Inverse Perfect Domination in Graphs

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Abstract

Let $G$ be a connected simple graph. A dominating set $S \subseteq V(G)$ is called a perfect dominating set of $G$ if each $u \in V(G) \setminus S$ is dominated by exactly one element of $S$. The perfect domination number of $G$, denoted by $\gamma_p(G)$, is the minimum cardinality of a perfect dominating set of $G$. Let $D$ be a minimum perfect dominating set of $G$. A perfect dominating set $S \subseteq (V(G) \setminus D)$ is called an inverse perfect dominating set of $G$ with respect to $D$. The inverse perfect domination number of $G$ denoted by $\gamma_p^{-1}(G)$ is the minimum cardinality of an inverse perfect dominating set of $G$. An inverse perfect dominating set of cardinality $\gamma_p^{-1}(G)$ is called $\gamma_p^{-1}$-set.

In this paper, we show that every integers $k$ and $n$ with $1 \leq k < n$ is realizable as inverse perfect domination number and order of $G$ respectively. Further, we give the characterization of the inverse perfect dominating set with inverse perfect domination numbers of one and two and give some important results.

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1. Introduction

In literature, the concept of domination in graphs introduced by Claude Berge in 1958 and Oystein Ore in 1962 [6] is currently receiving much attention. Following the article of Ernie Cockayne and Stephen Hedetniemi [2], the domination in graphs became an area of study by many researchers. One type of domination in graphs is the perfect domination. This was introduced by Cockayne et al. [1] in the paper Perfect domination in graphs. The Inverse domination in a graph was first found in the paper of Kulli [7] and further read in [3, 5]. Moreover, for the general concepts not mentioned, readers may refer to [4].

Let \( G = (V(G), E(G)) \) be a connected simple graph and \( v \in V(G) \). The neighborhood of \( v \) is the set \( N_G(v) = N(v) = \{ u \in V(G) : uv \in E(G) \} \). If \( S \subseteq V(G) \), then the open neighborhood of \( S \) is the set \( N_G(S) = N(S) = \bigcup_{v \in S} N_G(v) \). The closed neighborhood of \( S \) is \( N_G[S] = N[S] = S \cup N(S) \). A subset \( S \) of \( V(G) \) is a dominating set of \( G \) if for every \( v \in (V(G) \setminus S) \), there exists \( x \in S \) such that \( xv \in E(G) \), i.e., \( N[S] = V(G) \). The domination number \( \gamma(G) \) of \( G \) is the smallest cardinality of a dominating set of \( G \).

A dominating set \( S \subseteq V(G) \) is called a perfect dominating set of \( G \) if each \( u \in V(G) \setminus S \) is dominated by exactly one element of \( S \). The perfect domination number of \( G \), denoted by \( \gamma_p(G) \), is the minimum cardinality of a perfect dominating set of \( G \). Let \( D \) be a minimum dominating set of \( G \). The dominating set \( S \subseteq V(G) \setminus D \) is called an inverse dominating set with respect to \( D \). The minimum cardinality of inverse dominating set is called an inverse domination number of \( G \) and is denoted by \( \gamma^{-1}(G) \). An inverse dominating set of cardinality \( \gamma^{-1}(G) \) is called \( \gamma^{-1} \)-set of \( G \).

Motivated by the definition of inverse dominating set, we define the following variant of inverse domination in graphs. Let \( D \) be a minimum perfect dominating set of \( G \). A perfect dominating set \( S \subseteq (V(G) \setminus D) \) is called an inverse perfect dominating set of \( G \) with respect to \( D \). The inverse perfect domination number of \( G \) denoted by \( \gamma^{-1}_p(G) \) is the minimum cardinality of an inverse perfect dominating set of \( G \). An inverse perfect dominating set of cardinality \( \gamma^{-1}_p(G) \) is called \( \gamma^{-1}_p \)-set.

2. Results

One of the classical result in the domination theory which was introduced by Ore in 1962 state the following theorem:

**Theorem 2.1.** [6] Let \( G \) be a graph with no isolated vertex. If \( S \subseteq V(G) \) is a \( \gamma \)-set, then \( V(G) \setminus S \) is also a dominating set in \( G \).

