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Primes in Geometric-Arithmetic Progression

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Abstract

A geometric-arithmetic progression of primes is a set of k primes (denoted by GAP-k) of the form $p_1 r^j + jd$ for fixed p_1, r and d and consecutive j, *i.e*, $\{p_1,\ p_1 r + d,\ p_1 r^2 + 2d,\ p_1 r^3 + 3d,\ \ldots\}$. We study the conditions under which, for $k \geq 2$, a GAP-k is a set of k primes in geometric-arithmetic progression. Computational data (along with the MATHEMATICA codes) containing progressions up to GAP-13 is presented. Integer sequences for the sets of differences d corresponding to the GAPs of orders up to 12 are also presented.

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1. Introduction

Primes in arithmetic progression (denoted by AP-k, $k \geq 3$) refers to k prime numbers that are consecutive terms of an arithmetic progression. For example, 5, 11, 17, 23, 29 is an AP-5, a five-term arithmetic progression of primes with the common difference 6. In this example of five primes in arithmetic progression, the primes are not *consecutive* primes. CPAP-k denotes k consecutive primes in arithmetic progression. An example of CPAP-3 is 47, 53, 59 with the common difference 6. Primes in arithmetic progression have been extensively studied both analytically (see the comprehensive account in [3]) and numerically (see, [6, 7]). The largest known sequences contain up to 26 terms, *i.e*, AP-26 and 10 consecutive primes *i.e*, CPAP-10 (see [1, 8] for the AP-k records and [1, 9] for the CPAP-k records).

The *geometric-arithmetic progression* refers to

$$a,\ ar+d,\ ar^2+2d,\ ar^3+3d,\ \dots . \tag{1.1}$$

The sequence in (1.1) is not be confused with the *arithmetic-geometric progression*, a, $(a+d)r$, $(a+2d)r^2$, $(a+3d)r^3, \dots$, whose terms are composite by construction. Primes in geometric-arithmetic progression is a set of k primes (denoted by GAP-k) that are the consecutive terms of a geometric-arithmetic progression in (1.1). For example 3, 17, 79 is a 3-term geometric-arithmetic progression (*i.e*, a GAP-3) with $a = p_1 = 3$, $r = 5$ and $d = 2$. An example of GAP-5 is, 7, 47, 199, 911, 4423, with $p_1 = 7$, $r = 5$ and $d = 12$. The first term of the GAP-k is called the *start*, r the *ratio* and d the *difference*. The special case of GAP-2 shall be discussed separately.

For $r = 1$, GAP-k reduces to AP-k; in this sense, GAP-k is a generalization of the AP-k. It is possible to generate GAPs with $p_1 = 1$, in which a case the first term of the sequence is 1 and has to be excluded when computing the order (k) as 1 is excluded from the set of primes. Example of one such GAP-5 with $p_1 = 1$, $r = 7$ and $d = 720$ is 1, 727, 1489, 2503, 5281, 20407. One can also have GAPs with $p_1 = r$; an example for a GAP-5, is 5, 139, 353, 967, 3581, with $p_1 = r = 5$ and $d = 114$. There can be GAPs with composite r; an example of such a GAP-3 is 7, 107, 1579 with $p_1 = 7$, a composite $r = 15$ and $d = 2$; and an example for GAP-5 is 11, 919, 14543, 473227, 16509011 with $p_1 = 11$, a composite $r = 35$ and $d = 534$. Relevant examples are presented in Table-1 and Table-2 respectively.

In this section, we shall state the theorems. The proofs of these theorems with the related discussions and analysis shall be done in the next Section-2. The conjectures and open problems shall be presented in the third and concluding section. Following are the statements of the theorems:

Theorem 1.1. Let GAP-k denote the set of k primes forming the sequence $\left\{p_1 r^j + jd\right\}_{j=0}^{k-1}$, for fixed p_1, r and d, where d and r are positive integers, and k is an integer > 1. Then it is necessary that d is even; p_1 is an odd-prime coprime to d; r is an odd-number coprime to d. When $p_1 \neq 1$ and $r \neq 1$, the maximum possible order-k of the set is lesser of the two fixed numbers p_1 and the smallest prime factor of r. When $r = 1$, the maximum order of the set is p_1. When $p_1 = 1$, the maximum order of the set is less than the smallest prime factor of r.

Theorem 1.2. [Factors of d]

1. The values of the differences d for each of the minimal GAP-k, $k \geq 5$ are multiples of a k-dependent factor denoted along with the order k by ($k : \dots$#), where # is the primorial. They are (5 : 3#),
   (6 − 7 : 3#),
   (8 − 11 : 5#),
   (12 − 13 : 7#),
   (14 − 17 : 5#),
   (18 : 7#),

Table 1: Primes in Geometric-Arithmetic Progression with *minimal start* p_1, *minimal ratio* r and the *minimal difference* d.

