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Stability of Functional Equations In MPN Space Via Fixed Point And Direct Approach

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Abstract

The purpose of this paper is to establish the Hyers-Ulam-Rassias stability of the Quadratic functional equations
$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x)$
and $f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x)$
in Menger Probablistic normed spaces. We investigate the stability of above equations using fixed point and direct approach.

Keywords: Hyers-Ulam-Rassias stability, Quadratic functional equations, MPN-space.

1. Introduction

One of the interesting question in the theory of non-linear functional analysis involved is the stability problem of functional equations as follows: "When is it true that a mathematical object satisfying a certain property approximately must be close to an object satisfying the property exactly?" The concept of stability problem of functional equations originated from a question of S.M. Ulam [18], concerning the stability of group homomorphism:

Let G_1 be a group and let G_2 be a metric group with metric d (.,.). Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a mapping h: $G_1 \rightarrow G_2$ satisfies the inequalityd $(h(xy), h(x)h(y)) < \delta$ for all x, y $\in G_1$. Then there exists a homomorphism H: $G_1 \rightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? If the answer is affirmative, we would say that equation of homomorphism $H(xy) = H(x) H(y)$ is stable.

In the next year D.H. Hyres [7] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach Spaces. Subsequently the result of Hyers [7] was generalized by Th.M. Rassias [19] for linear mapping by considering an unbounded Cauchy difference.

In 1996, Hyers, G. Isac and Th.M. Rassias [6] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. Stability problems of different type of functional equations have been investigated by a number of researchers (see [5], [17], [20]) using fixed point method.

The theory of probabilistic normed space was introduced by Serstnev in1963 ([1], [2]). In [4] Alsina, Schweizer and Sklar gave a new definition of probabilistic normed spaces which includes Serstnev's as a special case and leads naturally to the identification of the principle class of probabilistic normed spaces, the Menger spaces. The notions of probabilistic metric spaces was introduced by K. Menger [12]. The idea of Menger was to use distribution function instead of non-negative real numbers as values of the metric. We know only Menger proposed the probabilistic concept of distance by replacing the number d (p, q) as distance between points (p, q) by a distribution function F (p, q). This idea leads to large development of probalistic analysis ([3], [13]).

The functional equations

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x) \tag{1.1}$$

and

$$f(2x + y) + f(2x - y) = f(x + y) + f(x - y) + 6f(x) \tag{1.2}$$

are called quadratic functional equations, since the function f (x)= cx^2 is a solution of these quadratic functional equations. In 2011, H.A.Kenary et. al.[10] proved the generalized Hyers-Ulam-Rassias stability of quadratic functional equation (1.1) using fixed point method. Further in 2011, W.G.Park [21] investigated the stability of approximate additive mappings and approximate quadratic mappings in 2-Banach spaces.

In 2002, I. S. Chang and H. M. Kim[11] established the general solution and proved the stability of quadratic functional equation (1.2). In 2013, M. Kumar et.al. [15] proved the Hyers-Ulam-Rassias stability of quadratic functional equations (1.1) and (1.2) in 2-Banach spaces.

2. Preliminaries

This section, adopt some definitions and preliminaries of Menger Probabilistic space, fixed point approach etc.

Definition 2.1 [9]: A function F: R $\to$ [0, 1] is called a distribution function if it is non-decreasing and left continuous, with $\sup_{t\in R} F(t) = 1$ and $\inf_{t\in R} F(t) = 0$.

The class of all distribution functions F with F (0) = 0 is denoted by D_+. ε_0 is the element of D_+ defined by

$$\varepsilon_0 = \begin{cases} 1 & t > 0 \\ 0 & t \le 0 \end{cases}$$

Definition 2.2 [16]: A binary operation *: [0, 1] × [0, 1] $\to$ [0, 1] is said to be t-norm if it satisfies the following conditions:

(1) * is commutative and associative;

(2) * is continuous;

(3) $a * 1 = a$ for all $a \in [0, 1]$;

(4) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.3 [14]: Let X be a real vector space, F a mapping from X toD_+ (for any $x \in X$, F (x) is denoted by F_x) and * a t-norm. The triple (X, F, *) is called a Menger probabilistic normed space (briefly Menger PN-space). If the following conditions are satisfied:

(1) $F_x(0) = 0$, for all $x \in X$;

(2) $F_x(0) = \varepsilon_0$ iff $x = \theta$;

(3) $F_{\alpha x}(t) = F_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in R$, $\alpha \neq 0$ and $x \in X$;

(4) $F_{x+y}(t_1 + t_2) \geq F_x(t_1) * F_x(t_2)$ for all $x, y \in X$ and $t_1, t_2 > 0$.