This motivate us to introduce a variant of domination in graphs, the inverse perfect domination in graphs. Theorem 2.1 guarantees the existence of \( \gamma^{-1}_p \)-set in some graph \( G \). Since the inverse perfect dominating set of any graph \( G \) of order \( n \) cannot be \( V(G) \), it follows that \( \gamma^{-1}_p(G) \neq n \) and hence \( \gamma^{-1}_p(G) < n \).
Since $\gamma_p^{-1}(G)$ does not always exist in a connected nontrivial graph $G$, we denote by $\mathcal{P}(G)$ be a family of all graphs with inverse perfect dominating set. Thus, for the purpose of this study, it is assumed that all connected nontrivial graphs considered belong to the family $\mathcal{P}(G)$. From the definitions, the following result is immediate.

**Remark 2.2.** Let $G$ be a connected graph of order $n \geq 4$. Then

(i) $1 \leq \gamma_p^{-1}(G) < n$.

(ii) $D \cap S = \emptyset$ where $D$ is a $\gamma_p$-set and $S$ is a $\gamma_p^{-1}$-set of $G$.

The next result says that the value of the parameter $\gamma_p^{-1}$ ranges over all positive integers.

**Theorem 2.3.** Given positive integers $k$ and $n$ such that $n \geq 2$ and $1 \leq k < n$, there exists a connected nontrivial graph $G$ with $|V(G)| = n$ and $\gamma_p^{-1}(G) = k$.

**Proof.** Let $H$ be a connected graph of order $k$ and let $D$ of order $k$ be a collections of graphs $I$ with $\gamma(I) = 1$ and $r = \sum_{I \in D} |V(I)|$.

If $V(G) = \bigcup_{v \in V(H), I \in D} V(v + I)$ and $n = k + r$, then the set $D = V(H)$ is a $\gamma_p$-set and the set $S = \bigcup_{x \in V(I), I \in D} \{x\}$, where $\{x\}$ is a dominating set of $I$, is a $\gamma_p^{-1}$-set of $G$.

Thus, $\gamma_p(G) = |D| = k$ and $\gamma_p^{-1}(G) = |S| = |D| = k$.

Moreover,

$$|V(G)| = \sum_{v \in V(H), I \in D} |V(v + I)| = |V(H)| + \sum_{I \in D} |V(I)| = k + r = n.$$  

This proves the assertion.

**Corollary 2.4.** Let $H$ be a connected graph of order $k$ and let $D$ of order $k$ be a collections of graphs $I$ with $\gamma(I) = 1$ and $r = \sum_{I \in D} |V(I)|$. If $V(G) = \bigcup_{v \in V(H), I \in D} V(v + I)$ of order $n$ then $2\gamma_p^{-1} \leq n$. 
Proof. Suppose that \( V(G) = \bigcup_{v \in V(H), I \in \mathcal{D}} V(v + I) \) of order \( n \). Then by proof of Theorem 2.3, \( \gamma_p^{-1}(G) = k \). Now,

\[
\begin{align*}
\sum_{I \in \mathcal{D}} |V(I)|
&= \sum_{I \in \mathcal{D}} |\{x\} \cup (V(I) \setminus \{x\})| \\
&= \sum_{I \in \mathcal{D}} |\{x\}| + \sum_{I \in \mathcal{D}} |V(I) \setminus \{x\}| \\
&= k + \sum_{I \in \mathcal{D}} |V(I) \setminus \{x\}|
\end{align*}
\]

\[\geq k.\]

Note that \( r = k \) if each graph \( I \) is trivial. Moreover, in view of the proof in Theorem 2.3, \( n = k + r \geq k + k = 2k = 2\gamma_p^{-1}(G). \) ■

Now, consider a path \( P_n = [x_1, x_2, \ldots, x_n] \). If \( n = 3k - 1 \) for some \( k \in \mathbb{N} \), then the sets \( D = \{x_{3j-1} : j = 1, 2, \ldots, \frac{n+1}{3}\} \) and \( S = \{x_{3j-2} : j = 1, 2, \ldots, \frac{n+1}{3}\} \) are \( \gamma_p \)-set and \( \gamma_p^{-1} \)-set. Thus, \( \gamma_p^{-1}(G) = \frac{n+1}{3} \). If \( n = 3k - 2 \) for some \( k \in \mathbb{N} \setminus \{1\} \), then the sets \( D = \{x_{3j-2} : j = 1, 2, \ldots, \frac{n+2}{3}\} \) and \( S = \{x_{3j-1} : j = 1, \ldots, \frac{n-1}{3}\} \cup \{x_n-1\} \) are \( \gamma_p \)-set and \( \gamma_p^{-1} \)-set. Thus, \( \gamma^{-1}(G) = \frac{n-1}{3} + 1 = \frac{n+2}{3} \). Thus the following remark holds.