k	p_1	r	d	Primes of the form, $p_1 * r^n + nd$, for n = 0 to $k-1$	Digits of First	Digits of Last
2	2	2	1	$2 * 2^n + n$	1	1
3	3	3	2	$3 * 3^n + 2n$	1	2
4	5	5	3(2#)	$5 * 5^n + 3(2\#)n$	1	3
5	5	5	14(3#)	$5 * 5^n + 14(3\#)n$	1	4
6	7	7	24(3#)	$7 * 7^n + 24(3\#)n$	1	6
7	7	7	554(3#)	$7 * 7^n + 554(3\#)n$	1	6
8	11	11	2087(5#)	$11 * 11^n + 2087(5\#)n$	2	9
9	11	11	30101(5#)	$11 * 11^n + 30101(5\#)n$	2	10
10	11	11	30101(5#)	$11 * 11^n + 30101(5\#)n$	2	11
11	11	11	14789586(5#)	$11 * 11^n + 14789586(5\#)n$	2	12
12	13	13	388796002(7#)	$13 * 13^n + 388796002(7\#)n$	2	14
13	13	13	$> 160 * 10^7 \times (7\#)$			
$14-17$	17	17	$> 10 * 10^7 \times (5\#)$			
18	19	19	$> 18 * 10^7 \times (7\#)$			
19	19	19	$> 10 * 10^7 \times (11\#)$			
$20-23$	23	23	$> 10 * 10^7 \times (11\#)$			
$24-29$	29	29	$> 11 * 10^7 \times (13\#)$			
$30-31$	31	31	$> 11 * 10^7 \times (13\#)$			
$32-37$	37	37	$> 6 * 10^7 \times (19 * 11\#)$			

$n\#$ is the primorial, $2 \cdot 3 \cdot 5 \cdots p$, where $p \leq n$.
For example, $10\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210$.
The symbol $>$ indicates the computationally obtained lower bound of
the *minimal difference*, d in search for the GAPs of the corresponding orders.

Table 2: Miscellaneous examples of Primes in Geometric-Arithmetic Progression

k	p_1	r	d	Primes of the form, $p_1 * r^n + nd$, for n = 0 to $k-1$	Digits of First	Digits of Last
2	2	5	7	$2 * 5^n + 7n$	1	2
2	13	80	53	$13 * 80^n + 53n$	1	4
2	$2^{4253} - 1$	3	$679 * (2)$	$(2^{4423} - 1) * 7^n + 679 * 2n$	1281	1281
2	$2^{4423} - 1$	7	802	$(2^{4423} - 1) * 7^n + 802n$	1332	1333
3	5	7	$4(3\#)$	$5 * 7^n + 4(3\#)n$	1	3
3	$2^{127} - 1$	3	$3695 * 2$	$(2^{127} - 1) * 3^n + 3695 * 2n$	39	40
3	$2^{521} - 1$	3	$553 * 2$	$(2^{521} - 1) * 3^n + 553 * 2n$	157	158
3	$2^{521} - 1$	19	$4365(3\#)(2^{127} - 1)$	$(2^{521} - 1) * 19^n + 4365(3\#)(2^{127} - 1)n$	157	160
3	$2^{4253} - 1$	19	$347610(3\#)$	$(2^{4253} - 1) * 19^n + 347610(3\#)n$	1281	1283
3	$2^{4423} - 1$	7	$388357 * (3\#)$	$(2^{4423} - 1) * 7^n + 388357(3\#)n$	1332	1334
4	11	35	$12(2\#)$	$11 * 35^n + 12(2\#)n$	2	6
4	$2^{521} - 1$	5	$33936(2\#)$	$(2^{521} - 1) * 5^n + 33936(2\#)n$	157	159
5	47	$2^{31} - 1$	$13554(3\#)$	$47 * (2^{31} - 1)^n + 13554(3\#)n$	2	39
6	19	13	$11(3\#)$	$19 * 13^n + 11(3\#)n$	2	7
6	$2^{31} - 1$	31	$4527(5\#)$	$(2^{31} - 1) * 31^n + 4527(5\#)n$	10	17
7	17	13	$98(3\#)$	$17 * 13^n + 98(3\#)n$	2	8
7	99538463	11	$9785(5\#)$	$99538463 * 11^n + 9785(5\#)n$	8	15
7	$2^{31} - 1$	31	$1732487(5\#)$	$(2^{31} - 1) * 31^n + 1732487(5\#)n$	10	19

Table 2: (*continued*)

k	p_1	r	d	Primes of the form, $p_1 * r^n + nd$, for n = 0 to $k-1$	Digits of First	Digits of Last
8	11	13	71190(3#)	$11 * 13^n + 71190(3\#)n$	2	9
8	13	11	124(5#)	$13 * 11^n + 124(5\#)n$	2	10
8	$2^{31}-1$	31	12730851(5#)	$(2^{31}-1) * 31^n + 12730851(5\#)n$	10	20
9	11	13	84(5#)	$11 * 13^n + 84(5\#)n$	2	10
9	$2^{31}-1$	31	14022681(5#)	$(2^{31}-1) * 31^n + 14022681(5\#)n$	10	22
10	13	11	11459(5#)	$13 * 11^n + 11459(5\#)n$	2	11
10	17	17	7700(5#)	$17 * 17^n + 7700(5\#)n$	2	13
10	$2^{31}-1$	31	45644719(5#)	$(2^{31}-1) * 31^n + 45644719(5\#)n$	10	23
11	13	11	129262(5#)	$13 * 11^n + 129262(5\#)n$	2	12
11	13	13	983234(7#)	$13 * 13^n + 983234(7\#)n$	2	13
11	17	17	10374441(5#)	$17 * 17^n + 10374441(5\#)n$	2	14
11	23	23	253393657(5#)	$23 * 23^n + 253393657(5\#)n$	2	15
11	79	79	253393657(5#)	$79 * 79^n + 868186398403324(5\#)n$	2	21
11	103	103	215704123382748(5#)	$103 * 103^n + 215704123382748(5\#)n$	3	23
11	$2^{19}-1$	19	71022143(5#)	$(2^{19}-1) * 19^n + 71022143(5\#)n$	6	19
12	19	19	719565847(5#)	$19 * 19^n + 719565847(5\#)n$	2	16
12	29	29	2764953906(7#)	$29 * 29^n + 2764953906(7\#)n$	2	18
12	31	31	70238079912(5#)	$31 * 31^n + 70238079912(5\#)n$	2	18
13	19	19	719565847(5#)	$19 * 19^n + 719565847(5\#)n$	2	17

$M_{19} = 2^{19} - 1$, $M_{31} = 2^{31} - 1$, $M_{127} = 2^{127} - 1$, $M_{521} = 2^{521} - 1$, $M_{4253} = 2^{4253} - 1$ and $M_{4423} = 2^{4423} - 1$ are Mersenne primes [10, 11].

