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Countable extreme Gibbs states in a one-dimensional model with a unique ground state and uniqueness conditions in one-dimensional models

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Abstract

We construct a one-dimensional model with a countably infinite spin space and a unique ground state having infinitely many extreme limit Gibbs states. We also discuss uniqueness conditions in one-dimensional models.

AMS Subject Classification: 82-xx.

Key Words: Hamiltonian, Ground State, Gibbs State, Markov Chain.

1. Introduction

The problem of phase transitions in one-dimensional models has attracted the interest of many researchers recently. It is well known that in usual cases, the limit Gibbs state is unique. But we will present several models with different properties which admit phase transitions.

Many authors dealt with this problem. For example, in Kerimov [10,11], the random fields take values in a countably infinite set. Also, the potential function of nearest neighbors is symmetric with respect to the two arguments and symmetric with respect to the point $x = -1/2$ and the external field is symmetric with respect to the point $x = 1/2$. While in Dyson [4,5] and Kerimov [9] the set of values of random fields is $\{0,1\}$, $\{-1,1\}$, $\{-1,1\}$ respectively, and the interaction is of long range. Kalikow [8] and Spitzer [18] considered the case when the random fields take values in a countably infinite set and the interaction is of nearest neighbors. Finally and in Miyamoto [14] and Sullivan [19] the random fields take values in $\{-1,1\}$ and the interaction is of nearest neighbors but it is spatially inhomogeneous.

In this paper, we construct a model which is an extension to the model in Kerimov and Mallak [13]. We will consider the case when the spin space is countably infinite.

The model will be presented in section 3 while the uniqueness condition will be treated in section 2.

2. Uniqueness Conditions

It is well known that the condition $\sum_{x\in Z} | xU(x)|$, $(U(x)$ is a pair potential of long range) implies uniqueness of limit Gibbs states, see [1], [2], [3], [15] and [16].

In [9] it is proved that in one dimensional anti-ferromagnetic model with the Hamiltonian

$$H(\varphi(x)) = \sum_{x,y\in Z;x>y} U(x-y)\varphi_x\varphi_y - \mu\sum_{x\in Z}\varphi_x$$

where μ is the external field, the potential $U(x)$ is a nonnegative convex function which is extendible to a twice continuously differentiable function such that

$$U(x) \sim Ax^{-\gamma},\ U' \sim -A\gamma x^{-\gamma-1},\ U'' \sim A\gamma(\gamma+1)x^{-\gamma-2},\ \ as\ x\to\infty,\ \ \gamma>1,$$

A is a strong constant, has a unique ground state at low temperatures. It is also proved that this model does not admit phase transitions.

The main question is that: given a one-dimensional model, under what conditions is the limit Gibbs state unique? In [10] the author formulated the following conjecture: Any one dimensional model with discrete (at most countable) spin space and with a unique ground state has a unique Gibbs state if the spin space of this model is finite or the potential of this model is translationally invariant.

In [12] it was shown that under some conditions this conjecture is correct. Now suppose that the model has a unique ground state φ^{gr} satisfying the following stability condition: for any finite set $A\subset Z^1$ with length $|A|$, $H(\varphi^{/}(x) - H(\varphi^{gr}(x)) \geq t\ |A|$ where $t>0$, $|A|$ is the number of sites of A and $\varphi^{/}(x)$ is a perturbation of the ground state $\varphi^{gr}(x)$ on the finite set A.Also suppose that the potential $U(B)$ satisfies the following natural decreasing condition: for any fixed interval I with length n, the expression $\sum_{B\subset Z^1;B\cap I\neq\phi,B\cap(Z^1-I)\neq\phi}U(B)$ grows no faster than n^{α}, $0<\alpha<1$. Then the model has a unique limit Gibbs state at low temperatures as given in [12].

These additional conditions are essential since in [13] a one-dimensional model with two spins and a unique ground state having infinitely many extreme limit Gibbs states was constructed. This model disproves the formulated conjecture in [10].

In the next section we extend the Hamiltonian defined by Kerimov and Mallak [13] to the case when the spin space is countably infinite.