**Remark 2.5.** Let \( G = P_n \in \mathcal{P}(G) \). Then

\[
\gamma_p^{-1}(P_n) = \begin{cases} 
\frac{n+1}{3}, & \text{if } n = 3k - 1 \text{ for some } k \in \mathbb{N} \\
\frac{n+2}{3}, & \text{if } n = 3k - 2 \text{ for some } k \in \mathbb{N} \setminus \{1\}.
\end{cases}
\]

It is worth noting that if the order of \( G = P_n \) is a multiple of 3, then we cannot find an inverse perfect dominating set in \( G \). Thus, the following remark holds.

**Remark 2.6.** Let \( G = P_n \). Then \( G \notin \mathcal{P}(G) \) if \( n = 3k \) where \( k \in \mathbb{N} \).

Next, consider a path \( C_n = [x_1, x_2, \ldots, x_n, x_1] \). If \( n = 3k \) for some \( k \in \mathbb{N} \), then the sets \( D = \{x_{3j-2} : j = 1, 2, \ldots, \frac{n}{3}\} \) and \( S = \{x_{3j-1} : j = 1, 2, \ldots, \frac{n}{3}\} \) are \( \gamma_p \)-set and \( \gamma_p^{-1} \)-set of \( G \). Thus, \( \gamma_p^{-1}(G) = \frac{n}{3} \). If \( n = 3k + 1 \) for some \( k \in \mathbb{N} \), then
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the sets \( D = \left\{ x_{3j-2} : j = 1, 2, \ldots, \frac{n+2}{3} \right\} \) and \( S = \left\{ x_{3j-1} : j = 1, \ldots, \frac{n-1}{3} \right\} \cup \{x_{n-1}\} \) are \( \gamma_p \)-set and \( \gamma_p^{-1} \)-set of \( G \). Thus, \( \gamma^{-1}(G) = \frac{n-1}{3} + 1 = \frac{n+2}{3} \). If \( n = 6k + 2 \) for some \( k \in \mathbb{N} \), then the sets \( D = \left\{ x_{3j-2} : j = 1, 2, \ldots, \frac{n+4}{6} \right\} \cup \left\{ \frac{n+10}{6}, \frac{n+16}{6}, \ldots, \frac{n+4}{3} \right\} \) and \( S = \left\{ x_{3j-1} : j = 1, 2, \ldots, \frac{n-2}{6} \right\} \cup \{x_{n-1}\} \) are \( \gamma_p \)-set and \( \gamma_p^{-1} \)-set of \( G \). Thus, \( \gamma^{-1}(G) = \frac{n+1}{3} + 1 = \frac{n+4}{3} \). Thus the following remark holds.

**Remark 2.7.** Let \( G = C_n \in \mathcal{P}(G) \). Then

\[
\gamma_p^{-1}(C_n) = \begin{cases} 
\frac{n}{3}, & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \\
\frac{n+2}{3}, & \text{if } n = 3k + 1 \text{ for some } k \in \mathbb{N} \\
\frac{n+4}{3}, & \text{if } n = 6k + 2 \text{ for some } k \in \mathbb{N}
\end{cases}
\]

It is worth mentioning that if the order of \( G = C_n \) is \( n = 6k - 1 \) where \( k \in \mathbb{N} \), then we cannot find an inverse perfect dominating set in \( G \). Thus, the following remark holds.

**Remark 2.8.** Let \( G = C_n \). Then \( G \notin \mathcal{P}(G) \) if and only if \( n = 6k - 1 \) where \( k \in N \).

Further observation shows that some special graphs have no inverse perfect dominating set. The following remark holds.

**Remark 2.9.** Let \( F_n = K_1 + P_{n-1} \) and \( W_n = K_1 + C_{n-1} \). Then \( F_n, W_n \notin \mathcal{P}(G) \) if the order \( n \geq 5 \).

Since \( \gamma_p(G) \) is the order of the minimum perfect dominating set of \( G \), it follows that \( \gamma_p(G) \leq \gamma_p^{-1}(G) \). The following remark holds.