$(19:\ 11\#)$,
$(20-23:\ 11\#)$,
$(24-29:\ 13\#)$,
$(30-31:\ 13\#)$,
$(32-37:\ 19*11\#)$,
$(38-41:\ 13\#)$,
$(42:\ 17\#)$,
$(43:\ 19\#)$,
$(44-47:\ 23*17\#)$,
$(48-53:\ 17\#)$,
$(54:\ 29*13\#)$,
$(55-58:\ 29*19*13\#)$,
$(59:\ 29*19\#)$,
$(60-61:\ 31*19*13\#)$,
$(62-67:\ 31*19\#)$,
$(68-71:\ 17\#)$,
$(72-73:\ 37*23*17\#)$,
$(74-79:\ 23\#)$,
$(80-83:\ 41*19\#)$,
$(84-89:\ 31*23\#)$,
$(90-97:\ 23\#)$,
$(98-99:\ 23\#)$,
$(100-101:\ 31*23\#)$,
$(102-103:\ 23\#)$ respectively.

2. The values of the differences d for all minimal GAP-k, $k \geq 5$ are multiples of $(3\#)$.

3. The values of the differences d for all minimal GAP-k, $k \geq 8$ are multiples of $(5\#)$.

4. The values of the differences d for all minimal GAP-k, $k \geq 18$ are multiples of $(7\#)$.

5. The values of the differences d for all minimal GAP-k, $k \geq 19$ are multiples of $(11\#)$.

6. The values of the differences d for all minimal GAP-k, $k \geq 38$ are multiples of $(13\#)$.

Theorem 1.3. Let $\left\{p_1 r^j + jd\right\}_{j=0}^{p'-1}$ be a GAP-p' of order p', where p' is smaller of the two primes p_1 and r_1 the smallest prime factor of r. Then the infinite sequence $\left\{p_1 r^j + jd\right\}_{j=p'}^{\infty}$ does not have any GAPs of order $\geq p'$.

Theorem 1.1 summarizes the conditions on a geometric-arithmetic progression to be a candidate for GAP-k. While Theorem 1.1 restricts the values of the differences d for any GAP-k ($k \geq 3$) to be even, Theorem 1.2 provides additional restrictions on the values of the differences d for minimal GAPs of a given order. Further restrictions on the order of the GAP are provided by Theorem 1.3.

2. Proofs of the Theorems

The following three subsections have the proofs of the theorems along with discussion and analysis, starting with Theorem 1.1.

2.1. Proof of Theorem-1

When d is odd, the alternate terms of the sequence $\left\{p_1 r^j + jd\right\}_{j=0}^{k-1}$, take even values. Hence, d cannot be odd. When $p_1 r^j$ is even then again the alternate terms of the sequence are even. So, it is necessary that d is even and $p_1 r^j$ is odd, ensuring that all the terms of the sequence are odd, a prerequisite for them to be prime. The first term of the sequence is p_1. So, p_1 is necessarily an odd-prime. Since, $p_1 r^j$ is odd it is necessary that r is also odd. For $p_1 r^j + jd$ to be prime it is necessary that p_1 and r are both coprime to d. This proves the theorem except for the part related to the order of the set.

First we consider the scenario $p_1 \neq 1$ and $r \neq 1$. The $(p_1+1)^{\text{th}}$ term of the sequence (obtained for $j = p_1$) is $p_1 r^{p_1} + p_1 d$, which is composite. Hence, $k \leq p_1$. Let r_1 be the smallest prime factor of r. The $(r_1 + 1)^{\text{th}}$ term of the sequence (obtained for $j = r_1$) is $p_1 r^{r_1} + r_1 d = p_1 r_1^{r_1} (r/r_1)^{r_1} + r_1 d$, which is composite. Hence $k \leq r_1$. When $r = 1$ and $p_1 \neq 1$, the sequence simplifies to $\{p_1 + jd\}$, whose $(p_1 + 1)^{\text{th}}$ term is $p_1 + p_1 d$, which is composite. Hence, $k \leq p_1$. When $p_1 = 1$ and $r \neq 1$, the sequence becomes $r^j + jd$, whose first term is 1 (for $j = 0$) and the $(r_1 + 1)^{\text{th}}$ term is $r^{r_1} + r_1 d = r_1^{r_1} (r/r_1)^{r_1} + r_1 d$, which is composite. Since, number 1 is not among the primes, $k \leq (r_1 - 1)$. The case $p_1 = 1$ and $r = 1$ is trivial and generates only one GAP (uniquely fixed with $d = 2$), which is the GAP-3, 3, 5, 7. This completes the proof of the theorem. ■

For every integer, $j \geq 2$, there exists a prime p such that $j < p < 2j$ (see for instance, [4]). The elements of GAPs ($r \neq 1$) grow faster than $2j$. Consequently, GAPs cannot have consecutive primes as its members. Hence, we do not have consecutive primes in geometric-arithmetic progression.