3. The Model

Consider a model of the classical mechanics on the one –dimensional integer lattice Z^1 with the Hamiltonian

$$H(\varphi(x)) = \sum_{x\in Z^1; x<0} U_1(\varphi(x), \varphi(B_{-n(x)-1}) - \sum_{x\in Z^1; x\geq 0} U_2(\varphi(x)) \qquad (1)$$

where the spin variable $\varphi(x)$ takes countable number of values 1, 2, 3, …, m, …, and $\varphi(B_{-n(x)-1})$ is the restriction of the configuration $\varphi(x)$ to the set $B_{-n(x)}$, B_{-n}, $n = 1, 2, \ldots$ in a half-open interval $[-c_n, -c_{-n-1})$, where $c_1 = 0$, $c_n = \sum_{i=1}^{n} 10^{3i+1}$ when $n > 1$, the value of $n(x)$ in (1) is defined by the condition $x \in B_{-n(x)}$.

In order to define the interaction potential U_1 of the model, we set two sequences $a_k = \frac{2}{3} + \sum_{i=1}^{k-1} (\frac{1}{4})^i$ and $b_k = \frac{2}{3} + \sum_{i=1}^{k} (\frac{1}{4})^i$, then we define the sequence of half-open intervals I_k, $k = 1, 2, \ldots$, $I_k = [a_k, b_k)$ and the sequence of positive numbers $P_k = \frac{a_k + b_k}{2}$.

The interaction in the model (1) takes place between points x and the left neighboring intervals $B_{n(x)-1}$. Here we define the interaction potential $U_1(\varphi(x), \varphi(B_{n(x)-1}))$, which specifies the interaction between the spin variable $\varphi(x)$ at the point x and the restriction of the configuration $\varphi(x)$ to the interval $B_{n(x)-1}$ is defined by the relations:

$$U_1(\varphi(x) = 1, \varphi(B_{-n(x)-1})) = \begin{cases} 0, & if \quad \sum\limits_{x\in Z^1;\ x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} = 1; \\ -\ln P_k, & if \quad \sum\limits_{x\in Z^1;\ x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} \in I_k, \ \exists k. \\ -\ln \frac{2}{3}, & if \quad \sum\limits_{x\in Z^1;\ x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} \notin I_k, \ \forall k \end{cases}$$

And for $m \neq 1$,

$$U_1(\varphi(x) = m, \varphi(B_{-n(x)-1})) = \begin{cases} \infty, & if \quad \sum\limits_{x\in Z^1; x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} = 1; \\ \left(\frac{\ln(1-P_k)}{\ln(1-P_k) - 1}\right)^{m-1}, & if \quad \sum\limits_{x\in Z^1; x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} \in I_k; \ \exists k. \\ \left(\frac{\ln 3}{1+\ln 3}\right)^{m-1}, & if \quad \sum\limits_{x\in Z^1; x\in B_{-n(x)-1}} \frac{\varphi(x)}{c_n - c_{n-1}} \notin I_k, \forall k \end{cases}$$

The function $U_2(\varphi(x)$ that plays the role of the external field is defined as

$$U_2(\varphi(x)) = \begin{cases} \ln 2 & , if \quad \varphi(x) = 1 \\ \ln(\tfrac{1}{2})^{m-1} & , if \quad \varphi(x) = m \neq 1. \end{cases}$$

Let $I_V = [-V, V]$ and $[-V, -1] = \bigcup_{i=1}^{r} B_{-i}$. Suppose that the boundary conditions $\varphi^k(x)$, $x \in Z^1 - I_V$ are fixed. The Hamiltonian on the subset I_V is given by

$$H_V(\varphi(x) \mid \varphi^k(x)) = \sum_{x=-V}^{-1} U_1(\varphi(x), \varphi(B_{-n(x)-1})) - \sum_{x=0}^{V} U_2(\varphi(x)).$$

The restriction of the configuration $\varphi(x)$ to the interval I_V will be denoted by $\varphi_V(x)$ and the set of all configurations $\varphi_V(x)$ will be denoted by $\phi(V)$.