**Remark 2.10.** Let \( G \) be a connected nontrivial graph. Then \( \gamma_p(G) \leq \gamma_p^{-1}(G) \).

Suppose that \( \gamma(G) = 1 \) where \( G \) is nontrivial. Let \( S = \{x\} \) be a \( \gamma \)-set of \( G \). Since every \( u \in V(G) \setminus S \) is dominated by exactly one element in \( S \), it follows that \( S \) is a minimum perfect dominating set of \( G \), that is, \( \gamma_p(G) = \gamma(G) \). Moreover, if \( G \in \mathcal{P}(G) \), then it can be verified that \( \gamma_p^{-1}(G) = 1 \), that is, \( \gamma(G) = \gamma_p(G) = \gamma_p^{-1}(G) \). Thus, for \( G \in \mathcal{P}(G) \), the following remark holds.

**Remark 2.11.** Let \( G \) be a connected nontrivial graph. Then \( \gamma(G) = \gamma_p(G) = \gamma_p^{-1}(G) = 1 \).
Theorem 2.12. Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma_p^{-1}(G) = 1$ if and only if $G = K_1 + H$ where $\gamma(H) = 1$.

Proof. Suppose that $\gamma_p^{-1}(G) = 1$. Let $S = V(K_1)$ be a $\gamma_p^{-1}$-set of $G$. Set $V(H) = V(G) \setminus S$. Since $\gamma_p(G) \leq \gamma_p^{-1}(G) = 1$ by Remark 2.10, it follows that $\gamma_p(G) = 1$. Let $D = \{x\}$ be a $\gamma_p$-set of $G$. Since $D \cap S = \emptyset$ by Remark 2.2, it follows that $D \subset V(H)$, that is, $\gamma(H) = 1$. Therefore, $G = K_1 + H$ where $\gamma(H) = 1$.

For the converse, suppose that $G = K_1 + H$ where $\gamma(H) = 1$. Clearly, $\gamma_p(G) = 1$. Let $D = V(K_1)$ be a $\gamma_p$-set of $G$ and $S = \{x\}$ be a dominating set in $H$. Then, $S$ is also a minimum dominating set of $G$ and $S$ is a perfect dominating set of $G$ by definition. Since $S \cap D = \emptyset$, it follows that $S \subseteq (V(G) \setminus D)$, that is, $S$ is a $\gamma_p^{-1}$-set in $G$. Hence, $\gamma_p^{-1}(G) = 1$.

The following results are direct consequences of Theorem 2.12.

Corollary 2.13. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_p^{-1}(G) = 1$ if and only if $G = K_2 + H$ for a subgraph $H$.

Suppose that $\gamma(H_1) = 1 = \gamma(H_2)$. Let $S_1 = \{a\}$ and $S_2 = \{b\}$ be dominating sets of $H_1$ and $H_2$ respectively. Then the graph $G = H_1 + H_2$ may be expressed as $G = (\langle S_1 \rangle + J) + (\langle S_2 \rangle + I)$ where $V(J) = V(H_1) \setminus S_1$ and $V(I) = V(H_2) \setminus S_2$. Thus, $G = (S_1) + (S_2 + J + I) = K_1 + H$ where $\gamma(H) = 1$. Thus the following result is a direct consequence of Theorem 2.12.

Corollary 2.14. Let $G$ and $H$ be connected graphs. Then $\gamma_p^{-1}(G + H) = 1$ if and only if $\gamma(G) = 1 = \gamma(H)$.

Remark 2.15. If $G$ is a complete graph of order $n \geq 2$, then $\gamma_p^{-1}(G) = 1$.

The following lemmas and remark are needed for the characterization of inverse perfect domination number equal to two.

Lemma 2.16. If $D = \{x, y\}$ is a perfect dominating set of $G$, then $N(x) \cap N(y) = \emptyset$ and $N[x] \cup N[y] = V(G)$.

Proof. Suppose that $N(x) \cap N(y) \neq \emptyset$. Let $a \in N(x) \cap N(y)$. Then, $a \in N(x)$ and $a \in N(y)$. This implies that $a \in V(G) \setminus D$ is dominated by $x$ and $y$ contrary to the fact that $D$ is a perfect dominating set of $G$. Thus, $N(x) \cap N(y) = \emptyset$. Suppose that $N[x] \cup N[y] \neq V(G)$. Then there exists $z \in V(G)$ such that $z \notin N[x] \cup N[y]$. Thus, $z \notin N[x]$ and $z \notin N[y]$. This implies that $z$ is not dominated by any element of $D$ contrary to the fact that $D$ is a dominating set of $G$. Thus, $N[x] \cup N[y] = V(G)$. 