Theorem 1.1 tells us the necessary conditions on p_1, r and d, for a geometric-arithmetic progression to be a candidate for GAP-k. The theorem is *nonconstructive*, giving no clues for a recipe to generate the GAP-k. A recipe is required to choose 'good' triplets (p_1, r, d) in order to generate GAP-k with larger values of k, and GAPs of a given order with large number of digits.

GAP-2 is a pair of primes of the form (p_1, $p_1 r + d$) and consequently structurally much simpler than the larger GAP-k. For GAP-2, theorem 1.1 simplifies to the condition that, $p_1 r$ and d are coprime. For example, with $p_1 = 2$, $r = 6$ and $d = 5$, we have (2, 17); with $p_1 = 3$, $r = 2$ and $d = 1$, (3, 7); and $p_1 = 7$, $r = 100$ and $d = 211$, (7, 911) respectively. In the world of primes, *titanic* is 1000+ digits [1]. Example of a titanic GAP-2 is obtained with $p_1 = M_{4253}$, $r = 7$ and $d = 1422$ as $(M_{4253},\ 7M_{4253} + 1422)$. Here, $M_{4253} = 2^{4253} - 1$ is the 19th Mersenne prime containing 1281 digits. Mersenne primes were chosen, as they are well-known and easy to express [10, 11]. Pairs of primes with specific properties have been extensively studied. For instance, *Sophie Germain primes* have the form $(p,\ 2p + 1)$. With $r = 1$ the GAP-2 further simplifies to the pair $(p,\ p + d)$. Prime pairs such as *twin primes*, $(p,\ p+2)$; *cousin primes*, $(p,\ p+4)$; *sexy primes*, $(p,\ p+6)$, among others have been extensively studied [1].

A GAP-*k* is said to be *minimal* if the minimal start p_1 and the minimal ratio r are equal, *i.e*, $p_1 = r = p$, where p is the smallest prime $\geq k$. Such GAPs have the form $\left\{p * p^j + jd\right\}_{j=0}^{k-1}$. Minimal GAPs with different differences, d do exist. For example, the minimal GAP-5 ($p_1 = r = 5$) has the possible differences, 84, 114, 138, 168, ... and the minimal GAP-6 ($p_1 = r = 7$) has the possible differences, 144, 1494, 1740, 2040, ... A minimal GAP-*k* is further said to be *absolutely minimal* if the difference d is minimum. All the GAPs up to $k = 12$ in Table-1 are absolutely minimal. Computationally obtained lower bounds of d in search for higher-order minimal GAPs are also presented in Table-1. In the context of the absolutely minimal GAPs, it is interesting to note that the absolutely minimal GAP-9 and the absolutely minimal GAP-10 occur for the same value of $d = 903030 = 31 * 971 * (5\#) = 30101 * (5\#)$, where $n\#$ is the primorial, $2 \cdot 3 \cdot 5 \cdots p$, where $p \leq n$. For example, $10\# = 2 \cdot 3 \cdot 5 \cdot 7 = 210$. Consequently, GAP-9 is a complete subset of GAP-10 (in this particular instance, since they have the same d). An individual GAP-9 occurs for a higher $d = 1004250 = (5^2) * 13 * 103 * (5\#) = 33475 * (5\#)$. This is analogous to the situation of AP-4 and AP-5 with the minimal start (which is 5). The corresponding sequence is $\{5 + jd\}$. For $d = 6$ (which is the minimum difference), the AP-4 and AP-5 are 5, 11, 17, 23, and 5, 11, 17, 23, 29 respectively. The next AP-4 and AP-5 again occur at $d = 12$. The individual AP-4 occurs only at $d = 18$, which is 5, 23, 41, 59.

A given pair of start p_1 and ratio r, in general generates a GAP-*k* of a certain order *k* for different values of the difference d. In this note, we shall focus on the set of differences corresponding to the minimal GAPs. The minimal GAP-2, $\left\{2 * 2^j + jd\right\}_{j=0}^{1}$ is a pair of primes, $(2,\ 4 + d) \equiv (2,\ p - 4)$, where p is any prime. Consequently, d belongs to the sequence $\{p - 4\}$, where $\{p\}$ is the infinite sequence of primes. Since, the sequence $\{p\}$ is infinite, the sequence $\{p - 4\}$ is also infinite. We shall cite various integer sequences from *The On-Line Encyclopedia of Integer Sequences* (OEIS) created and maintained by Neil Sloane. For example, the sequence of primes, $\{p\}$ is identified by A000040 in [12]. The infinite sequence $\{p - 4\}$: 1, 3, 7, 9, 13, 15, 19, 25, ..., is A172367 [13]. The integer sequences of the differences d, corresponding to the minimal GAPs of each order 3 to 12 are presented in [14]-[23]. In general, there are no reasons to believe that the sequence of the differences d corresponding to any GAP (minimal or non-minimal) are finite. Analogous sequences for the differences also exist for the primes in arithmetic progression. See [24]-[25] for the sequences of differences corresponding to the primes in arithmetic progression with the minimal start. A study of these integer sequences may provide a pattern, which will potentially guide us in our search for higher order GAPs and APs.

From the computed sequences, we note that the set of differences for a given minimal GAP-k have a *common* k-dependent multiplicative factor. This factor has been indicated as (...#) in Table-1. We have included only the common factor. Individual differences d do have additional factors. For instance, the first difference for the minimal GAP-11 is $443687580 = 2112798(7\#) = 14789586(5\#)$. The second difference is not a multiple of (7#) and hence we have shown the first difference as 14789586(5#) in Table-1.

