infty}\frac{r\ exp\ (-rx+(\frac{\alpha}{2}r^2-\frac{\beta^2}{2\alpha})t)}{r+\beta\alpha^{-1}}\ \mathrm{d}r, \qquad 4.14$$

where $a^2 \equiv \frac{2}{\alpha}(\sigma+\frac{\beta^2}{2\alpha})$, and $H(a)$ is the right branch of the hyperbola $\sigma_1^2-\sigma_2^2 = a^2$. Since the function in the integrand is analytic in the region of $\sigma_1 > -\frac{\beta^2}{2\alpha}$, the value of the integral in (4. 14) does not change if the path $H(a)$ is replaced with the path $\mathrm{Re}r = a$. Now Let us define $\zeta \equiv r+\frac{\beta}{\alpha} = a+\frac{\beta}{\alpha}+i\zeta'$, and the right-most-hand side of (4. 14) is equivalent to the following term:

$$\int_{a+\beta\alpha^{-1}-i\infty}^{a+\beta\alpha^{-1}+i\infty}\frac{exp\ (I_0(x,\zeta,t;a))}{\zeta}(\zeta-\frac{\beta}{\alpha})\ \mathrm{d}\zeta \equiv I_1(x,\zeta,t;a)+I_2(x,\zeta,t;a), \qquad 4.15$$

where

$$I_0(x,\zeta,t;a) = -(\zeta-\frac{\beta}{\alpha})x+\{\frac{\alpha}{2}(\zeta-\frac{\beta}{\alpha})^2-\frac{\beta^2}{2\alpha}\}t.$$

In (4. 15), due to the analiticity of the integrand, I_1 does not vary as a tends to $\beta\alpha^{-1}$, and the following equality holds.

$$\lim_{a\to\beta\alpha^{-1}} I_1\,(x,\zeta,t;a) = i\sqrt{\frac{2\pi}{\alpha t}}\exp(\frac{\beta x}{\alpha}-\frac{(x+\beta t)^2}{2\alpha t}) = iJ(x,t).$$

Next, for I_2 in (4. 15), we have the following relationship:

$$\frac{\partial I_2}{\partial x} = \int_{a+\beta\alpha^{-1}-i\infty}^{a+\beta\alpha^{-1}+i\infty}\frac{\beta}{\alpha\zeta}(\zeta-\frac{\beta}{\alpha})\exp(I_0(x,\zeta,t;a))\ \mathrm{d}\zeta$$

$$= \frac{\beta}{\alpha}(iJ+I_2). \qquad 4.16$$

In conjunction with the obvious fact $\lim_{x\to\infty} I_2\,(x,t) = 0$, (4. 16) has a solution

$$I_2(x,t) = -\frac{\beta}{\alpha}\int_x^\infty iJ(z,t)\exp(\frac{\beta}{\alpha}(x-z))\ \mathrm{d}z = iK(x,t).$$

These facts together with (4. 15) yield

$$L_t^{-1}[\frac{exp\,(-rx)}{\alpha r+\beta}]=\frac{1}{2\pi i}(I_1+I_2)=G(x,t),$$

and consequently we have the desired result.

4. 4. Proof of Theorem 4. 1

In this subsection, we prove Theorem 4. 1 on the basis of Lemmas 4. 1–4. 4 introduced in the previous subsection. As we have stated, we first calculate the explicit representation of $w(x,t)$ using $R(t)$, and then $R(t)$ by using $w(0,t)=0$. First, we divide (2. 1) $_1$–(2. 1) $_4$ into the following three problems:

$$\begin{cases}-Lw_\phi=\Lambda R_\infty\phi(x) \quad \text{in } \boldsymbol{R}_+, (37)\\ Bw_\phi|_{x=0}=w_\phi(0)=0, (38)\\ \lim_{x\to\infty} w_\phi(x)=0.\end{cases} \qquad 4.17$$