Definition 2.4 [8]: Let X be a set. A function $d: X \times X \to [0, \infty]$ is called a generalized metric on X if it satisfies the following conditions:

(1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$;

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

Theorem 2.1 [8]: Let (X, d) be a complete generalized metric space and $J: X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for all $x \in X$, either $d(J^n x, J^{n+1}x) = \infty$ for all non-negative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty$ for all $n \geq n_0$;

(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X; d(J^{n_0} x, y) < \infty\}$

(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Definition 2.5 [9]: Let (X, F, *) be a Menger PN-space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} F_{x_{n-x}}(t) = 1$. For all $t > 0$. In this case, x is called the limit of $\{x_n\}$.

Definition 2.6 [16]: The sequence $\{x_n\}$ in Menger PN-space (X, F, *) is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$ there exists some n_0 such that $F_{x_n - x_m}(\delta) < 1-\varepsilon$ for all $m, n > n_0$.

Clearly, every convergent sequence in Menger PN-space is Cauchy. If each Cauchy sequence is convergent sequence in Menger PN-space (X, F, *) then (X, F, *) is called Menger-Probabilistic Banach space (briefly, Menger PB-space).

3. Main Result

3.1 Stability of Quadratic Functional Equations Using Fixed Point Approach

In this section, we prove the Hyers-Ulam-Rassias stability of quadratic functional equations in Menger Probabilistic normed space, using fixed point method.

Definition 3.1: Let (X, F, *) be a Menger PN-spaces and (Y, G, *) be a MengerPB-spaces. A mapping f: X $\rightarrow$ Y is said to be P-approximately quadratic if

$$G_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-16f(x)}(t+s) \geq F_x(t) * F_y(t) \qquad (3.2)$$

for all t, s > 0.

Theorem 3.1: Let f: X $\rightarrow$ Y be a P-approximately quadratic functional equation and there exist $0 < \alpha < \frac{1}{9}$ such that $F_x(3t) \geq F_{\alpha x}(t)$ (3.3)

Then there exists a unique quadratic mapping J: X $\rightarrow$ Y such that

$$G_{f(x)-J(x)}(t) = F_x\left(\frac{1-9\alpha}{3\alpha}\right) \qquad (3.4)$$

for all x ϵ X and t >0.

Proof: Taking y = 0 and s = 5t in (3.2)

$$G_{2f(3x)-18f(x)}(6t) \geq F_x(t)$$

$$G_{f(3x)-9f(x)}(3t) \geq F_x(t) \qquad (3.5)$$

Now, replacing x by $\frac{x}{3}$ in (3.5), we get

$$G_{f(x)-9f\left(\frac{x}{3}\right)}(3t) \geq F_{\frac{x}{3}}(t) \qquad (3.6)$$

Using the Definition (2.3) and replacing t by $\frac{t}{3}$ in (3.6), we obtain

$$G_{f(x)-9f\left(\frac{x}{3}\right)}(t) \geq F_x(t) \qquad (3.7)$$

for all x $\in$ X, t > 0. Let us assume the set P = {m: X $\rightarrow$ Y: m (0) = 0} and the generalized metric in P defined by

$$d(n, m) = \inf\{c \in [0, \infty]/G_{m(x)-n(x)}(ct) \geq F_x(t)\} \qquad (3.8)$$

where inf ϕ = + ∞. It is not difficult to prove that (P, d) is complete [see [19], lemma [2.1]. Now, we suppose a linear mapping T : P $\rightarrow$ P such thatTn (x) = $9n\left(\frac{x}{3}\right)$ for all x $\in$ X. We show that T is strictly contractive mapping with the Lipschtiz Constant 9α. In fact, m, n $\in$ P be such that d (m, n) < c. Then we get,

$$G_{m(x)-n(x)}(ct) \geq F_x(t) \qquad (3.9)$$

Whence

$$G_{Tm(x)-Tn(x)}(9\alpha ct) = G_{9m\left(\frac{x}{3}\right)-9n\left(\frac{x}{3}\right)}(9\alpha ct)$$

$$= G_{m\left(\frac{x}{3}\right)-n\left(\frac{x}{3}\right)}(\alpha ct) \tag{3.10}$$