2.2. Proof of Theorem-2

The proof of the theorem 1.2 is based on modular arithmetic and is presented in Appendix-A. Factors up to $k = 103$ are presented in Table-3. In an arithmetic progression (AP-k) with the minimal start, the pattern of the differences is known to be a multiple of $k\#$, where k is the largest prime $\leq k$ (if k is not a prime). If k is a prime than the common difference is a multiple of $(k-1)\#$ (see [6, 7]). Unlike in the case of the AP-k with the minimal start, there is no obvious pattern in the case of the minimal GAP-k. The factor (...#) is not even monotonic. In none of the cases, it has been possible to establish the factors containing the higher powers of 2, 3, 5, or 7. Theorem 1.2 only gives the restrictions on the common difference d in order for the generating sequence to be a candidate for minimal GAP. The existence of the minimal GAP-k, $k \geq 12$ is yet to be established (numerically or otherwise). The extension of the theorem 1.2 to non-minimal GAPs is also discussed in Appendix-A.

So far, we have considered the GAPs from the sequence $\left\{p_1 r^j + jd\right\}_{j=0}^{k-1}$. The sequence, $\left\{p_1 r^j + jd\right\}$ can have sets of primes for consecutive j, not necessarily starting with $j = 0$. For example, the sequence, $\left\{5 * 5^j + 4j\right\}_{j=7}^{j=9}$ generates the GAP-3, 390653, 1953157, 9765661. Another example is the sequence, $\left\{13 * 13^j + 156497 * (11\#) j\right\}_{j=3}^{j=12}$ generating the GAP-10, 1084552771, 1446403573, 1812367159, 2231796937, 3346287211, 13496563933, 141112064479, 1795775474737, 23302061711251, 302879444689093. Such sets, not starting with $j = 0$, cannot be put in the form $\left\{P_1 R^j + jD\right\}_{j=0}^{j=k-1}$, where P_1, R and D are derived from p_1, r and d.

2.3. Proof of Theorem-3

Theorem 1.1 requires p_1 to be an odd-prime and r to be any odd number, $p' \geq 3$. When $r = 1$, the GAP is reduced to an AP and we have $p' = p_1 \geq 3$. Let us recall that theorem 1.1 forbids GAPs of orders greater than p' throughout the interval $[j = 0, \infty)$. While proving theorem 1.1, we saw that $(p_1+1)^{\text{th}}$ and the $(r_1+1)^{\text{th}}$ terms of the sequence, $\left\{p_1 r^j + jd\right\}_{j=0}^{\infty}$ are composite. The $(np_1 + 1)^{\text{th}}$ term (where, $n = 1, 2, 3, \ldots$) of the sequence (obtained for $j = np_1$) is $p_1 r^{np_1} + np_1 d$. This term is composite and belongs to the interval $[j = p_1, \infty)$. There are only $(p_1 - 1)$ terms between any two successive $(np_1 + 1)^{\text{th}}$ and $(\overline{n+1} p_1 + 1)^{\text{th}}$ terms. So, the interval $[j = p_1, \infty)$ cannot have any

Table 3: The differences d for the minimal GAP of each order are multiples of a common factor

Order-k	Generating Prime p	Common Factor
2	2	1
3	3	2
4	5	2
5	5	3#
6–7	7	3#
8–11	11	5#
12–13	13	7#
14–17	17	5#
18	19	7#
19	19	11#
20–23	23	11#
24–29	29	13#
30–31	31	13#
32–37	37	$19 * 11\#$
38–41	41	13#
42	43	17#
43	43	19#
44–47	47	$23 * 17\#$
48–53	53	17#
54	59	$29 * 13\#$
55–58	59	$29 * 19 * 13\#$
59	59	$29 * 19\#$
60–61	61	$31 * 19 * 13\#$
62–67	67	$31 * 19\#$
68–71	71	17#
72–73	73	$37 * 23 * 17\#$
74–79	79	23#
80–83	83	$41 * 19\#$
84–89	89	$31 * 23\#$
90–97	97	23#
98–99	101	23#
100–101	101	$31 * 23\#$
102–103	103	23#

GAPs of order more than $(p_1 - 1)$. The $(nr_1 + 1)^{\text{th}}$ term (obtained for $j = nr_1$) of the sequence is $p_1 r^{nr_1} + nr_1 d = p_1 r_1^{nr_1}(r/r_1)^{nr_1} + r_1 d$. This is also composite. Similar arguments forbid the GAPs of order more than $(r_1 - 1)$ in the interval $[j = p_1, \infty)$. This proves the theorem. In passing we note that, when $r = 1$, the GAP is reduced to an AP. Then theorem 1.3 tells us that the sequence, $\{p_1 + jd\}_{j=p_1}^{\infty}$ does not have any APs of order $\geq p_1$.

3. Final Comments and Suggestions for Future Research

For a given triplet, (p_1, r, d), the sequence, $\left\{p_1 r^j + jd\right\}_{j=0}^{N}$, may not always generate very many primes. For example, the sequence, $\left\{5 * 3^j + 2j\right\}$, takes prime values for, $j = 0, 1, 7, 29, 49, 83, 436, 536, 1274, \ldots$. The sequence, $\left\{7 * 13^j + 36j\right\}_{j=0}^{1000}$, has only a single pair (*i.e*, a GAP-2) for $j = 0, 1$, which is $(7, 127)$. Similar is the situation for a wide range of (p_1, r, d), making it very hard to find GAPs. GAP-3 and GAP-4 with 159 digits were obtained using the Mersenne primes. Titanic GAP-3 were also obtained.