$$\begin{cases}(\frac{\partial}{\partial t}-L)u=\Lambda\overline{R}(t)\phi(x) \quad \text{in } \boldsymbol{R}\times\boldsymbol{R}_+, (40)\\ u(x,0)=w_0(x)-w_\phi(x) \quad \text{on } \boldsymbol{R},\end{cases} \qquad 4.18$$

$$\begin{cases}(\frac{\partial}{\partial t}-L)v=0 \quad \text{in } \boldsymbol{R}_+\times\boldsymbol{R}_+, (42)\\ Bv|_{x=0}(t)=\bar{\psi}(t)-Bu|_{x=0}(t) \quad \text{on } \boldsymbol{R}_+, (43)\\ v(x,0)=0 \quad \text{on } \boldsymbol{R}_+.\end{cases} \qquad 4.19$$

The solution to (4. 17) after the spacial Laplace transform is provided by

$$\tilde{\tilde{w}}_\phi(\eta)=\frac{\Lambda R_\infty(\tilde{\tilde{\phi}}(\eta)-1)}{\eta(\beta-\frac{\alpha\eta}{2})},$$

which leads to

$$w_\phi(x)=\exp(\frac{\beta x}{\alpha})*_x(H*_x\phi-H).$$

When $\phi(x)=\delta(x-d)$ and $w_0(x)=\delta(x-x_0)$, w_ϕ equals to w_d defined in the previous subsection, right after (4. 2). w_d is the same function to that derived by Gelenbe [8] and Takahashi [17]. Since $w(x,t)=w_\phi(x)+u(x,t)+v(x,t)$ satisfies problem (2. 1), we only have to construct solutions to problems (4. 18) and (4. 19). Now, thanks to Lemma 4. 3, we first have

$$u(x,t)=\Lambda\Gamma_1*(\overline{R}\phi^*(x))+\Gamma_1*_x(w_0-w_\phi)^*.$$

Next, Lemma 4. 4 yields

$$v(x,t)=-2\exp(\frac{\beta x}{\alpha})G*_t(\bar{\psi}-Bu|_{x=0}).$$

Next we calculate $R(t)$. Due to (2. 1) $_5$,

$$w(0,t)=u(0,t)+v(0,t)=0$$

holds, which yields

$$u(0,t)-2\exp(\frac{\beta x}{\alpha})G*_t\{\frac{d}{dt}R(t)+\Lambda\overline{R}(t)-Bu\}|_{x=0}=0.$$

Since $R(0)=0$ by virtue of Lemma 4. 2, applying the Laplace transform to this yields

$$\{\Lambda\tilde{\tilde{R}}(s)C_1(s)+C_2(s)\}-2\tilde{G}(0,s)$$

$$\times \{(s+\Lambda)\tilde{\overline{R}}(s) + R_\infty - \frac{\alpha r - \beta}{2}(\Lambda\tilde{\overline{R}}(s)C_1(s) + C_2(s))\} = 0.$$

Thus, we obtain

$$\tilde{\overline{R}}(s) = \frac{R_\infty - \alpha r C_2(s)}{\alpha r \Lambda C_1(s) - (s+\Lambda)}.$$

It is also to be seen that $R_\infty = 1 - \int_{\boldsymbol{R}_+} w_\phi(x)\ \mathrm{d}x$. Actually, from the representation of $\tilde{\overline{R}}(s)$ above, we have

$$\lim_{t\to\infty} \overline{R}(t) = \lim_{s\to 0} s\,\tilde{\overline{R}}(s) = 1 - R_\infty - \int_{\boldsymbol{R}_+} w_\phi(x)\ \mathrm{d}x,$$

whose left-most-hand side vanishes due to its definition.