The finite-volume Gibbs state in $\phi(V)$ at inverse temperature $\beta = T^{-1}$ and boundary conditions $\varphi^k(x)$ are defined by

$$P_V^k(\varphi_V(x) \mid \varphi^k(x)) = \mathrm{E}_V^{-1} \exp(-\beta H_V(\varphi_V(x) \mid \varphi^k(x)))$$

where $\mathrm{E}_V = \sum_{\varphi_V(x) \in \phi(V)} \exp(-\beta H_V(\varphi_V(x) \mid \varphi^k(x)))$ is the partition function.

An extreme limit Gibbs state is the weak limit of finite-volume Gibbs states. It is well known that the set of all limit Gibbs states coincides with the closed convex hull of the set of weak limits of finite-volume Gibbs states [6].

A configuration $\varphi^{gr}(x)$ is said to be a ground state, if for any perturbation $\varphi^{/}(x)$ of the configuration $\varphi^{gr}(x)$, the expression $H(\varphi^{/}(x)) - H(\varphi^{gr}(x))$ is non-negative.

It follows from the construction of the Hamiltonian that the model (1) can be interpreted as an inhomogeneous Markov chain with countable states 1, 2, 3, ...(see [6], [17]) starting at minus infinity, whose transition probabilities are defined by the following rule:

If the point x belongs to the block $B_{-n(x)}$, then the probability for the variable $\varphi(x)$ depends on the spin variable $\varphi(x)$ belonging to the previous block $B_{-n(x)-1}$; namely, at $\beta = 1$.

If the density of particles in $B_{-n(x)-1}$ is 1, then the probability that $\varphi(x) = 1$ is 1.

If the density of particles in $B_{-n(x)-1}$ belongs to the interval I_k, then the probability that $\varphi(x) = 1$ is P_k.

If the density of particles in $B_{-n(x)-1}$ does not belong to any interval I_k, then the probability that $\varphi(x) = 1$ is $\frac{2}{3}$.

If the point belongs to the interval $[0, \infty)$ then the probability that $\varphi(x) = 1$ is $\frac{2}{3}$.

It can be readily verified that the configuration $\varphi^{gr}(x)=1, \forall x \in Z^1$ is the only ground state of model (1). It follows directly if we notice that, in each case the probability that $\varphi(x)=1$ is $> \frac{1}{2}$.

Obviously, for each k, there exists a configuration $\varphi^k(x)$, such that the value of the density of the particles in each block B_n for all sufficiently large $n=n(k)$ belongs to the interval I_k :

$$\sum_{x\in Z^1; x\in B_{-n}} \frac{\varphi(x)}{c_n - c_{n-1}} \in I_k .$$

Let the value of $\beta=1$. A limit Gibbs state corresponding to the boundary conditions $\varphi^k(x)$ will be denoted by P^k. In spite of the fact that model (1) has a unique ground state, the set of limit Gibbs states of the model is very rich.

The following theorem shows the existence of 'density' limit Gibbs states characterized by the densities of particles in typical configurations.

Theorem 1. At $\beta=1$ model (1) has a countable number of extreme limit Gibbs states P^k.

Proof:

Let P^k be a limit Gibbs state corresponding to the boundary conditions $\varphi^k(x)$. In order to prove the theorem, we show that P^l can not be represented as a finite linear combination of limit Gibbs states P^{l_i}: for any collections $l_1, \dots, l_s$ and $\mu_1, \dots, \mu_s$, where $l_i \neq l$ and $0 < \mu_i \leq 1$, $\forall i = 1,\dots,s$, $P^l \neq \sum_{i=1}^{s} \mu_i P^{l_i}$. For this reason we show that there exists an interval B_{-n}, such that the restriction of the measures P^l and $\sum_{i=1}^{s} \mu_i P^{l_i}$ on B_{-n} are different:

$$P^l[B_{-n}] \neq \sum_{i=1}^{s} \mu_i P^{l_i}[B_{-n}]. \qquad (2)$$

We define B_{-n} as an interval satisfying the condition $n > l_i$, $n > l$ and the densities of particles in the restrictions of the configurations $\varphi^{l_i}(x)$ and $\varphi^l(x)$ to B_{-n} belong to the intervals I_{l_i} and I_l respectively; that is,

$$\sum_{x\in Z^1; x\in B_{-n}} \frac{\varphi^l(x)}{c_n - c_{n-1}} \in I_l$$

$$\sum_{x\in Z^1; x\in B_{-n}} \frac{\varphi^{l_i}(x)}{c_n - c_{n-1}} \in I_{l_i}.$$