Numerical data in this article was computed initially (up to GAP-6 in Table-1), using the Microsoft EXCEL [2, 26–31]. MS EXCEL has also found applications in specific problems such as the study of quadratic surfaces in the laboratory [32]; chemical physics [33]; and resistor networks [34]. The primality of the numbers generated by EXCEL was checked using the database of primes at *The Prime Pages* [1] and the *Sequence A000040*, from *The On-Line Encyclopedia of Integer Sequences* (OEIS), created and maintained by Neil Sloane [12]. For higher orders, we are using the *MATHEMATICA* [5]. Search for GAPs with ever larger k and geometric-arithmetic progressions containing larger primes is in progress.

From theorem 1.1, it is evident that the order, k of any GAP-k does not exceed both the starting prime p_1 and the smallest prime factor of the ratio r. Equipped with this fact and the numerical data, we have the following two conjectures.

Conjecture 3.1. [Minimal Start] The minimal starting prime, p_1 in a GAP-k is the smallest prime $\geq k$.

Conjecture 3.2. [Minimal Start and Minimal Ratio] The minimal starting prime, p_1 and minimal ratio, r in a GAP-k is the smallest prime $\geq k$ and $p_1 = r$.

Computational data in Table-1 supports these conjectures up to $k = 12$.

Theorem 1.3 is for the restricted case of GAPs, whose order is any prime $p' \geq 3$ and forbids the existence of GAPs of order p' in the infinite interval $[j = p', \infty)$. Moreover, the theorem is silent about the absence (or existence) of GAPs of orders lower than $(p' - 1)$ in the interval $[j = p_1, \infty)$. It is interesting to note that the ten sequences $\left\{p * p^j + jd\right\}_{j=k}^{1000}$, (for $k = 3$ to 12), for the choice of the absolutely minimal triplets (p, p, d) in Table-1 do not have any GAPs of order 3 to p respectively. Some of them do have one or two GAP-2 in the interval, $[j = k,\ 1000]$ respectively. This numerical data provides room for extending the theorem 1.3 to cases, when the order of minimal GAP is not a prime. This leads to the conjecture.

Conjecture 3.3. [GAP free] Whenever the sequence $\left\{p * p^j + jd\right\}_{j=0}^{k-1}$ has the minimal GAP-k, the rest of the infinite sequence, $\left\{p * p^j + jd\right\}_{j=k}^{\infty}$ does not have any GAPs of orders ≥ 3.

We end this note with several open questions, similar to the ones, which exist for the primes in arithmetic progression [3].

Problem 3.4. Are there arbitrarily long geometric-arithmetic progressions of primes?

Problem 3.5. Are there infinitely many k-term geometric-arithmetic progressions consisting of k primes?

Problem 3.6. Do the prime numbers contain infinitely many geometric-arithmetic progressions of length k for all k?

Problem 3.7. Are there infinitely many GAP-k for any k?

We conjecture that the answer to all the above questions is in the affirmative!

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Appendix-A: Proof of Theorem 1.2

Theorem 1.1 states that the common factor of all differences of any GAP-k, $k \geq 3$ is 2 (*i.e*, d is even). Theorem 1.2 states additional factors of all the differences d of a given minimal GAP-k, $k \geq 5$. The theorem essentially consists of two parts: part-1 is for the specific GAP-k with k up to 103; part-2 has global statements giving the common factor of all differences of any minimal GAP, whose order exceeds a particular number. Its proof is based on modular arithmetic. As we shall soon see, it suffices to demonstrate the procedure of proving the statements in a few specific cases of both part-1 and part-2 respectively. Rest of the statements in the theorem for higher orders can be proved closely following the procedures established for lower orders. In fact the methodology presented can be used to derive results and extend the theorem to still higher orders and importantly to the non-minimal GAPs.

The minimal GAP-5 is defined by the sequence $\left\{5 * 5^j + jd\right\}_{j=0}^{4}$ and the common difference d is restricted in such a way that the defining sequence has 5 primes. The first four terms of this sequence belong to GAP-4. The residues of this sequence $\pmod 3$ are $\{2,\ 1+d,\ 2+2d,\ 1,\ 2+d\}$. Primality requires the second residue $1+d$ such that $d \not\equiv 2 \pmod 3$ and the fifth residue $2+d$ such that $d \not\equiv 1 \pmod 3$. Consequently, $d \equiv 0 \pmod 3$. Otherwise, the second and fifth terms in the defining sequence would be multiples of 3 and not prime. Hence, d is *necessarily* a multiple of 3. Theorem 1.1 restricts the values of the differences d of any GAP-k, $k \geq 3$ to be multiples of 2. Consequently, the values of the differences for GAP-5 are restricted to be multiples of (3#). The fifth residue does not belong to GAP-4 and hence the result is not applicable to GAP-4. The third residue $2+2d$ is degenerate as it gives the same information as the second residue. Among the five residues, the first was *numeric* (*i.e*, free of d) and the third was degenerate.