$\geq F_{\frac{x}{3}}(\alpha t)$

$\geq F_x(t)$

for all $x \in X$, $t > 0$. Then

$d(Tm, Tn) < 9\alpha c$

This means that

$$d(Tm, Tn) \leq 9\alpha d(m, n) \tag{3.11}$$

for all $m, n \in P$. It follows from (3.7) that

$$d(f, Tf) \leq 3\alpha \tag{3.12}$$

Now, using Theorem (2.1) there exist a mapping $J: X \to Y$ satisfying the following:

(a) J is fixed point of T, means

$$J\left(\frac{x}{3}\right) = \frac{1}{9}J(x) \tag{3.13}$$

for all $x \in X$. The mapping J is unique fixed point of T in the set $\psi = \{n \in P: d(m, n) < \infty\}$. This implies that J is unique mapping satisfying (3.13) such that there exist $c \in [0, \infty]$ satisfying

$G_{f(x)-T(x)}(ct) \geq F_x(t)$ for all $x \in X$, $t > 0$. (3.14)

(b) $d(T^n f, J) \to 0$ as $n \to \infty$

This means the equality

$$\lim_{n\to\infty} 9^n f\left(\frac{x}{3n}\right) = J(x) \text{ for all } x \in X. \tag{3.15}$$

(c) $d(f, J) \leq \dfrac{d(f, Jf)}{1-L} \leq \dfrac{3\alpha}{1-9\alpha}$ with $f \in \psi$ and so

$$G_{f(x)-J(x)}\left(\frac{3\alpha t}{1-9\alpha}\right) \geq F_x(t) \tag{3.16}$$

This implies that

$$G_{f(x)-J(x)}(t) \geq F_x\left(\frac{1-9\alpha}{3\alpha}\right) \tag{3.17}$$

Then the inequality (3.4) holds. On the other hand

$$G_{9^n\left[f\left(\frac{3x}{3^n}+\frac{y}{3^n}\right)+f\left(\frac{3x}{3^n}-\frac{y}{3^n}\right)-f\left(\frac{x}{3^n}+\frac{y}{3^n}\right)-f\left(\frac{x}{3^n}-\frac{y}{3^n}\right)-16f\left(\frac{x}{3^n}\right)\right]}\ t+s\ \geq F_{\frac{x}{3^n}}(t) * F_{\frac{y}{3^n}}(s) \tag{3.18}$$

By Definition (2.3) and equation (3.3), we have

$F_{\frac{x}{3^n}}(t) \geq F_{\alpha^n x}(t), F_{\frac{y}{3^n}}(s) \geq F_{\alpha^n y}(s)$

$$F_{\frac{x}{3^n}}(t).F_{\frac{y}{3^n}}(s) \geq F_{\alpha^n x}(t) * F_{\alpha^n y}(s) \text{ for all } x \in X \text{ and } t, s > 0. \tag{3.19}$$

Now since $\lim_{n\to\infty} F_{\alpha^n x}(t).F_{\alpha^n y}(s) = 1$, we have

$$G_{J(3x+y)+J(3x-y)-J(x+y)-J(x-y)-16J(x)}(t+s) = 1 \tag{3.20}$$

for all $x, y \in X$. This complete the proof.

Definition 3.2 Let (X, F, *) be a menger PN-space and (Y, G, *) be a Menger PB-spaces. A mapping f: X→ Y is said to be P-approximately quadratic if

$$G_{f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)}(t+s) \geq F_x(t) * F_y(s) \tag{3.21}$$

for all t, s > 0.

Theorem 3.2: Let f:X→Y be a P-approximately functional equation and there exists

$$0 < \alpha < \frac{1}{4} \text{ such that } F_x(2t) \geq F_{\alpha x}(t) \tag{3.22}$$

Then there exists a unique quadratic mapping J:X→Y such that

$$G_{f(x)-J(x)}(t) = F_x\left(\frac{1-4\alpha}{2\alpha}\right) \tag{3.23}$$

for all x ϵ X and t > 0.