Lemma 2.17. If $S = \{a, b\}$ is an inverse perfect dominating set of $G$, then $N(a) \cap
$N(b) = \emptyset$ and $N(a) \cup N(b) = V(G)$.

**Proof.** Since $S = \{a, b\}$ is also a perfect dominating set of $G$, the desired result follows by applying Lemma 2.16.

**Remark 2.18.** If $S = \{a, b\}$ is an inverse perfect dominating set of $G$, then there always exists $D = \{x, y\}$ such that $D$ is a perfect dominating set of $G$ and vertices $x, y, a$ and $b$ are distinct. (Note that $G \in \mathcal{P}(G)$).

**Theorem 2.19.** Let $G$ be a connected non-complete graph of order $n \geq 4$. Then $\gamma^{-1}_p(G) = 2$ if and only if $G \neq K_2 + H$ for any subgraph $H$ and there exists distinct vertices $x, y, a$ and $b$ such that $\{x, y\}$ is a dominating set of $G$, $N(x) \cap N(y) = \emptyset$, $N(a) \cap N(b) = \emptyset$ and one of the following holds.

(i) The vertices $x$ and $y$ are adjacent and

(a) $\gamma((N(x) \setminus \{y\})) = 1$ and $\gamma((N(y) \setminus \{x\})) = 1$; or

(b) $\gamma((N(x) \setminus \{y\})) = 1$ and $\gamma((N(y) \setminus \{x\}) \setminus N(a)) = 1$ and $N(y) \cap N(a) \neq \emptyset$ for some $a$ that dominate $(N(x) \setminus \{y\})$; or

(c) $\gamma((N(x) \setminus \{y\}) \setminus N(b)) = 1$ and $N(x) \cap N(b) \neq \emptyset$ for some $b$ that dominate $(N(y) \setminus \{x\})$ and $\gamma((N(y) \setminus \{x\}) = 1$; or

(d) $\gamma((N(y) \setminus \{x\}) \setminus N(a)) = 1$ and $N(y) \cap N(a) \neq \emptyset$ for some $a$ that dominate $(N(x) \setminus \{y\}) \setminus N(b)$, and $\gamma((N(x) \setminus \{y\}) \setminus N(b)) = 1$ and $N(x) \cap N(b) \neq \emptyset$ for some $b$ that dominate $(N(y) \setminus \{x\}) \setminus N(a)$.

(ii) The vertices $x$ and $y$ are not adjacent and

(a) $\gamma((N(x))) = 1$ and $\gamma((N(y))) = 1$; or

(b) $\gamma((N[y] \setminus N(a)) \cup \{b\}) = 1$ and $N(y) \cap N(a) \neq \emptyset$ for some $a$ that dominate $\langle N(x) \rangle$ and $\gamma((N(x))) = 1$; or

(c) $\gamma((N[x] \setminus N(b)) \cup \{a\}) = 1$ and $N(x) \cap N(b) \neq \emptyset$ for some $b$ that dominate $\langle N(y) \rangle$ and $\gamma((N(y))) = 1$; or

(d) $\gamma((N[x] \setminus N(b)) \cup \{a\}) = 1$ and $N(x) \cap N(b) \neq \emptyset$ for some $b \in N(y)$ and $\gamma((N[y] \setminus N(a)) \cup \{b\}) = 1$ and $N(y) \cap N(a) \neq \emptyset$ for some $a \in N(x)$; or

**Proof.** Suppose that $\gamma^{-1}_p(G) = 2$. Then $\gamma_p(G) \leq 2$ by Remark 2.10. If $\gamma_p(G) = 1$ then $\gamma^{-1}_p(G) = 1$ by Remark 2.11 contrary to our assumption. Thus, $\gamma_p(G) = 2$. Let $D = \{x, y\}$ be a $\gamma_p$-set. Then $D$ is a dominating set of $G$ and $N(x) \cap N(y) = \emptyset$ by Lemma 2.16. Further, let $S = \{a, b\}$ be a $\gamma^{-1}_p$-set. Then $S$ is a dominating set of $G$ and $N(a) \cap N(b) = \emptyset$ by Lemma 2.17. Moreover, vertices $x$, $y$, $a$ and $b$ are distinct by Remark 2.18, that is $D \cap S = \emptyset$. Let $a \in N(x)$ and $b \in N(y)$. Consider the following cases.
Case 1. Suppose that \( xy \in E(G) \).