Prior to the verification of the existence and uniqueness of the global-in-time solution, we prepare the following lemma. For arbitrary $T > 0$ and functions $f \in W_2^{\frac{2+l}{2}}(0,T)$ and $g \in W_2^l(G)$ in general, the following inequality holds with a constant $C_l > 0$ dependent only on l.

$$\| fg \|_{W_2^{l,\frac{l}{2}}(G_T)} \le \| f \|_{W_2^{\frac{l}{2}}(0,T)} \| g \|_{W_2^l(G)},$$

$$\left\| \int_0^t \exp(\Lambda(t-\tau)) f(\tau)\ \mathrm{d}\tau \right\|_{W_2^{\frac{l}{2}}(0,T)} \le C_l T^{\frac{2-l}{2}} \| f \|_{L_2(0,T)}.$$

For the proof of Lemma 4. 5, we only refer the reader to [21] and omit it here.

We now show the existence and uniqueness of a global-in-time solution in the desired regularity to problem (2. 1). Since the regularity of $R(t)$ as a continuous function is uncertain from the representation (4. 1), it is necessary to calculate it. First, we note the representation

$$R(t) = \frac{\alpha}{2}\int_0^t \exp(-\Lambda(t-\tau))\frac{\partial w}{\partial x}(0,\tau)\ \mathrm{d}\tau, \qquad 4.20$$

which is derived from (2. 1) $_3$. Indeed, by virtue of (2. 1) $_3$ and (2. 1) $_4$, it is easily seen that

$$\frac{\alpha}{2}\frac{\partial w}{\partial x}|_{x=0}(t) = \frac{d}{dt}R(t) + \Lambda R(t),$$

which leads to (4. 20).

Then we consider the successive approximation of w. Define $w_{(0)} = w_0$, and for $m \ge 0$, $w_{(m+1)}$ are recursively defined as a solution to the following problem:

$$\begin{cases} (\frac{\partial}{\partial t} - L)w_{(m+1)} = \frac{\alpha\Lambda\phi(x)}{2}\int_0^t \exp(-\Lambda(t-\tau))\frac{\partial w_{(m)}}{\partial x}(0,\tau)\ \mathrm{d}\tau (46) \\ (x,t) \in \boldsymbol{R}_+^2, (47) \\ w_{(m+1)}(x,0) = w_0(x) \quad x > 0, (48) \\ \lim_{x\to\infty} w_{(m+1)}(x,t) = 0 \quad t > 0, (49) \\ w_{(m+1)}(0,t) = 0 \quad t > 0. \end{cases} \qquad 4.21$$

Making use of Lemmas 4. 3–4. 4, we already have the solution to (4. 21) on a time interval $(0,T)$ with an arbitrary $T > 0$, and the estimate

$$\| w_{(m+1)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+\times(0,T))}$$
$$\leq C_0(\| w_0 \|_{W_2^{1+l}(\boldsymbol{R}_+)} + \left\| \phi(x) \int_0^t \exp(-\Lambda(t-\tau)) \frac{\partial w_{(m)}}{\partial x}(0,\tau)\, \mathrm{d}\tau \right\|_{W_2^{l,\frac{l}{2}}(\boldsymbol{R}_+\times(0,T))})$$

holds with $C_0 > 0$ depending only on constant parameters due to the representation of the solutions. Take a positive constant $M > 0$ such that

$$C_0 \| w_0 \|_{W_2^{1+l}(\boldsymbol{R}^2)} + \| w_{(m)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}^2\times(0,T_{m-1}))} \leq M.$$

Then, by virtue of the inequality

$$\left\| \phi(x) \int_0^t \exp(-\Lambda(t-\tau)) \frac{\partial w_{(m)}}{\partial x}(0,\tau)\, \mathrm{d}\tau \right\|_{W_2^{l,\frac{l}{2}}(\boldsymbol{R}_+\times(0,T))}$$
$$\leq CT^{\frac{2-l}{2}} \| \phi \|_{W_2^l(\boldsymbol{R}_+)} \| w_{(m)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+\times(0,T))}$$

if we take $T' \in (0,T]$ small enough so that

$$T'^{\frac{2-l}{2}} \| \phi \|_{W_2^l(\boldsymbol{R}_+)} \| w_{(m)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+\times(0,T'))} + C_0 \| w_0 \|_{W_2^{1+l}(\boldsymbol{R}_+)} \leq M$$

holds, the sequence $\{w_{(m)}\}$ is bounded in $W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_{T'}^2)$ with T' independent of m. Next, we consider the problem for $\widetilde{w}_{(m+1)} \equiv w_{(m+1)} - w_{(m)}$ obtained by subtracting (4. 21) from itself with m replaced by $m-1$.