Let us define a random variable

$$\chi_{-n} = \sum_{x \in Z^1; x \in B_{-n}} \frac{\varphi(x)}{c_n - c_{n-1}}.$$

We prove relation (2) by showing that for any k and n, $n > k$ and at sufficiently large V,

$$P_V^k(\chi_{-n} \in I_k) > \frac{3}{4} \tag{3}$$

where P_V^k is the Gibbs distribution corresponding to the boundary conditions $\varphi^k(x), x \in Z^1 - [-V, V]$.

Indeed, relation (3) implies (2), since from (3) it follows that if $n > l,$ and $n > \max_i(l_i),$ then $P_V^k(\chi_{-n} \in I_k) > \frac{3}{4}$ and

$$\sum_{i=1}^{s} \mu_i P^{l_i}(\chi_{-n} \in I_l) < 1 - \sum_{i=1}^{s} \mu_i P^{l_i}(\chi_{-n} \in I_{l_i}) < \frac{1}{4}.$$

Suppose that $[-V, -1] = \bigcup_{i=1}^{r} B_{-i}$. It follows from the definition of the potential that all spin variables $\varphi(x), x \in [0, \infty)$ are independent. They take 1, 2, 3, ...with respective probabilities $\frac{2}{3}, \frac{1}{3(2)^1}, \frac{1}{3(2)^2}, \ldots$.Hence the restriction of the Gibbs distribution P_V^k to the set $\varphi(x)$, $x \in [-V, -1]$ can be treated as a one-sided inhomogeneous Markov chain with countably infinite states starting at minus infinity, see [6] and [17].

Thus,

$$P_V^k(\chi_{-n} \in I_k) \geq P_V^k\left(\bigcap_{i=n}^{r} \chi_{-i} \in I_k\right) = P_V^k(\chi_{-r} \in I_k) \prod_{i=r-1}^{n} P_V^k(\chi_{-i} \in I_k \mid \chi_{-i-1} \in I_k).$$

Now we estimate the expression inside the product. By the definition of the potential

$$P_V^k(\varphi(x) = 1 \mid \chi_{-i-1} \in I_k) = P_k.$$

Let us define the sequence of positive numbers $\varepsilon_k = \frac{1}{2(4)^k}$. By the law of large numbers,

$$\begin{aligned}
&P_V^k(\chi_{-i} \in I_k \mid \chi_{-i-1} \in I_k) \\
&\geq P_V^k(|\chi_{-i} - P_k| < \varepsilon_k \mid \chi_{-i-1} \in I_k) \\
&\geq 1 - \frac{1}{|B_{-i}| \varepsilon_k^2} \\
&= 1 - \frac{4^{2k+1}}{10^{3n+1}} \\
&> 1 - \frac{1}{10^{3n-2k}}
\end{aligned}$$

and since $n > k,$ this means $P_V^k(\chi_{-i} \in I_k \mid \chi_{-i-1} \in I_k) > 1 - 10^{-n}$. Hence,

$$P_V^k(\chi_{-r} \in I_k)\prod_{i=r-1}^{n} P_V^k(\chi_{-i} \in I_k \mid \chi_{-i-1} \in I_k) > \prod_{i=r-1}^{n}(1-10^{-i}) > \prod_{i=1}^{\infty}(1-10^{-i}) > \frac{3}{4}.$$

Completing the proof.

Thus the model given by (1) has at least a countable number of limit Gibbs states corresponding to the boundary conditions $\varphi^k(x)$. Since the Gibbs measure P_V^k corresponding to the volume V and the boundary conditions $\varphi^k(x)$ by the definition of the potential depends just on the density of particles outside $[-V,V]$ and in the definition of the potential the set of all possible densities is partitioned into the countable number of classes, we can conclude that the set of all extreme limit Gibbs states is countable.

4. Conclusion

The main result of this work is that we could extend the Hamiltonian defined by Kerimov and Mallak [13] to the case when the spin space is countably infinite keeping the ground state unique and getting a countable number of extreme limit Gibbs states.

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