Subcase 1. If \( S_a = \{a\} \) is a dominating set of \( N(x) \) and \( S_b = \{b\} \) is dominating set of \( N(y) \), then \( \gamma((N(x) \setminus \{y\})) = 1 \) and \( \gamma((N(y) \setminus \{x\})) = 1 \). This proves (i).

Subcase 2. If \( S_a = \{a\} \) is a dominating set of \( N(x) \) and \( S_b = \{b\} \) is not a dominating set of \( N(y) \), then \( \gamma((N(x) \setminus \{y\})) = 1 \) and \( b \in N(y) \) does not dominate \( N(y) \). This implies that there exists \( d \in N(y) \) such that \( d \notin N(b) \). Since \( S = \{a, b\} \) is a dominating set of \( G \), it follows that \( d \in N(a) \). Thus, \( S_b \) is a dominating set of \( N(y) \), that is, \( \gamma((N(y) \setminus \{x\}) \setminus N(a)) = 1 \). Since \( d \in N(y) \cap N(a) \), \( N(y) \cap N(a) \neq \emptyset \) for some \( a \) that dominate \( N(x) \). This proves (ii).

Subcase 3. If \( S_a = \{a\} \) is not a dominating set of \( N(x) \) and \( S_b = \{b\} \) is dominating set of \( N(y) \). Then \( a \in N(x) \) does not dominate \( N(x) \) and \( b \in N(y) \) does not dominate \( N(y) \). This implies that there exists \( c \in N(x) \) such that \( c \notin N(a) \), and there exists \( d \in N(y) \) such that \( d \notin N(b) \). Since \( S = \{a, b\} \) is a dominating set of \( G \), it follows that \( c \in N(b) \). Thus, \( S_a \) is a dominating set of \( N(x) \setminus \{y\} \setminus N(b) \), that is, \( \gamma((N(x) \setminus \{y\}) \setminus N(b)) = 1 \) and \( S_b \) is a dominating set of \( N(y) \setminus \{x\} \setminus N(a) \), that is, \( \gamma((N(y) \setminus \{x\}) \setminus N(a)) = 1 \). Since \( c \in N(x) \cap N(b) \), \( N(x) \cap N(b) \neq \emptyset \) and since \( d \in N(y) \cap N(a) \), \( N(y) \cap N(a) \neq \emptyset \) for some \( a \) that dominate \( N(x) \) and for some \( b \) that dominate \( N(y) \). This proves (iii).

Case 2. Suppose that \( xy \notin E(G) \).

Subcase 1. If \( S_a = \{a\} \) is a dominating set of \( N(x) \) and \( S_b = \{b\} \) is a dominating set of \( N(y) \), then \( \gamma((N(x))) = 1 \) and \( \gamma((N(y))) = 1 \). This proves (iii).
Since $\gamma_p(G)$ exists, there exists a dominating set of $G$ such that $d \notin N(a)$. Since $S = \{a, b\}$ is a dominating set of $G$, it follows that $d \notin N(b)$. Thus, $S_a$ is a dominating set of $\langle (N[x] \setminus N(b)) \cup \{a\} \rangle$, that is, $\gamma(\langle (N[x] \setminus N(b)) \cup \{a\} \rangle) = 1$. Since $d \notin N(b)$, $N(x) \cap N(b) \neq \emptyset$ for some $b$ that dominate $\langle N(y) \rangle$. This proves $(iic)$.

**Subcase 4.** If $S_a = \{a\}$ is not a dominating set of $\langle N(x) \rangle$ and $S_b = \{b\}$ is not a dominating set of $\langle N(y) \rangle$. This implies that there exists $d \in V(G)$ such that $d \notin N(a)$ and there exists $c \in V(G)$ such that $c \notin N(b)$. Since $S = \{a, b\}$ is a dominating set of $G$, it follows that $d \in N(b)$ and $c \in N(a)$. Thus, $S_a$ is a dominating set of $\langle (N[x] \setminus N(b)) \cup \{a\} \rangle$, that is, $\gamma(\langle (N[x] \setminus N(b)) \cup \{a\} \rangle) = 1$ and $S_b$ is a dominating set of $\langle (N[y] \setminus N(a)) \cup \{b\} \rangle$, that is, $\gamma(\langle (N[y] \setminus N(a)) \cup \{b\} \rangle) = 1$. Since $d \notin N(b)$ and $c \in N(y)$, it follows that $N(x) \cap N(b) \neq \emptyset$ for some $b \in V(G)$ and $N(y) \cap N(a) \neq \emptyset$ for some $a \in V(G)$. This proves $(iid)$.