Proof: Putting y = 0 and s =3t in (3.21), we have

$$G_{2f(2x)-8f(x)}(4t) \geq F_x(t) \tag{3.24}$$

$$G_{f(2x)-4f(x)}(2t) \geq F_x(t) \tag{3.25}$$

Replacing x by $\frac{x}{2}$ in (3.25), we have

$$G_{f(x)-4f(\frac{x}{2})}(2t) \geq F_{\frac{x}{2}}(t) \tag{3.26}$$

Using the Definition (2.3) and replacing t by $\frac{t}{2}$ in (3.26), we have

$$G_{f(2x)-4f\left(\frac{x}{2}\right)}(t) \geq F_x(t) \tag{3.27}$$

for all $x \in X,\ t > 0$.

Let us assume the set P = {m: X → Y: m (0) = 0} and the generalized metric in P defined by

$$d(n, m) = \inf\{c \in [0, \infty] / G_{m(x)-n(x)}(ct) \geq F_x(t)\} \tag{3.28}$$

where inf ϕ = + ∞. It is not difficult to prove that (P, d) is complete [see [19], lemma. 2.1]. Now, we suppose a linear mapping T: P → P such that $Tn(x) = 4n\left(\frac{x}{2}\right)$ for all $x \in X$. We show that T is strictly contractive mapping with the Lipschtiz Constant 4α. In fact, m, n ∈ P be such that d(m, n) < C. Then we get,

$$G_{m(x)-n(x)}(ct) \geq F_x(t) \tag{3.29}$$

Whence

$$G_{Tm(x)-Tn(x)}(4\alpha ct) = G_{4m\left(\frac{x}{2}\right)-4n\left(\frac{x}{2}\right)}(4\alpha ct)$$

$$= G_{m\left(\frac{x}{2}\right)-n\left(\frac{x}{2}\right)}(\alpha ct) \tag{3.30}$$

$$\geq F_{\frac{x}{2}}(\alpha t)$$

$$\geq F_x(t)$$

for all $x \in X$, $t > 0$. Then
$d(Tm, Tn) < 4\alpha c$
This means that
$$d(Tm, Tn) \le 4\alpha d(m, n) \tag{3.31}$$
for all $m, n \in P$. It follows from (3.27) that
$$d(f, Tf) \le 2\alpha \tag{3.32}$$
Now, using Theorem (2.1) there exist a mapping J: $X \to Y$ satisfying the following:
(a) J is fixed point of T, means
$$J\left(\frac{x}{2}\right) = \frac{1}{4}J(x) \tag{3.33}$$
for all $x \in X$. The mapping J is unique fixed point of T in the set $\psi = \{n \in P: d(m, n) < \infty\}$.This implies that J is unique mapping satisfying (3.33) such that there exist $c \in [0, \infty]$ satisfying
$$G_{f(x) - T(x)}(ct) \ge F_x(t) \text{ for all } x \in X, t > 0. \tag{3.34}$$
(b) $d(T^n f, J) \to 0$ as $n \to \infty$
This means the equality
$$\lim_{n\to\infty} 4^n f\left(\frac{x}{2^n}\right) = J(x) \text{ for all } x \in X. \tag{3.35}$$
(c) $d(f, J) \le \frac{d(f, Jf)}{1-L} \le \frac{2\alpha}{1-4\alpha}$ with $f \in \psi$ and so
$$G_{f(x)-J(x)}\left(\frac{2\alpha t}{1-4\alpha}\right) \ge F_x(t) \tag{3.36}$$
This implies that
$$G_{f(x)-J(x)}(t) \ge F_x\left(\frac{1-4\alpha}{2\alpha}\right) \tag{3.37}$$
Then the inequality (3.23) holds. On the other hand
$$G_{4^n\left[f\left(\frac{2x}{2^n}+\frac{y}{2^n}\right)+f\left(\frac{2x}{2^n}-\frac{y}{2^n}\right)-f\left(\frac{x}{2^n}+\frac{y}{2^n}\right)-f\left(\frac{x}{2^n}-\frac{y}{2^n}\right)-6f\left(\frac{x}{2^n}\right)\right]}\ t+s\ \ge F_{\frac{x}{2^n}}(t) * F_{\frac{y}{2^n}}(s) \tag{3.38}$$
Using Definition (2.3) and equation (3.3), we have
$$F_{\frac{x}{2^n}}(t) \ge F_{\alpha^n x}(t), F_{\frac{y}{2^n}}(s) \ge F_{\alpha^n y}(s)$$
$$F_{\frac{x}{2^n}}(t).F_{\frac{y}{2^n}}(s) \ge F_{\alpha^n x}(t) * F_{\alpha^n y}(s) \text{ for all } x \in X \text{ and } t, s > 0. \tag{3.39}$$
Now since $\lim_{n\to\infty} F_{\alpha^n x}(t).F_{\alpha^n y}(s) = 1$, we have
$$G_{J(2x+y)+J(2x-y)-J(x+y)-J(x-y)-6J(x)}(t+s) = 1 \tag{3.40}$$
for all $x, y \in X$. This complete the proof.