$$\begin{cases} (\frac{\partial}{\partial t} - L)\widetilde{w}_{(m+1)} = \frac{\alpha\Lambda\phi(x)}{2} \int_0^t \exp(-\Lambda(t-\tau)) \frac{\partial \widetilde{w}_{(m)}}{\partial x}(0,\tau)\, \mathrm{d}\tau (x,t) \in \boldsymbol{R}_+^2, \\ \widetilde{w}_{(m+1)}(x,0) = 0 \quad x > 0, \\ \lim_{x\to\infty} \widetilde{w}_{(m+1)}(x,t) = 0 \quad t > 0, \\ \widetilde{w}_{(m+1)}(0,t) = 0 \quad t > 0, \end{cases}$$

In the similar manner as above, by taking $T'' \in (0,T']$ small enough, we obtain

$$\| \widetilde{w}_{(m+1)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+\times(0,T''))} < \epsilon' \| \widetilde{w}_{(m)} \|_{W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+\times(0,T''))}$$

with $\epsilon' < 1$, and therefore the sequence $\{w_{(m)}\}_{m=0}^{\infty}$ forms a Cauchy sequence in $W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+ \times (0,T''))$. Thus, the limit $w = \lim_{m\to\infty} w_{(m)}$ exists, which is the desired solution. Up to now, we have constructed the local-in-time solution w. Actually, T'' is determined independently of the data, and we can construct the global solution by the continuation argument [15]. Uniqueness of the global solution is obvious, and we omit the proof of it. Thus, we have the unique existence of a global solution $w \in W_2^{2+l,\frac{2+l}{2}}(\boldsymbol{R}_+^2)$ to (2. 1).

We shall then show it has additional regularity in t and actually belongs to

$$C^\infty(\boldsymbol{R}_+; W_2^{2+l}(\boldsymbol{R}_+)).$$

This is derived by substituting (4. 20) into the representation of w in the statement of Theorem 4. 1. Consequently, again due to (4. 20), $R \in C^{\infty}(\boldsymbol{R}_{+})$ holds. This completes the proof of Theorem 4. 1. Corollary 4. 1 is derived directly from Theorem 4. 1. By making use of (2.1), it is easily seen that

$$\int_0^{\infty} (u+v)(x,t)\ \mathrm{d}x + \int_0^{\infty} w_{\phi}\,(x)\ \mathrm{d}x = 1 - R(t) \quad \forall t > 0$$

holds. When (ϕ, w_0) in (2.1) is replaced with $(\delta(x-l), \delta(x-x_0))$, if we assume $x_0 = 0$, which means $R(0) = 1$, which is inconsistent with (4. 4). Thus, $x_0 > 0$ is a necessary condition.

Stationary solution

In this section, we provide the explicit form of the stationary solution to (2. 1) with sufficient conditions for its existence.

Explicit representation of stationary solution

The following is the second main result of this paper: In addition to the assumptions in Theorem 4. 1, let us assume the following conditions:

(i) $\int_0^{\infty} \overline{R}\,(t)\ \mathrm{d}t < \infty$;

(ii) The solution $w_{\phi}(x)$ to (4. 17) satisfies $w_{\phi}(x) \in C(\boldsymbol{R}_{+}) \cap L_2\,(\boldsymbol{R}_{+})$;

(iii) The spacial Laplace transform of ϕ, denoted by $\tilde{\tilde{\phi}}(\eta)$, is continuously differentiable at $\eta = 0$, and the derivative has a finite negative value at $\eta = 0$: $\frac{d}{d\eta}\tilde{\tilde{\phi}}(\eta)|_{\eta=0} = -d < 0$.