For the converse, suppose that there exist distinct vertices $x, y, a,$ and $b$ such that $\{x, y\}$ is a dominating set of $G$, $N(x) \cap N(y) = \emptyset$, $N(a) \cap N(b) = \emptyset$, and $(i)$ or $(ii)$ holds. Let $D = \{x, y\}$ be a dominating set of $G$. Since $N(x) \cap N(y) = \emptyset$, every vertex $u \in V(G) \setminus D$ is dominated by exactly one vertex in $D$. This implies that $D$ is a perfect dominating set of $G$, that is, $\gamma_p(G) \leq |D|$. Since $G \neq K_2 + H$ for any graph $H$, $\gamma^{-1}_p(G) \neq 1$ by Corollary 2.13. This implies that $\gamma^{-1}_p(G) \geq 2$ and hence $D$ is a $\gamma_p$-set of $G$ by Remark 2.10.

Suppose first that $x$ and $y$ are adjacent and that $(ia)$ holds. Let $S_a = \{a\}$ be a dominating set in $\langle N(x) \setminus \{y\} \rangle$ and $S_b = \{b\}$ be a dominating set in $\langle N(y) \setminus \{x\} \rangle$. Then, $N[a] \supseteq N[x] \setminus \{y\}$ and $N[b] \supseteq N[y] \setminus \{x\}$. Thus,

$$N[a] \cup N[b] \supseteq (N[x] \setminus \{y\}) \cup (N[y] \setminus \{x\}) = N[x] \cup N[y] = V(G).$$

Since $N[a] \cup N[b] \subseteq V(G)$, it follows that $N[a] \cup N[b] = V(G)$, that is, $S = \{a, b\}$ is a dominating set of $G$. Since $N(a) \cap N(b) = \emptyset$, every element $u \in V(G) \setminus S$ is dominated by exactly one element of $S$. Thus, $S$ is a perfect dominating set in $G$. Since $x, y, a,$ and $b$ are distinct vertices in $G$, $D \cap S = \emptyset$, that is, $S \subseteq V(G) \setminus D$ where $D$ is a $\gamma_p$-set of $G$. Thus, $S$ is an inverse perfect dominating set of $G$ with respect to $D$. Since $2 = \gamma_p(G) \leq \gamma^{-1}_p \leq |S| = 2$, it follows that $\gamma^{-1}_p(G) = 2$.

Next, suppose that $(ib)$ holds. Let $\{a\}$ be a dominating set in $\langle N(x) \setminus \{y\} \rangle$ and $\{b\}$ be a dominating set in $\langle (N(y) \setminus \{x\}) \setminus N(a) \rangle$.

$$N[a] \cup N[b] \supseteq [(N[x] \setminus \{y\}) \cup N(a)] \cup [(N[y] \setminus \{x\}) \setminus N(a)] = (N[x] \setminus \{y\}) \cup (N[y] \setminus \{x\}) = N[x] \cup N[y] = V(G).$$
Since $N[a] \cup N[b] \subseteq V(G)$, it follows that $N[a] \cup N[b] = V(G)$. This implies that $S = \{a, b\}$ is a dominating set of $G$. By similar arguments above, $\gamma_p^{-1}(G) = 2$.

Similarly, if any of the conditions (ic) or (id) holds, then it can be shown that $\gamma_p^{-1}(G) = 2$.

Further, if $x$ and $y$ are non-adjacent such that (iia), (iib), (icc), or (iid) holds, then by applying similar arguments used in (i), $\gamma_p^{-1}(G) = 2$. The proof is completed. ■

The following result follows immediately from Theorem 2.19.

**Corollary 2.20.** Let $G$ be a connected graph of order $n \geq 4$. Then $\gamma_p^{-1}(G) = 2$ if $G = K_2 \circ H$ with $\gamma(H) = 1$.

Generally, if $G = H_1 \circ H_2$ where $H_1$ is a connected graph of order $m$ ($m \geq 1$) and $\gamma(H_2) = 1$, then $\gamma_p^{-1}(G) = m$.

**References**