3.2 Stability of Quadratic Functional Equations (1.1) and (1.2) Using Direct Approach

Theorem 3.3: Let (X, F, *) be a Menger PN-space and (Y, G, *) be a Menger PB-space. A mapping f: $X \to Y$ be a P-approximately quadratic if

$G_{f(3x+y)+f(3x-y)-f(x+y)-f(x-y)-16f(x)}(t+s) \geq F_x(t) * F_y(t)$

for all $x, y \in X$ and t, x ϵ $[0,\infty)$. (3.41)

Then there exists a unique quadratic mapping Q: X $\rightarrow$ Y such that

$G_{Q(x)-f(x)}(t) \geq F_x(t), \quad \forall x, y \in X$ and t >0. (3.42)

Proof: Put y=0 and s = 5t in (3.41) to obtain

$G_{2f(3x)-18f(x)}(6t) \geq F_x(t)$

$G_{f(3x)-9f(x)}(3t) \geq F_x(t)$ (3.43)

Replacing x by $3^n x$ in (3.43), we see that

$G_{f(3^{n+1}x)-9f(3^n x)}(3t) \geq F_{3^n x}(t).$

It follows

$G_{f(3^{n+1}x)-9f(3^n x)}(3^{n+1}t) \geq F_x(t).$

Whence

$G_{\frac{f(3^{n+1}x)}{9^{n+1}}-\frac{f(3^n x)}{9^n}}(3^{-n-1}t) \geq F_x(t).$

If n > m > 0, then

$$G_{\frac{f(3^n x)}{9^n}-\frac{f(3^m x)}{9^m}}\left(\sum_{k=m+1}^{n} 3^{-k-1}t\right)$$

$$\geq G_{\sum_{k=m+1}^{n}\left(\frac{f(3^k x)}{9^k}-\frac{f(3^{k-1}x)}{9^{k-1}}\right)}\left(\sum_{k=m+1}^{n} 3^{-k-1}t\right) \quad (3.44)$$

$$\geq \prod_{k=m+1}^{n} G_{\frac{f(3^k x)}{9^k}-\frac{f(3^{k-1}x)}{9^{k-1}}} \left(3^{-k-1}t\right) \geq F_x(t).$$

Let c > 0 and ε be given. Since $\lim_{k\rightarrow\infty} F_x(t) = 1$, there is some $t_0 > 0$ such that $F_x(t_0) \geq 1 - \varepsilon$. Fix some $t > t_0$. The convergence of the series $\sum_{n=1}^{\infty} 3^{-n-1}t$ shows that there exists some $n_0 \geq 0$ such that for each $n > m \geq n_0$, the inequality $\sum_{k=m+1}^{n} 3^{-k-1}t < c$ holds. It follows that,

$$G_{\frac{f(3^n x)}{9^n}-\frac{f(3^m x)}{9^m}}(c) \geq G_{\frac{f(3^n x)}{9^n}-\frac{f(3^m x)}{9^m}}\left(\sum_{k=m+1}^{n} 3^{-k-1}t_0\right) \geq F_x(t_0) \geq 1-\varepsilon.$$