(iv) Then, the solution $w(x,t)$ in Theorem 4. 1 converges to a function $w_{\phi}(x)$ as $t \to \infty$. When $\phi(x)$ is a function in an ordinary sense, condition (iii) above is equal to the condition $\int_0^{\infty} x\,\phi(x)\ \mathrm{d}x = d,$ which implies that the mean distance of the reflection from the point $x = 0$ exists and equals d. When ϕ is the Dirac's delta function, (iii) is satisfied by $\phi = \delta(x-d)$.

Proof. We calculate the stationary solution by using the well known formula of the Laplace transform

$$\lim_{t\to\infty} f\,(t) = \lim_{s\to 0} s\,\tilde{f}(s).$$

Note that the temporal Laplace transform of $\Gamma_1(x,t)$ is provided by

$$\tilde{\Gamma}_1(x,s) = \begin{cases} \dfrac{1}{\alpha r}\exp(\,(\dfrac{\beta}{\alpha} - r)x) & (x \geq 0), \\ \dfrac{1}{\alpha r}\exp(\,(\dfrac{\beta}{\alpha} + r)x)) & (x < 0), \end{cases}$$

with r defined in Section 4,

$$r = \sqrt{\frac{2}{\alpha}(s + \frac{\beta^2}{2\alpha})}.$$

It is obvious that

$$\lim_{t\to\infty} u\,(x,t) = 0 \quad \forall x > 0$$

holds. Next, we consider $\lim_{t\to\infty} v\,(x,t)$. Note that

$$s\tilde{v}(x,s) = -\frac{2s(\tilde{\psi}(s) - L_t[Bu|_{x=0}](s))}{\alpha r + \beta}\exp((\frac{\beta}{\alpha} - r)x),$$

where

$$Bu = \Lambda B[\Gamma_1 * (\overline{R}\phi^*)] + B[\Gamma_1 *_x (w_0 - w_d)^*].$$

Using the representation of $\tilde{\Gamma}_1(x,s)$ above, we have

$$L_t[Bu|_{x=0}](s) = \{\frac{\alpha}{2}(\frac{\beta}{\alpha} + r) - \beta\}(\Lambda\tilde{\overline{R}}(s)C_1(s) + C_2(s)).$$

Thus, we have

$$\lim_{s\to 0} s\,\hat{v}(x,s) = -2\lim_{s\to 0}\frac{\alpha r - \beta}{2\alpha}\exp(\frac{2\beta x}{\alpha})$$

$$\times\{s\tilde{\overline{R}}(s) - \overline{R}(0) + \Lambda\tilde{\overline{R}}(s) - \frac{\alpha r - \beta}{2}(\Lambda\tilde{\overline{R}}C_1(s) + C_2(s))\}.$$

It is easily seen that

$$\lim_{s\to 0} C_1\,(s) = -\frac{1}{\beta},$$

$$\lim_{s\to 0} C_2\,(s) = \frac{1}{\beta}(1 - \int_{\boldsymbol{R}_+} w_\phi\,(x)\;\mathrm{d}x),$$

hold, which leads to

$$\lim_{s\to 0} s\,\hat{v}(x,s) = \frac{2\beta}{\alpha}\exp(\frac{2\beta x}{\alpha})(R_\infty - 1 + \int_{\boldsymbol{R}_+} w_\phi\,(x)\;\mathrm{d}x) = 0.$$

Here, we have made use of the formula

$$\lim_{s\to 0} s\,\tilde{f}(s) = \lim_{t\to\infty} f\,(t),$$

and $R(0) = 0$. This is the desired result. □

The result above implies $\lim_{t\to\infty} w\,(x,t) = w_\phi(x)$, which coincides the existing results.