The defining sequence for the minimal GAP-7 is $\left\{7 * 7^j + jd\right\}_{j=0}^{6}$. The corresponding residues $\pmod 3$ are $\{1,\ 1+d,\ 1+2d,\ 1,\ 1+d,\ 1+2d,\ 1\}$. Since, GAP-6 is defined by the same sequence except for the index, its residues are the same as the first six residues for GAP-7. The second and third residues are sufficient to establish that d is a multiple of 3 for both GAP-6 and GAP-7. Consequently, the values of the differences for GAP-6 and GAP-7 are restricted to be multiples of (3#). The residues $\pmod 5$ are $\{2,\ 4+d,\ 3+2d,\ 1+3d,\ 2+4d,\ 4,\ 3+d\}$. The first and sixth residues are numeric. The second, fourth and fifth residues require $d \not\equiv 1 \pmod 5$, $d \not\equiv 3 \pmod 5$, and $d \not\equiv 2 \pmod 5$ respectively. The third and seventh residues are degenerate. The case, $4 \pmod 5$ remains unaddressed and hence d need not be multiple of 5.

The presence of numeric and degenerate residues of a given defining sequence hinders the larger factors. Rest of the results in part-1 of the theorem are proved closely following the procedure used for GAP-4 to GAP-7. The procedure is straightforward but becomes laborious as the order-k grows. We have used the MATHEMATICA to compute the residues [5]. Following are the residues for GAP-7 $\pmod 5$

```
In[1]:= Clear[p];
p = 7;
PolynomialMod[{p, p*p + d, p*p^2 + 2*d, p*p^3 + 3*d,
p*p^4 + 4*d, p*p^5 + 5*d, p*p^6 + 6*d}, 5]

Out[3]= {2, 4 + d, 3 + 2 d, 1 + 3 d, 2 + 4 d, 4, 3 + d}
```

In part-1 of the theorem, the results are for individual GAP-k, $k \leq 103$. We have demonstrated the procedure up to $k = 7$ and it is straightforward to extend it to higher orders. Part-2 onwards the statements are global and the procedure is as follows. The differences for the minimal GAP-5 are divisible by 3. Since, $5 \equiv 2 \pmod 3$ the result is applicable to all those primes > 5, whose residues $\pmod 3$ are 2. The differences for GAP-7 are divisible by 3 and $7 \equiv 1 \pmod 3$. The result is again applicable to all those primes > 7, whose residues $\pmod 3$ are 1. The result was individually proved for GAP-6 in part-1. The non-zero residues of 3 are $\{1, 2\}$. Consequently the differences for all GAP-k, $k \geq 5$ are divisible by (3#) with the factor 2 coming from theorem 1.1.

The non-zero residues of 5 are $\{1, 2, 3, 4\}$, and the corresponding 4 primes with these residues are $\{11, 17, 13, 19\}$. Note, that $7 \equiv 2 \pmod 5$ but its differences are not divisible by 5 as seen in part-1. We now individually examine the four generating sequences for GAP-11, GAP-17, GAP-13 and GAP-19 respectively and conclude that the differences for each of them are multiples of 5. The factor of (3#) is already established, so we conclude that all GAP-k, $k \geq 19$ have their differences as multiples of (5#). The inequality $k \geq 19$, is refined by using the results in part-1. The factor of (5#) was established for the lower order GAP-k, $k = 8$ to $k = 18$ in part-1. Hence, the values of the differences d for all minimal GAP-k, $k \geq 8$ are multiples of (5#). The set $\{11, 17, 13, 19\}$ was only a *candidate* set. Had the differences for any of the GAP-11, GAP-17, GAP-13 or GAP-19 failed to be divisible by 5, we would have examined the GAPs of higher orders, corresponding to that residue.

The non-zero residues of 7 are $\{1, 2, 3, 4, 5, 6\}$ and the corresponding 6 primes are $\{29, 23, 31, 53, 19, 13\}$. The primes $11 \equiv 4 \pmod 7$ and $17 \equiv 3 \pmod 7$ are not relevant in view of the results in part-1. Following the procedure used for establishing the factors (3#) and (5#), we conclude that the differences d for all minimal GAP-k, $k \geq 18$ are multiples of (7#).

The candidate set of 10 primes corresponding to the non-zero residues of 11 is $\{23, 79, 47, 37, 71, 61, 29, 19, 31, 43\}$. The differences d for each of the GAPs of these orders are divisible by (11#). The largest prime in this set is 79 and a spontaneous result is that all GAP-k, $k \geq 79$ have their differences as multiples of (11#). Using the results from part-1, we refine the inequality and conclude that the differences d for all minimal GAP-k, $k \geq 19$ are multiples of (11#).

The candidate set of 12 primes corresponding to the non-zero residues of 13 is $\{53, 41, 29, 43, 31, 71, 59, 47, 61, 101, 89, 103\}$. The largest prime in this set is 103. Hence, the results in part-1 are up to $k = 103$. The candidate set successfully works and we conclude that the differences d for all minimal GAP-k, $k \geq 38$ are multiples of (13#).

The candidate set of 16 primes corresponding to the non-zero residues of 17 is $\{103, 53, 71, 89, 73, 193, 109, 127, 43, 163, 79, 97, 47, 167, 83, 101\}$. The largest prime in this set is 193.

The candidate set of 18 primes corresponding to the non-zero residues of 19 is $\{191, 59, 79, 61, 43, 101, 83, 103, 199, 67, 163, 107, 89, 109, 167, 149, 131, 37\}$. The largest prime in this set is 199.