Hence $\left\{\frac{f(3^n x)}{9^n}\right\}$ is a Cauchy sequence in (Y, G, *). Since (Y, G, *) is a MengerPB-space, this sequence converges to some Q (x) $\in$ Y. Hence, we can define a mapping Q: X $\rightarrow$ Y such that $\lim_{n\rightarrow\infty} G_{\frac{f(3^n x)}{9^n}-Q(x)}(t) = 1$. Moreover, if we put m = 0 in (3.44) we observe that

$$G_{\frac{f(3^n x)}{9^n}-f(x)}\left(\sum_{k=1}^{n} 3^{-k-1}t\right) \geq F_x(t).$$

Therefore,

$$G_{\frac{f(3^n x)}{9^n}-f(x)}(t) \geq F_x\left(\frac{t}{\sum_{k=1}^{n} 3^{-k-1}}\right). \qquad (3.45)$$

Next we will show that Q is quadratic. Let x, y ∈ X, then we have

$$G_{Q(3x+y)+Q(3x-y)-Q(x+y)-Q(x-y)-16Q(x)}(t)$$

$$\geq G_{Q(3x+y)-\frac{f(3^n(3x+y))}{9^n}}\left(\frac{t}{6}\right) * G_{Q(3x-y)-\frac{f(3^n(3x-y))}{9^n}}\left(\frac{t}{6}\right)$$

$$* G_{\frac{f(3^n(x+y))}{9^n}-Q(x+y)}\left(\frac{t}{6}\right) * G_{\frac{f(3^n(x-y))}{9^n}-Q(x-y)}\left(\frac{t}{6}\right) * G_{16\frac{f(3^n x)}{9^n}-16Q(x)}\left(\frac{t}{6}\right)$$

$$* G_{\frac{f(3^n(3x+y))}{9^n}+\frac{f(3^n(3x-y))}{9^n}-\frac{f(3^n(x+y))}{9^n}-\frac{f(3^n(x-y)}{9^n}-16\frac{f(3^n x)}{9^n}}\left(\frac{t}{6}\right).$$

In above inequality, we can see that the first five terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, $t \to \infty$ and the sixth term, by (3.41) is greater than or equal to $F_{3^n x}\left(\frac{9^n t}{12}\right) * F_{3^n y}\left(\frac{9^n t}{12}\right) = F_x\left(\frac{3^n t}{12}\right) * F_y\left(\frac{3^n t}{12}\right)$, which tends to 1 as $n \to \infty$. Therefore $Q(3x+y)+Q(3x-y) = Q(x+y)+Q(x-y)+16Q(x)$. Now, we approximate the difference between f and Q. For every x ∈ X and t > 0, by (3.45) for large enough n, we obtain

$$G_{Q(x)-f(x)}(t) \geq G_{Q(x)-\frac{f(3^n x)}{9^n}}\left(\frac{t}{2}\right) * G_{\frac{f(3^n x)}{9^n}-f(x)}\left(\frac{t}{2}\right) \geq F_x(t).$$

Let Q´ be another quadratic function from X to Y which satisfies (3.42). We have

$$G_{Q(x)-Q'(x)}(t) \geq G_{Q(x)-f(x)}\left(\frac{t}{2}\right) * G_{f(x)-Q'(x)}\left(\frac{t}{2}\right) \geq F_x(t)$$

for each t > 0. Therefore, we can say that the mapping Q = Q´.

Theorem 3.4: Let (X, F, *) be a Menger PN-space and (Y, G, *) be a Menger PB-space. A mapping f: X → Y be a P-approximately quadratic if

$$G_{f(2x+y)+f(2x-y)-f(x+y)-f(x-y)-6f(x)}(t+s) \geq F_x(t) * F_y(s)$$

$$\forall x, y \in X \text{ and } t, x\epsilon[0, \infty) . \qquad (3.46)$$

Then there exists a unique quadratic mapping Q: X → Y such that

$$G_{Q(x)-f(x)}(t) \geq F_x(t), \qquad \forall x, y \in X \text{ and } t > 0. \qquad (3.47)$$