Boundedness of solution

We finally verify that the solution is non-negative, which has to be satisfied since it represents the probability distribution function at each time. The following proposition guarantees that if (2. 1) has a solution of sufficient regularity, it is non-negative at each time under some assumptions on data. Let $T \in (0,\infty]$ and $l > 1$ be arbitrarily provided. Then it satisfies $w(x,t) \geq 0\ \forall(x,t) \in \boldsymbol{R}_+^2$.

Proof. Due to (4. 1), we have

$$\lim_{t\to 0} R\,(t) = \lim_{t\to 0}\frac{d}{dt}R(t) = 0,$$

$$\lim_{t\to 0} D_t^2\,R(t) = +\infty.$$

Therefore, for $\epsilon > 0$ sufficiently small, $R(t) > 0$ holds on $(0,\epsilon]$. Now let us assume $R(t)$ vanishes on (ϵ,∞), and

$$t_0 \equiv \min\{\, t \in (\epsilon, \infty) | R(t) = 0\}.$$

By virtue of (4. 20), we then have

$$\frac{\partial w}{\partial x}(0,t) \leq 0$$

on a certain interval $[t_1, t_2] \in [\epsilon, t_0]$ with $t_2 < t_0$. Then, since $\lim_{x\to\infty} w\,(x,t) = 0$, there exists a point $(x_3, t_3) \in (0,\infty) \times [t_1, t_2]$ where $w(x,t)$ takes its negative minimum value in this region. Now the condition $l > 1$ and Theorem 4. 1 lead to

$$w(x,t) \in C^{\infty}(\boldsymbol{R}_+; W_2^{3+\alpha}(\boldsymbol{R}_+)) \bigcap C\,(\boldsymbol{R}_+^2)$$

with a certain $\alpha > 0$. Then, thanks to the Sobolev embedding theorem [6] [21], we may deal with it as a regular function, to which the maximum principle [6] is applicable. At (x_3, t_3),

$$\frac{\partial w}{\partial t} \leq 0,\ \frac{\partial w}{\partial x} = 0, D_x^2 w \geq 0.$$

This and $(2.\ 1)_1$ imply $R(t_3) \leq 0$, which contradicts the definition of t_0. From these considerations, we have $R(t) > 0$ on $(0,\infty)$, that is, $\int_0^{\infty} w\,(x,t)\ \mathrm{d}x \leq 1$. Then, applying the maximum principle again, $w(x,t) \geq 0$ holds on $\boldsymbol{R}_+^2$. □

Thanks to Proposition 6. 1, we are able to see that the solution to (2. 1) is non-negative without any lengthy calculations.

Appendix: ***About Compatibility condition***

This appendix focuses on the compatibility condition of initial boundary value problems in a usual sense [19]. Let us consider a general form of the initial boundary value problem of the partial differential equation of parabolic type:

$$\begin{cases} (\dfrac{\partial}{\partial t} - L)w = f \quad (x,t) \in \boldsymbol{R}_+^2, \\ Bw(t)|_{x=0} = g \quad t \in \boldsymbol{R}_+, \\ w(x,0) = w_0 \quad x \in \boldsymbol{R}_+, \\ \lim_{x\to+\infty} w\,(x,t) = 0 \quad t \in \boldsymbol{R}_+. \end{cases}$$

The desired condition consist in the fact that the derivatives

$$D_t^k w|_{t=0}(x) \quad (k = 0,1,2,\dots),$$

which are determined for $t = 0$ by the equation and initial condition, satisfy the boundary condition at $x = 0$. Let us introduce the notations

$$w^{(k)}(x) = \frac{\partial^k w}{\partial t^k}|_{t=0}(x) \quad (k = 0,1,2,\dots).$$

They are determined recursively as follows:

$$w^{(0)}(x) = w_0(x), \quad w^{(k)}(x) = Lw^{(k-1)}(x) + D_t^k f|_{t=0}(x),$$
$$(k = 1,2,\dots).$$

Then, we say that the compatibility conditions of order m with $m \geq 0$ are fulfilled if

$$Bw^{(k)}|_{x=0} = D_t^k g|_{t=0} \quad (0 \leq k \leq m)$$

hold.

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