The procedure of proving this theorem can be applied to the non-minimal GAPs. The common factor ...# of the differences d of a GAP-k with the start p_1, and ratio r shall be denoted by $(k : p_1,\ r,\ \ldots\#)$. The examples are $(3 : 5, 7, 3\#)$,
$(3 : 2^{521} - 1, 19, 3\#)$,
$(3 : M_{4253} = 2^{4253} - 1, 19, 3\#)$,
$(3 : M_{4423} = 2^{4423} - 1, 7, 3\#)$,
$(3 : M_{4423} = 2^{4423} - 1, 19, 3\#)$,
$(3 : M_{4423} = 2^{4423} - 1, 2^{4423} - 1, 3\#)$,
$(4 : 11, 35, 2\#)$,
$(4 : 2^{521} - 1, 5, 2\#)$,

$(5 : 17, 13, 3\#)$,
$(5 : 47, 2^{31} - 1, 3\#)$,
$(5 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(5 : M_{521} = 2^{521} - 1, 11, 5\#)$,
$(5 : M_{521} = 2^{521} - 1, 521, 5\#)$,
$(5 : M_{607} = 2^{607} - 1, 5, 3\#)$,
$(5 : M_{607} = 2^{607} - 1, 7, 3\#)$,
$(5 : M_{607} = 2^{607} - 1, 11, 5\#)$,
$(5 : M_{607} = 2^{607} - 1, 13, 3\#)$,
$(6 : 19, 13, 3\#)$,
$(6 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(7 : 7, 11, 5\#)$,
$(7 : 7, 13, 3\#)$,
$(7 : 7, 17, 3\#)$,
$(7 : 7, 19, 3\#)$,
$(7 : 11, 7, 3\#)$,
$(7 : 11, 13, 3\#)$,
$(7 : 11, 17, 3\#)$,
$(7 : 13, 7, 3\#)$,
$(7 : 17, 7, \#)$,
$(7 : 19, 7, 3\#)$,
$(7 : 19, 23, 3\#)$,
$(7 : 23, 19, 3\#)$,
$(7 : 99538463, 11, 5\#)$,
$(7 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(7 : M_{521} = 2^{521} - 1, 11, 5\#)$,
$(8 : 11, 13, 3\#)$,
$(8 : 13, 11, 5\#)$,
$(8 : 31, 13, 3\#)$,
$(8 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(9 : 11, 13, 5\#)$,
$(9 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(10 : 13, 11, 5\#)$,
$(10 : M_{31} = 2^{31} - 1, 31, 5\#)$,
$(11 : 11, 13, 7\#)$
$(11 : 13, 11, 5\#)$,
$(11 : 17, 17, 5\#)$,
$(11 : 19, 19, 5\#)$,
$(11 : 23, 23, 5\#)$,
$(11 : 79, 79, 5\#)$,
$(11 : 101, 19, 5\#)$,
$(11 : 101, 101, 5\#)$,
$(11 : 103, 103, 5\#)$,

$(11 : M_{19} = 2^{19} - 1, 19, 5\#)$,
$(11 : 99538463, 11, 5\#)$,
$(11 : 99538463, 13, 7\#)$,
$(11 : 99538463, 19, 5\#)$,
$(11 : M_{31} = 2^{31} - 1, 31, 5\#)$ and
$(11 : M_{521} = 2^{521} - 1, 19, 5\#)$.

The choice of the $(k : p_1, r, ...\#)$ in the above examples includes the cases covered in Table-2.

Appendix-B: MATHEMATICA Codes

Most of the data in this article was computed using the versatile package *MATHEMATICA* [5]. The following program searches for the values of the differences d for the minimal GAP-5, in the range [0, 1000].

```
In[1]:= Clear[p]; p = 5;
gapset5d = {};
Do[If[PrimeQ[{p, p*p + d, p*p^2 + 2*d, p*p^3 + 3*d,
     p*p^4 + 4*d}] == {True, True, True, True, True},
  AppendTo[gapset5d, d]], {d, 0, 10^3}]; gapset5d // Timing

Out[4]= {6.50521*10^-19, {84, 114, 138, 168, 258, 324, 348,
  462, 552, 588, 684, 714, 744, 798, 882, 894, 972}}
```

The output is the set of 17 values of d: {84, 114, 138, . . . , 972}. The above program (christened *runner*) picks the values of d but skips the finer details. The following program (christened *walker*) gives the complete GAP sets corresponding to each d.

```
In[5]:= f[n_, m_] := (5)*(5)^n + n*m;
Column[Table[{m,
   Cases[Table[{f[n, m], f[n + 1, m], f[n + 2, m],
        f[n + 3, m], f[n + 4, m]}, {n, 0, 5}],
        {a1_, a2_, a3_, a4_, a5_} /;
       PrimeQ[{a1, a2, a3, a4, a5}] == {True, True, True,
        True, True}]}, {m, 114, 114}]] // Timing

Out[6]= {4.33681*10^-19, {114, {{5, 139, 353, 967, 3581}}}
```

In the above program we choose the difference 114 and obtained the corresponding GAP-5: {5, 139, 353, 967, 3581}. As the name goes the walker is much slower than the runner and hence, not suitable for generating the sequence of differences d. The runner can be made into an *accelerator* by replacing $\{d, 0, 10\text{^}3\}$ with $\{d, 0, 10\text{^}3, \mathbf{2}\}$ and confining the search to multiples of 2 (as restricted by theorem 1.1). It can be further accelerated by the replacement of $\{d, 0, 10\text{^}3, \mathbf{2}\}$ with $\{d, 0, 10\text{^}3, \mathbf{6}\}$ and refining the search to multiple of $(3\#) = 6$ (as restricted by theorem 1.2). Such replacements are relevant as the numbers grow. It is straightforward to extend the above programs (for GAP-5) to higher orders.

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