Proof: Put x = y and s = 3t in (3.46) to obtain

$$G_{2f(x)-8f(x)}(4t) \geq F_x(t)$$

$$G_{f(2x)-4f(x)}(2t) \geq F_x(t) \qquad (3.48)$$

Replacing x by $2^n x$ in (3.48), we see that

$$G_{f(2^{n+1}x)-4f(2^n x)}(2t) \geq F_x(t).$$

It follows

$G_{f(2^{n+1}x)-4f(2^n x)}(2^{n+1}t) \geq F_x(t).$

Whence

$G_{\frac{f(2^{n+1}x)}{4^{n+1}}-\frac{f(2^n x)}{4^n}}(2^{-n-1}t) \geq F_x(t).$

If n > m > 0, then

$$G_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}\left(\sum_{k=m+1}^{n} 2^{-k-1}t\right)$$

$$\geq G_{\sum_{k=m+1}^{n}\left(\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}\right)}\left(\sum_{k=m+1}^{n} 2^{-k-1}t\right) \tag{3.49}$$

$$\geq \prod_{k=m+1}^{n} G_{\frac{f(2^k x)}{4^k}-\frac{f(2^{k-1}x)}{4^{k-1}}}\left(2^{-k-1}t\right) \geq F_x(t).$$

Let c > 0 and ε be given. Since $\lim_{k=m+1} F_x(t) = 1,$ there is some $t_0 > 0$ such that $F_x(t_0) \geq 1 - \varepsilon$. Fix some $t > t_0$. The convergence of the series $\sum_{n=1}^{\infty} 2^{-n-1}t$ show that there exists some $n_0 \geq 0$ such that for each $n > m \geq n_0$, the inequality $\sum_{k=m+1}^{n} 2^{-k-1}t < c$ holds. It follows that,

$$G_{\frac{f(2^n x)}{4^n}-\frac{f(2^m x)}{4^m}}(c) \geq G_{\frac{f(2^n x)}{4n}-\frac{f(2^m x)}{4^m}}\left(\sum_{k=m+1}^{n} 2^{-k-1}t_0\right) \geq F_x(t_0) \geq 1-\varepsilon.$$

Hence $\left\{\frac{f(2^n x)}{4^n}\right\}$ is a Cauchy sequence in (Y, G, *). Since (Y, G, *) is a Menger PB-space, this sequence converges to some Q (x) ∈ Y. Hence, we can define a mapping Q: X → Y such that $\lim_{n\to\infty} G_{\frac{f(2^n x)}{4^n}-Q(x)}(t) = 1.$ Moreover, if we put m = 0 in (3.49) we observe that

$$G_{\frac{f(2^n x)}{4^n}-f(x)}\left(\sum_{k=1}^{n} 2^{-k-1}t\right) \geq F_x(t).$$

Therefore,

$$G_{\frac{f(2^n x)}{4^n}-f(x)}(t) \geq F_x\left(\frac{t}{\sum_{k=1}^{n} 2^{-k-1}}\right). \tag{3.50}$$

Next we will show that Q is quadratic. Let x, y ∈ X, then we have

$$G_{Q(2x+y)+Q(2x-y)-Q(x+y)-Q(x-y)-6f(x)}(t)$$

$$\geq G_{Q(2x+y)-\frac{f(2^n(2x+y))}{4^n}}\left(\frac{t}{6}\right) * G_{Q(2x-y)-\frac{f(2^n(2x-y))}{4^n}}\left(\frac{t}{6}\right)$$

$$* G_{\frac{f(2^n(x+y))}{4^n}-2Q(x+y)}\left(\frac{t}{6}\right) * G_{\frac{f(2^n(x-y)}{4^n}-2Q(x-y)}\left(\frac{t}{6}\right) * G_{6\frac{f(2^n x)}{4^n}-6Q(x)}\left(\frac{t}{6}\right)$$

$$*G_{\frac{f(2^n(2x+y))}{4^n}+\frac{f(2^n(2x-y))}{4^n}-\frac{f(2^n(x+y))}{4^n}-\frac{f(2^n(x-y)}{4^n}-\frac{f(2^n x)}{4^n}}\left(\frac{t}{6}\right).$$

We can see that the first five terms on the right hand side of the above inequality tend to 1 as $n \to \infty$, $t \to \infty$ and the sixth term, by (3.46) is greater than or equal to $F_{2^n x}\left(\frac{4^n t}{10}\right) * F_{2^n y}\left(\frac{4^n t}{10}\right) = F_x\left(\frac{2^n t}{10}\right) * F_y\left(\frac{2^n t}{10}\right)$, which tends to 1 as $n \to \infty$. Therefore $Q(2x+y)+Q(2x-y)=Q(x+y)+Q(x-y)+6Q(x)$. Next we approximate the difference between f and Q. For every $x \in X$ and $t > 0$, by (3.50) for large enough n, we have

$$G_{Q(x)-f(x)}(t) \geq G_{Q(x)-\frac{f(2^n x)}{4^n}}\left(\frac{t}{2}\right) * G_{\frac{f(2^n x)}{4^n}-f(x)}\left(\frac{t}{2}\right) \geq F_x(t).$$

Let Q´ be another quadratic function from X to Y which satisfies (3.47). We have

$$G_{Q(x)-Q'(x)}(t) \geq G_{Q(x)-f(x)}\left(\frac{t}{2}\right) * G_{f(x)-Q'(x)}\left(\frac{t}{2}\right) \geq F_x(t)$$

for each $t > 0$. Therefore, we can say that Q = Q´.

References

1. A. N. Serstnev, On the motion of a random normed spaces, Doklady Akademii Nauk SSSR, (149) (2) (1963), 280-283, English translation in Soviet Math. Doklady (4) (1963), 388-390.
2. A. N. Serstnev, On the notion of a random normed space, Dokl. Akad. Nauk. (149) (1963), 280-283.
3. B. Schweizer and A. Sklar, Statistical metric spaces, Pacific Journal of Mathematics (10) (1960), 313-334.
4. C. Alsina, B. Schweizer and A. Sklar, On the definition of a probabilistic normed space, Aequationes Math., (46) (1993), 91-98.
5. D. Deses, On the representation of non-Archimedean objects, Topology Appl. (153) (5-6) (2005), 774-785.
6. D. H. Hyers, G. Isac, and Th. M. Rassias, Stability of Functional in Equations Several Ariables Birkhauser, Basel (1998).
7. D.H. Hyers, On the stability of linear functional equation, Proc. Nat. Acad. Sci. U.S.A. (27) (1941), 222-224.
8. E. Movahednia, Fixed point and Generalized Hyers-Ulam-Rassias stability of a Quadratic functional equations, J. of Math. and Comp. Sci., (6) (2013), 72-78.
9. E. Movahednia and S. Eshtehar, Fixed point and Hyers-Ulam-Rassias stability of a Quadratic functional equations in MPN space, J. of Math. and Comp. Sci., (5) (1) (2012), 22-27.
10. H. A. Kenary, D.Y. Shin, J. R. Lee and H. Hoseini, Fixed Point and Hyers-Ulam-Rassias stability of functional equations, (5) (37) (2011), 1827-1833.

11. I. S.chang and H. M. Kim, On the Hyers-Ulam-Rassias stability of quadratic functional equations, J.Ineq.Pure and Appl. Math., (3) (3) (2002), 1-12.
12. K. Mengar, Statistical metrics, Proc. Nat. Acad. Sci., (28) (1942), 535-537.
13. K. Menger, Statistical metrics, Porceedings of the National Academy of Sciences of the United States of America, (28) (1942), 535-537.
14. K. W. Jun And H. M. Kim, On the Hyers-Ulam-Rassias stability of a general cubic functional equation, Math. Inequal.Appl., (6) (1) (2003), 87-95.
15. M. Kumar and Ashish, Hyers-Ulam-Rassias stability of quadratic functional equations in 2-Banach spaces, Int. J. Comp. Appl., (63) (8) (2013), 1-4.
16. N. Eghbali and M. Ganji, Hyers-Ulam-Rassias stability of a Quadratic functional equations in Menger probabilistic normed space, J. of Appl. Anal. and Comp., (2) (2) (2012), 149-159.
17. P. Gvavruta, A generalization of them Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. (184) (3) (1994), 431-436.
18. S. M. Ulam, Problems in Modren Mathematics, Science ed., John Wiley and Sons, New York, (1960).
19. Th. M. Rassias, On the stability of linear mapping in Banach spaces, Proc. Amer. Math. Soc., (72) (1978), 297-300.
20. W. Fechner, Stability of a functional inquality associated with the Jordan-von Neumann functional equation, Aequationes Math. (71) (1-2) (2006), 149-161.
21. W. G. Park, approximate additive mapping in 2-Banach spaces and related topics, J. Math. Anal. Appl. (376) (2011), 193-202.